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On a class of differential equations in Banach space

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Abstract. In this paper we present some results on the Cauchy problem for the differential equation with lagged argument $x'(t) = f(t, x(g(t)))$ in Banach spaces. We assume that the function f satisfies some regularity conditions expressed in terms of the measure of non-compactness. We prove a theorem on the existence of a solution, a theorem on the existence of an extremal integral and a theorem on the continuous dependence of the extremal integral on initial data. The results extend those of the previous work [11].

In this paper we study the Cauchy problem for the differential equation with lagged argument $x'(t) = f(t, x(g(t)))$ in an infinitely dimensional Banach space. We deal with the problem using a method developed by Ambrosetti [1]. This method is based on the properties of the function $\mathscr{L}(\cdot)$ introduced by Kuratowski, which is a kind of "measure of non-compactness". The results of this paper extend the results of [11] and [12].

It is well known that neither the continuity, nor even the uniform continuity of the function f, does imply the existence of a solution of the Cauchy problem for the equation $x' = f(t, x)$ in a Banach space. In papers [1], [3], [4], [8] and [14] several existence theorems are proved in the cases where the bounded continuous function f satisfies some regularity conditions expressed in terms of the measure of non-compactness. In particular, an Ambrosetti type condition (termed here assumption (A)) will be used in this paper.

0. Let E be a Banach space and let X be a bounded subset of E. We define $\mathscr{L}(X)$ as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leqslant \varepsilon$ ([9], [10], p. 318). (The number $\mathscr{L}(X)$ is called the *measure of non-compactness of the set* X.) For properties of $\mathscr{L}(\cdot)$ see [1], [6], [7] and [8]. In particular, we have:

1° if $A \subset B$, then $\mathscr{L}(A) \leqslant \mathscr{L}(B)$;

2° $\mathscr{L}(A+B) \leqslant \mathscr{L}(A) + \mathscr{L}(B)$;

3° $\mathscr{L}(\operatorname{conv} A) = \mathscr{L}(A) = \mathscr{L}(\bar{A})$;

4° $\mathscr{L}\big(\bigcup_{0\leqslant q\leqslant h} q\cdot A\big) = h\cdot\mathscr{L}(A)$;

5° $\mathscr{L}(A) \leqslant \sup\{\|x\|:\ x\in A\}$;

6° $\mathscr{L}(A) = 0$ if and only if A is conditionally compact;

7° (theorem of Kuratowski) if $A_{n+1}\subset A_n$ for $n = 1, 2, \ldots$ and $\mathscr{L}(A_n)\to 0$ as $n\to\infty$, then $\bigcap_{n=1}^{\infty} A_n$ is a non-empty conditionally compact subset of E.

Denote by $C(J, E)$ the space of all continuous functions from a compact interval J to a Banach space E, with the usual supremum norm $|||\cdot|||$. For $X\subset C(J, E)$ we write $X(t) = \{x(t):\ x\in X\}$ and $\int_{t_0}^{t} X(s)\,ds = \{\int_{t_0}^{t} x(s)\,ds:\ x\in X\}$.

If a subset X of $C(J, E)$ is bounded and all the functions belonging to X are equicontinuous, then:

8° (Ambrosetti [1]) $\mathscr{L}(X) = \mathscr{L}\big(\bigcup\{X(t):\ t\in J\}\big) = \sup\{\mathscr{L}(X(t)):\ t\in J\}$.

1. Let $(E, \|\cdot\|)$ be a Banach space, and let $I_0 = [t_0, t_0+a]$, $I = [t_0, t_0+h]$, where $0 < h \leqslant a$. Assume that $g\colon I_0\to(-\infty, \infty)$ is a continuous function such that $g(t)\leqslant t$ for every $t\in I_0$. Let $m = \min\{g(t):\ t\in I_0\}$ and let $\varphi\colon [m, t_0]\to E$ be a continuous functions. Write

$$J = [m, t_0+h], \qquad B = \{x\in E:\ \|x-\varphi(t_0)\|\leqslant K+b\},$$

where $b > 0$ and $K = \sup\{\|\varphi(t)-\varphi(t_0)\|:\ m\leqslant t\leqslant t_0\}$.

Let us consider the Cauchy problem

$$\text{(PC)}\qquad x'(t) = f\big(t, x(g(t))\big),\quad x(t) = \varphi(t)\quad \text{for } m\leqslant t\leqslant t_0,$$

where $f\colon I_0\times B\to E$ is a given bounded continuous function, and $M = \sup\{\|f(t, x)\|:\ (t, x)\in I_0\times B\}$.

