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On the Wold-type decomposition of a pair of commuting isometries

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Abstract. In the paper the Wold-type decompositions of a pair of commuting isometries are considered. Some results about the existence of this decomposition and about connections beetwen it and the Wold decomposition of the discrete semi-group generated by these isometries are given. The uniqueness of this decomposition (if it exists) is proved and an example of the pair of commuting isometries without this decomposition is showed.

In what follows, H is a complex Hilbert space with inner product (x, y); $x, y \in H$, and norm $\|x\| = \sqrt{(x, x)}$; $x \in H$. $L(H)$ denotes the algebra of all linear bounded operators (shortly, operators) on H. For $T \in L(H)$, T^* is the adjoint of T. I_H, or shortly I, denotes the identity operator. $T|_K$ is the restriction of the operator T to the subspace K. Let S be a semi-group. The map $T\colon S \to L(H)$ is called a *semi-group of operators* if $T(t+s) = T(t)T(s)$ for every $t, s \in S$. Let $V(s)$, $s \in S$, be an abelian semi-group of isometries on H. The semi-group $V(s)$ is called:

(a) *unitary* if $V(s)$ is a unitary operator for every $s \in S$,

(b) *of type "e"* if $H = \bigvee_{s_2 - s_1 \notin -S} V(s_1)^* V(s_2) H$ and $V(s)$ has not the unitary part,

(c) *of type "s"* if there is a wandering subspace L for this semi-group such that

$$H = \bigoplus_{s \in S} V(s) L.$$

(The subspace L is called *wandering for a semi-group* $T(s)$, $s \in S$, if $T(s_1)L \perp T(s_2)L$ for every $s_1 \neq s_2$, $s_1, s_2 \in S$.)

Let G be an abelian group and S its subsemi-group such that $S \cap -S = \{0\}$. Then, as was proved by Suciu in [2], for every semi-group of isometries $V(s)$, $s \in S$, the space $L = [\bigvee_{s_2 - s_1 \notin -S} V(s_1)^* V(s_2) H]^{\perp}$ is its wandering subspace and there is the unique decomposition $H = H_u \oplus H_e \oplus H_s$ such that the spaces H_u, H_e, H_s reduce $V(s)$ for every $s \in S$ and $V(s)|_{H_u}$ is an unitary semi-group, $V(s)|_{H_e}$ is of type "e", $V(s)|_{H_s}$ is of type "s".

Moreover, we have the identities:

$$H_u = \{x \in H\colon \|V(s)^* x\| = \|x\| \text{ for every } s \in S\}, \tag{1}$$

$$H_s = \bigoplus_{s \in S} V(s) L. \tag{2}$$

The above decomposition will be called the *Suciu decomposition* of the semi-group of isometries. It is easy to see that the Wold decomposition of one isometry V is the same as the Suciu decomposition of the semi-group $V(n) = V^n$, where $n = 0, 1, 2, \ldots$ Note that in this case $H_e = \{0\}$.

Now suppose that V_1 and V_2 are commuting isometries on the space H. The Wold decomposition of a single isometry suggests the following definition: The decomposition $H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$ will be called the *Wold decomposition* of the pair of isometries V_1 and V_2 if the following conditions are satisfied:

$$\text{the spaces } H_{uu}, H_{us}, H_{su}, H_{ss} \text{ reduce } V_1 \text{ and } V_2, \tag{3}$$

$$V_1|_{H_{uu}} \text{ and } V_2|_{H_{uu}} \text{ are unitary operators}, \tag{4}$$

$$V_1|_{H_{us}} \text{ is unitary and } V_2|_{H_{us}} \text{ is a shift}, \tag{5}$$

$$V_1|_{H_{su}} \text{ is a shift and } V_2|_{H_{su}} \text{ is unitary}, \tag{6}$$

$$V_1|_{H_{ss}} \text{ and } V_2|_{H_{ss}} \text{ are shifts}. \tag{7}$$

This definition gives rise to the following questions:

Has every pair of commuting isometries a Wold decomposition? If it exists, is this decomposition unique?