The above assumptions concerning the sets I_0, I, J, E, B and the functions g, φ, f are valid throughout this paper and will not be repeated in formulations of particular theorems.

We introduce the following definitions:

Definition 1. Let ε be a positive number. A continuous function $v\colon J\to B$ is said to be an *ε-approximate solution* of the (PC) problem on the interval J, if it satisfies the following conditions:

(i) $v(t) = \varphi(t)$ for $m\leqslant t\leqslant t_0$;

(ii) $v(\cdot)$ has the right-hand derivative $D^+v(t)$ for $t_0\leqslant t < t_0+h$, and $v(t) = \varphi(t_0) + \int_{t_0}^{t} D^+v(s)\,ds$ for $t\in I$;

(iii) $\big\|D^+v(t) - f\big(t, v(g(t))\big)\big\| \leqslant \varepsilon$ for $t_0\leqslant t < t_0+h$.

DEFINITION 2. We call an *Euler polygonal line* for (PC) on J any function $v\colon J \to B$ of the form

$$v(t) = \begin{cases} \varphi(t) & \text{for } m \leqslant t \leqslant t_0, \\ \varphi(t_0)+(t-t_j)\cdot f\big(t_j, v(g(t_j))\big)+ & \\ \qquad + \sum_{k=0}^{j-1} (t_{j-k}-t_{j-k-1})\cdot f\big(t_{j-k-1}, v(g(t_{j-k-1}))\big) & \\ \qquad\qquad \text{for } t_j \leqslant t \leqslant t_{j+1},\ j = 0, 1, \ldots, m-1, & \end{cases}$$

where $t_0 < t_1 < \ldots < t_m = t_0+h$ is an arbitrary partition of the interval I.

DEFINITION 3. A continuous function $x\colon J \to B$ is said to be a *solution* of the (PC) problem on the interval J, if it is a differentiable function on I such that $x(t) = \varphi(t)$ for $m \leqslant t \leqslant t_0$ and $x'(t) = f\big(t, x(g(t))\big)$ for $t \in I$.

DEFINITION 4. By an *E-solution* of the (PC) problem on the interval J we mean a solution of (PC) which is the limit of a uniformly convergent (on J) sequence of Euler polygonal lines which are approximate solutions of this problem on J.

Moreover, we introduce the following assumption (cf. [1], [14]):

(A) *There exists a constant* $k \geqslant 0$ *such that* $\mathscr{L}(f[I_0 \times X]) \leqslant k\cdot\mathscr{L}(X)$ *for every subset* X *of* B.

2. It is known ([2], [3]) that for every $\varepsilon > 0$ there exists an ε-approximate solution of the Cauchy problem for the equation $x' = f(t, x)$. In [11] we proved the following theorem:

2.1. THEOREM. *Let* $h \leqslant \min\big(a, (b+K)/M\big)$ *and let the following condition be satisfied*:

(*) *there exists a compact subset* H *of* B *such that* $\varphi(t) \in H$ *for* $m \leqslant t \leqslant t_0$, *and* $\varphi(t_0)+(t-t_0)\cdot\operatorname{conv}(f[I \times H]) \subset H$ *for* $t \in I$.

Then for any $\varepsilon > 0$ *there exists an* ε*-approximate solution* $u_\varepsilon(\cdot)$ *of* (PC) *on* J *such that* $u_\varepsilon(t) \in H$ *for every* $t \in J$. *Moreover, this solution is an Euler polygonal line of the* (PC) *problem on* J.

For every positive integer n let us denote:

by S_n – the set of all $(1/n)$-approximate solutions of (PC) on the interval J;

by E_n – the set of all Euler polygonal lines $v \in S_n$ such that $v(t) \in H$ for every $t \in J$, where H is the set from Theorem 2.1.

It is easy to verify that all the functions belonging to S_n are uniformly bounded and equicontinuous. First, we prove

2.2. *Let condition* (*) *of Theorem* 2.1 *be satisfied. Then*:

(a) *If* $(u_k) \subset S_n$, $u_k[I] \subset H$ $(k = 1, 2, \ldots)$ *and* $|||u_k-u||| \to 0$ *as* $k \to \infty$;

then $u(t)=\varphi(t)$ *for* $m\leqslant t\leqslant t_0$ *and* $\left\|u(t)-\varphi(t_0)-\int_{t_0}^{t} f\big(s, u(g(s))\big)ds\right\| = n^{-1}|t-t_0|$ *for* $t\in I$.