Are there any connections between the Wold decomposition of the pair of isometries V_1 and V_2, and the Suciu decomposition of the semi-group $V(n, m) = V_1^n V_2^m$?

We begin from the following easy observation:

Remark 1. Suppose that the decomposition $H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$ is the Wold decomposition of a pair of isometries V_1 and V_2. If $H_1 = H_{uu} \oplus H_{us}$ and $H_2 = H_{su} \oplus H_{ss}$, then the decomposition $H = H_1 \oplus H_2$ is the Wold decomposition of the single isometry V_1.

Now we can prove

PROPOSITION 1. *If a Wold decomposition of a pair of commuting isometries exists, then it is unique.*

Proof. Suppose we have two decompositions $H = \bigoplus_{i=1}^{4} H_i$ and $H = \bigoplus_{i=1}^{4} H_i'$. Evidently, it is sufficient to show that $H_i \subset H_i'$ for $i = 1, \ldots, 4$. It follows, by Remark 1, that

$$H_1 \oplus H_2 = H_1' \oplus H_2' \perp H_3' \oplus H_4' = H_3 \oplus H_4. \tag{8}$$

If we apply Remark 1 to the second isometry, we get

$$H_1 \oplus H_3 = H_1' \oplus H_3' \perp H_2' \oplus H_4' = H_2 \oplus H_4. \tag{9}$$

Now, by (8) we have $H_1 \perp H_3'$ and by (9) $H_1 \subset H_1' + H_3'$. Consequently, $H_1 \subset H_1'$. Analogously we prove $H_i \subset H_i'$ for $i = 2, 3, 4$.

Consider the semi-group $V(n, m) = V_1^n V_2^m$, $n, m = 0, 1, \ldots$, where V_1 and V_2 are commuting isometries on the space H. We need some information about the Suciu decomposition of this semi-group. First we prove the following:

PROPOSITION 2. *Let the semi-group $V(n, m)$ be defined as above. Suppose that the decomposition $H = H_u \oplus H_e \oplus H_s$ is the Suciu decomposition of $V(n, m)$ and L is the wandering subspace for $V(n, m)$ such that $H_s = \bigoplus_{n,m\geqslant 0} V(n, m)L$. If $L_i = H \ominus V_i H$ $(i = 1, 2)$, then $L \subset L_1 \cap L_2$, $\bigoplus_{n=0}^{\infty} V_1^n L \subset L_2$ and $\bigoplus_{m=0}^{\infty} V_2^m L \subset L_1$. Moreover, if a space $L' \subset L_1 \cap L_2$ is such that $\bigoplus_{n=0}^{\infty} V_1^n L' \subset L_2$ or $\bigoplus_{m=0}^{\infty} V_2^m L \subset L_1$, then L' is a wandering subspace for $V(n, m)$.*

Proof. Let Z_+^2 denote the semi-group $Z_+^2 = \{(n, m)\colon n, m \in Z, n, m \geqslant 0\}$ with addition. Then (see [2])

$$\begin{aligned}
L &= [\bigvee_{(n,m)-(p,q)\notin -Z_+^2} V_1^{*p} V_2^{*q} V_1^n V_2^m H]^\perp = [\bigvee_{\substack{n,p,q,m\in Z\\ n-p>0 \text{ or } m-q>0}} V_1^{*p} V_2^{*q} V_1^n V_2^m H]^\perp \\
&= [\bigvee_{\substack{n,m,p,q\in Z\\ n-p>0}} V_1^{*p} V_2^{*q} V_1^n V_2^m H \vee \bigvee_{\substack{n,p,q,m\in Z\\ m-q>0}} V_1^{*p} V_2^{*q} V_1^n V_2^m H]^\perp \\
&= [\bigvee_{\substack{n,p,q,m\in Z\\ n-p>0}} V_2^{*q} V_1^{*p} V_1^n V_2^m H \vee \bigvee_{\substack{n,p,q,m\in Z\\ m-q>0}} V_1^{*p} V_2^{*q} V_2^m V_1^n H]^\perp \\
&= [\bigvee_{\substack{n,p,q,m\in Z\\ n-p>0}} V_2^{*q} V_1^{n-p} V_2^m H \vee \bigvee_{\substack{n,p,q,m\in Z\\ m-q>0}} V_1^{*p} V_2^{m-q} V_1^n H]^\perp \\
&= [\bigvee_{r,q,m\in Z, r>0} V_2^m V_2^{*q} V_1^r H \vee \bigvee_{s,p,n\in Z, s>0} V_1^{*p} V_1^n V_2^s H]^\perp.
\end{aligned}$$

Consequently, if $x \in L$, then

$x \perp V_2^{*q} V_2^m V_1^r H$ for $q, m, r \in Z$, $r > 0$ and

$x \perp V_1^{*p} V_1^n V_2^s H$ for $p, n, s \in Z$, $s > 0$.

If we put $r = s = 1$ and $n = m = p = q = 0$, we get $x \perp V_1 H$ and $x \perp V_2 H$. Hence, by definitions of L_1 and L_2, $x \in L_1 \cap L_2$. If we put $n = 0$, $r = 1$ and $q > 0$, then we get $x \perp V_2^{*q} V_1 H$. It follows $V_2^q x \perp V_1 H$ for $q > 0$ and consequently $\bigoplus_{n=0}^{\infty} V_2^n L \subset L_1$. Analogously we prove that $\bigoplus_{m=0}^{\infty} V_1^m L \subset L_2$.

Now, let L' be a subspace of H such that $L' \subset L_1 \cap L_2$ and $V_1^n L' \subset L_2$ for every $n \geqslant 0$. Suppose that $x, y \in L'$. Then $x \in L_2$ and $V_1^n y \in L_2$ for every $n \geqslant 0$. Consequently, for every $m > 0$, $V_2^m V_1^n y \in V_2^m L_2$. Since L_2 is the wandering subspace for V_2, we get $(V_1^n V_2^m y, x) = 0$ for $m > 0$, $n \geqslant 0$ and $x, y \in L'$. If $m = 0$ and $n > 0$ we have $(V_1^n V_2^m y, x) = (V_1^n y, x) = 0$ because $V_1^n y \in V_1^n L$ and $x \in L' \subset L_1$. Consequently, $V_1^n V_2^m L' \perp L'$ for every $n, m \geqslant 0$ and $n+m > 0$. Since V_1 and V_2 are commuting isometries, this implies that L' is a wandering subspace for the semi-group $V(n, m)$, which finishes the proof.

As an immediate consequence of this proposition we have

COROLLARY 1. *Suppose V_1 and V_2 are commuting isometries on the space H. If $L_1 \cap L_2 = \{0\}$, where $L_i = H \ominus V_i H$ $(i = 1, 2)$, then the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$ has not the part of type "s".*

COROLLARY 2. *If U is a unitary operator which commutes with an isometry V, then the semi-group $\{V^n U^m\}_{n,m\geqslant 0}$ has not the part of type "s".*

Now, we shall prove the characterization of semi-groups of type "s".

THEOREM 1. *Suppose that V_1 and V_2 are commuting isometries on the space H and write $L_i = H \ominus V_i H$ $(i = 1, 2)$. Then the following conditions are equivalent:*

(i) *the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$ is of type "s",*

(ii) *V_1 and V_2 are doubly commuting shifts,*

(iii) *V_2 is a shift and $L_2 = \bigoplus_{n=0}^{\infty} V_1^n(L_1 \cap L_2)$ or V_1 is a shift and $L_1 = \bigoplus_{m=0}^{\infty} V_2^m(L_1 \cap L_2)$,*

(iv) *$L_1 \cap L_2$ is a wandering subspace for the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$ and $H = \bigoplus_{n,m\geqslant 0}^{\infty} V_1^n V_2^m(L_1 \cap L_2)$.*

Proof. (i) $\Rightarrow$ (ii). Let L be a wandering subspace for the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$ such that $H = \bigoplus_{n,m\geqslant 0} V_1^n V_2^m L$. Let us define $L_1' = \bigoplus_{m=0}^{\infty} V_2^m L$ and $L_2' = \bigoplus_{n=0}^{\infty} V_1^n L$. Then $H = \bigoplus_{n=0}^{\infty} V_1^n L_1' = \bigoplus_{m=0}^{\infty} V_2^m L_2'$. It follows that V_1 and V_2 are commuting shifts and $L_i' = L_i$ $(i = 1, 2)$. Now we shall show that V_1^* commutes with V_2.

If $x \in H$, then $x = \sum_{m=0}^{\infty} V_1^m x_m$, where $x_m \in L_1$. Consequently

$$V_1^* V_2 x = \sum_{m=0}^{\infty} V_1^* V_2 V_1^m x_m = \sum_{m=0}^{\infty} V_1^* V_1^m V_2 x_m = \sum_{m=1}^{\infty} V_1^{m-1} V_2 x_m + V_1^* V_2 x_0.$$

We shall show that $V_1^* V_2 x_0 = 0$. Since $x_0 \in L_1 = \bigoplus_{m=0}^{\infty} V_2^m L$, we have

$V_2x_0 \in L_1$. It follows, by the definition of L, that $0 = (V_2x_0, V_1y) = (V_1^* V_2x_0, y)$ for every $y \in H$. Consequently $V_1^* V_2x_0 = 0$. On the other hand,

$$V_2 V_1^* x = \sum_{m=0}^{\infty} V_2 V_1^* V_1^m x_m = \sum_{m=1}^{\infty} V_2 V_1^{m-1} x_m + V_2 V_1^* x_0$$
$$= \sum_{m=1}^{\infty} V_1^{m-1} V_2 x_m + V_2 V_1^* x_0.$$

But $x_0 \in L_1$. It follows that for every $y \in H$ we have $(V_1^* x_0, y) = (x_0, V_1 y) = 0$. Then $V_2 V_1^* x_0 = 0$ and consequently V_1^* and V_2 commute.

(ii) $\Rightarrow$ (iii). We shall prove that $L_2 = \bigoplus_{n=0}^{\infty} V_1^n(L_1 \cap L_2)$. The second assertion can be obtained in the same way. First we shall show that L_2 reduces V_1. Let $x \in L_2$. Then $V_2^* x = 0$ and consequently for every $y \in H$ we have

$$(V_1 x, V_2 y) = (V_2^* V_1 x, y) = (V_1 V_2^* x, y) = 0$$

and

$$(V_1^* x, V_2 y) = (V_2^* V_1^* x, y) = (V_1^* V_2^* x, y) = 0.$$

It follows that $V_1^* x \in L_2$ and $V_1 x \in L_2$. Consequently L_2 reduces V_1. Hence $V_1^n(L_1 \cap L_2) \subset L_2$ for every $n \geqslant 0$. Evidently, $L_1 \cap L_2$, regarded as a subspace of L_1, is a wandering subspace for V_1. Then we have $\bigoplus_{n=0}^{\infty} V_1^n(L_1 \cap L_2) \subset L_2$.

Let $L = L_2 \ominus V_1 L_2$. If we prove that $L \subset L_1 \cap L_2$, then we get $L_2 = \bigoplus_{n=0}^{\infty} V_1^n L \subset \bigoplus_{n=0}^{\infty} V_1^n(L_1 \cap L_2) \subset L_2$, which finishes this part of the proof. Suppose that $x \in L$. Then $x \perp V_1 L_2$ and consequently $V_1^* x \perp L_2$. On the other hand $x \in L_2$. Since L_2 reduces V_1, we have $V_1^* x \in L_2$. This implies that $V_1^* x = 0$ and so $x \in L_1$. Since $x \in L_2$, our proof is complete.

(iii) $\Rightarrow$ (iv). Suppose that the first condition of (iii) is fulfilled. Since V_2 is a shift, we have $H = \bigoplus_{m=0}^{\infty} V_2^m L_2$. Then $H = \bigoplus_{m=0}^{\infty} V_2^m \big(\bigoplus_{n=0}^{\infty} V_1^n(L_1 \cap L_2)\big) = \bigoplus_{m,n \geqslant 0} V_1^n V_2^m (L_1 \cap L_2)$. In the second case the proof is the same.

(iv) $\Rightarrow$ (i). (i) follows (iv) immediately by the definition of a semi-group of type "s".

Now we can give an answer to the problem of connections between the Wold decomposition of a pair of commuting isometries and the Suciu decomposition of the semi-group generated by these isometries

THEOREM 2. *Suppose that a pair of commuting isometries V_1 and V_2 on the space H has the Wold decomposition $H = H_{uu} \oplus H_{us} H \oplus_{su} \oplus H_{ss}$.*

Let $H = H_u \oplus H_e \oplus H_s$ *be the Suciu decomposition of the semi-group* $\{V_1^n V_2^m\}_{n,m\geqslant 0}$. *Then*

$$H_{uu} = H_u, \quad H_{us}\oplus H_{su} \subset H_e \quad \text{and} \quad H_s \subset H_{ss}.$$

Moreover, $H_s = H_{ss}$ *if and only if* V_1 *and* V_2 *doubly commute.*

Proof. First we prove that $H_u = H_{uu}$. H_u reduces the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$ to the unitary semi-group. In particular, H_u reduces V_1 and V_2, and the operators $V_1|_{H_u}$, $V_2|_{H_u}$ are unitary. Since the Wold decomposition is unique (Proposition 1), we get $H_u \subset H_{uu}$. On the other hand, H_{uu} reduces the operators V_1 and V_2 to the unitary operators. Consequently the semi-group $\{V_1^n V_2^m|_{H_{uu}}\}_{n,m\geqslant 0}$ is unitary. Now, since the Suciu decomposition is unique, we get $H_{uu} \subset H_u$. Thus $H_u = H_{uu}$.

Now, it is easy to see that H_{us} reduces the semi-group $\{V_1^n V_2^m\}_{n,m\geqslant 0}$. As regards the semi-group $\{V_1^n V_2^m|_{H_{us}}\}_{n,m\geqslant 0}$, then by Corollary 2 we see that it is of type "e". This shows that $H_{us} \subset H_e$. Analigously we prove that $H_{su} \subset H_e$. Consequently $H_{us}\oplus H_{su} \subset H_e$.

The semi-group $\{V_1^n V_2^m|_{H_s}\}_{n,m\geqslant 0}$ is of type "s". It follows by Theorem 1 that $V_1|_{H_s}$ and $V_2|_{H_s}$ are commuting shifts. This implies that $H_s \subset H_{ss}$.

Now suppose that $H_{ss} = H_s$. It follows by Theorem 1 that V_1 and V_2 doubly commute on the space H_{ss}. It is easy to see that every operator which commutes with a unitary operator must commute with its adjoint. Hence V_1 and V_2 doubly commute on the spaces H_{uu}, H_{us} and H_{su}. This shows that V_1 and V_2 doubly commute.

Now, if V_1 and V_2 doubly commute, then $V_1|_{H_{ss}}$ and $V_2|_{H_{ss}}$ are doubly commuting shifts. It follows by Theorem 1 that the semi-group $\{V_1^n V_2^m|_{H_{ss}}\}_{n,m\geqslant 0}$ is of type "s". Consequently $H_{ss} \subset H_s$, which together with the inclusion $H_s \subset H_{ss}$ finishes the proof.

Now we shall consider the problem of the existence of the Wold decomposition of a pair of commuting isometries. Before that, we shall prove a lemma which is the key result in this part of the paper.

LEMMA 1. *Let* V *be an isometry on* H *with the Wold decomposition* $H = H_u \oplus H_s$. *Then* H_u *is an invariant subspace for every* $T \in L(H)$ *which commutes with* V.

Proof. Since V is unitary on H_u, we have $h = VV^*h$ for $h \in H_u$. It follows that for $h \in H_u$, $Th = TVV^*h = VTV^*h$. This implies that $V^*Th = V^*VTV^*h = TV^*h$ for every $h \in H_u$. It is easy to show by induction that $V^{*n}Th = TV^{*n}h$ for every $h \in H_u$ and $n = 0, 1, \ldots$ Consequently, for $h \in H_u$ and $n, m = 0, 1, \ldots$ we get

$$V^m V^{*n} Th = V^m T V^{*n} h = T V^m V^{*n} h$$

and

$$V^{*n} V^m Th = V^{*n} T V^m h = T V^{*n} V^m h.$$

Since the projection P_{H_u} is in the von Neumann algebra generated by V, we get in limit $P_{H_u}Th = TP_{H_u}h = Th$. Hence $TH_u \subset H_u$ and our proof is complete.

COROLLARY 3. *If T doubly commutes with V, then H_u reduces T.*

Now we can prove

PROPOSITION 3. *A pair of commuting isometries V_1 and V_2 has the Wold decomposition if and only if the space H_i reduces V_i $(i = 1, 2)$, where the decompositions $H = H_1 \oplus K_1$ and $K_1 = H_2 \oplus K_2$ are the Wold decompositions of the single isometries V_2 and $V_1|_{K_1}$, respectively.*

Proof. Suppose that H_i reduces V_i $(i = 1, 2)$. Let $H_1 = H_u \oplus H_s$ be the Wold decomposition of the single isometry $V_1|_{H_1}$. Since V_2 is unitary on H_1, $V_1|_{H_1}$ and $V_2|_{H_1}$ doubly commute. Hence, by Corollary 3, H_u reduces $V_2|_{H_1}$. Consequently H_u and H_s reduce V_1 and V_2. Evidently $V_1|_{H_u}$, $V_2|_{H_u}$ and $V_2|_{H_s}$ are unitary and $V_1|_{H_s}$ is a shift. It follows by our assumptions that H_2 and K_2 reduce V_1 and V_2. Evidently $V_1|_{H_2}$ is unitary and $V_2|_{H_2}$, $V_1|_{K_2}$ and $V_2|_{K_2}$ are shifts. Then the decomposition $H = H_u \oplus H_s \oplus H_2 \oplus K_2$ is the Wold decomposition of the pair V_1 and V_2.

On the other hand, if $H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$ is the Wold decomposition of the pair V_1 and V_2, then it is easy to see (see Remark 1) that $H_2 = H_{us}$ and $H_1 = H_{uu} \oplus H_{su}$. Consequently H_i reduces V_i $(i = 1, 2)$ and our proof is complete.

Before we prove the positive answer to our problem, we give an example of commuting isometries which have not the Wold decomposition.

EXAMPLE 1. Let $\{e_{i,j}\}_{(i,j)\in I}$ be a sequence of orthonormal vectors in a Hilbert space K, where $I = \{(i, j): i \geqslant 0 \text{ or } j \geqslant 0\}$. Let H be the closed span of $\{e_{i,j}\}_{(i,j)\in I}$. Suppose that V_1 and V_2 are operators on H such that $V_1 e_{i,j} = e_{i+1,j}$ and $V_2 e_{i,j} = e_{i,j+1}$ for $(i, j) \in I$. It is easy to see that V_1 and V_2 are commuting isometries on H. Let the decomposition $H = H_u \oplus H_s$ be the Wold decomposition of V_2. By Proposition 3 it is sufficient to show that H_u does not reduce V_1. An easy computation shows that H_u is equal to the closed span of $\{e_{i,j}\}$, $i \geqslant 0$. It follows that $e_{0,1} \in H_u$ and $e_{-1,1} \notin H_u$. But $V_1^* e_{0,1} = e_{-1,1}$. Hence H_u does not reduce V_1.

Combining Corollary 3 with Proposition 3 we get:

THEOREM 3. *Every pair of doubly commuting isometries has the Wold decomposition.*

Let $H = H_1 \oplus H_2$ be the Wold decomposition of an isometry V and let E be the spectral measure of its unitary part. Suppose that m is the Lebesgue measure on the unit circle. It is a known fact that the spaces $H_s = \{x \in H_1: (Ex, x) \text{ is singular with respect to } m\}$ and $H_a = \{x \in H_1: (Ex, x) \text{ is absolutely continuous with respect to } m\}$ reduce V and $H_1 = H_s \oplus H_a$. We shall call $V|_{H_s}$ and $V|_{H_a}$ the singular and the unitary absolutely continuous part of V, respectively. Theorem 2.1 of [1] implies the following observation:

Remark 2. If $T \in L(H)$ and commutes with V, then H_s reduces T.

This remark and Proposition 3 implies

THEOREM 4. *Suppose that V_1 and V_2 are commuting isometries without unitary absolutely continuous part. Then the pair of V_1 and V_2 has the Wold decomposition.*

Our last positive result is the following:

THEOREM 5. *Let V_1 and V_2 be a pair of commuting isometries. Suppose that V_2 has not the unitary absolutely continuous part and its shift part has finite multiplicity. Then the pair of V_1 and V_2 has the Wold decomposition.*

Before the proof of this theorem we prove one more lemma.

LEMMA 2. *Suppose that U is a unitary operator on the space H such that $\sigma_p(U) = \{0\}$. If there is a shift S on H which has finite multiplicity and commutes with U, then $H = \{0\}$.*

Proof of Lemma 2. Suppose that $H \neq \{0\}$. Then $L = H \ominus SH \neq 0$ and $\dim L < \infty$. First we shall show that L is an invariant subspace for U. Let $h \in L$. Then $h \perp SH$. Since U is unitary, we have $Uh \perp USH = SUH = SH$. Consequently $Uh \in L$. Now, by our assumption that $\dim L < \infty$, we see that $U|_L$ is an operator on a finite dimensional space. It follows that there are $\lambda \in C$ and $h \in L$ such that $h \neq 0$ and $\lambda h = U|_L h = Uh$. Consequently λ is in the point spectrum of U an we have a contradiction.

Proof of Theorem 5. Let the decomposition $H = H_1 \oplus K_1$ be the Wold decomposition of V_2. Since V_2 has not the unitary absolutely continuous part, by Remark 2 we get that H_1 and K_1 reduce V_1. Let the decomposition $K_1 = H_2 \oplus K_2$ be the Wold decomposition of $V_1|_{K_1}$. It follows by Proposition 3 that it is sufficient to show that H_2 reduces V_2. We shall prove that $V_1|_{K_1}$ has not the unitary absolutely continuous part; this, on account of Remark 2, will finish the proof. According to Remark 2 we can assume without loss of generality that $V_1|_{K_1}$ has not the singular part. Now, by Lemma 1 the space H_2 is invariant for $V_2|_{K_1}$. Evidently, if $H_2 \neq \{0\}$, then $V_2|_{H_2}$ is a shift of finite multiplicity. But $V_1|_{H_2}$ is absolutely continuous. It follows that its point spectrum is empty and by Lemma 2 we have a contradiction. Consequently $H_2 = \{0\}$ and our proof is complete.

References

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