(b) *If* $u\in\bigcap_{n=1}^{\infty}\overline{E}_n$, *then* $u(\cdot)$ *is an E-solution of* (PC) *on* J.

(c) *If* $E_n\neq\emptyset$ *for each* $n\geqslant 1$ *and* $\lim_{n\to\infty}\mathscr{L}(E_n)=0$, *then the set of all E-solutions of* (PC) *defined on* J *and taking values in the set* H *is a non-empty compact subset of* $C(J, E)$.

Proof. Obviously, $u(t)=\varphi(t)$ for $m\leqslant t\leqslant t_0$ and $u(t)\in H$ for $t\in I$. For each t in I we have

$$\begin{aligned}
&\left\|u(t)-\varphi(t_0)-\int_{t_0}^{t} f\big(s, u(g(s))\big)ds\right\| \\
&\qquad\leqslant |||u_k-u||| + \int_{t_0}^{t}\left\|D^+u_k(s)-f\big(s, u_k(g(s))\big)\right\|ds + \\
&\qquad\quad + \int_{t_0}^{t}\left\|f\big(s, u_k(g(s))\big)-f\big(s, u(g(s))\big)\right\|ds \\
&\qquad\leqslant |||u_k-u||| + n^{-1}|t-t_0| + \int_{t_0}^{t}\left\|f\big(s, u_k(g(s))\big)-f\big(s, u(g(s))\big)\right\|ds.
\end{aligned}$$

Since $f|_{I\times H}$ is a uniformly continuous function, we have $\lim_{k\to\infty}\left\|f\big(t, u_k(g(t))\big)-f\big(t, u(g(t))\big)\right\| \simeq 0$ for $t\in I$. Consequently,

$$\left\|u(t)-\varphi(t_0)-\int_{t_0}^{t} f\big(s, u(g(s))\big)ds\right\| \leqslant n^{-1}|t-t_0| \quad \text{for } t\in I.$$

The proof of (c) is similar to the proof given in [3] (cf. also [4], [15]), by the Kuratowski theorem, and 2.2 (b) can be easily obtained. This completes the proof of our theorem, since (a) implies (b).

Now we prove the following result:

2.3. *Let assumption* (A) *be satisfied,* $h\cdot k<1$, *and let* $S_n\neq\emptyset$ *for each* $n\geqslant 1$. *Then* $\mathscr{L}(S_n)\to 0$ *as* $n\to\infty$.

Proof. Using the Ambrosetti result we obtain

$$\mathscr{L}(S_n)=\sup\{\mathscr{L}(S_n(t)):\ t\in J\} = \mathscr{L}\big(\bigcup\{v[J]:\ v\in S_n\}\big).$$

Let us fix t in I. Obviously,

$$S_n(t) \subset \varphi(t_0) + \Big\{ \int_{t_0}^{t} \big[D^+ v(s) - f\big(s, v(g(s))\big)\big]\, ds \colon v \in S_n \Big\} + \\ + \Big\{ \int_{t_0}^{t} f\big(s, v(g(s))\big) ds \colon v \in S_n \Big\};$$

hence

$$\mathscr{L}(S_n(t)) \leqslant \mathscr{L}\Big(\Big\{ \int_{t_0}^{t} \big[D^+ v(s) - f\big(s, v(g(s))\big)\big]\, ds \colon v \in S_n \Big\}\Big) + \\ + \mathscr{L}\Big(\Big\{ \int_{t_0}^{t} f\big(s, v(g(s))\big) ds \colon v \in S_n \Big\}\Big).$$

We have

$$\mathscr{L}\Big(\Big\{ \int_{t_0}^{t} \big[D^+ v(s) - f\big(s, v(g(s))\big)\big]\, ds \colon v \in S_n \Big\}\Big) \\ \leqslant \sup \Big\{ \Big\| \int_{t_0}^{t} \big[D^+ v(s) - f\big(s, v(g(s))\big)\big]\, ds \Big\| \colon v \in S_n \Big\} \leqslant n^{-1} h$$

and, by the integral mean-value theorem and condition (A),

$$\mathscr{L}\Big(\Big\{ \int_{t_0}^{t} f\big(s, v(g(s))\big) ds \colon v \in S_n \Big\}\Big) \\ \leqslant \mathscr{L}\Big((t-t_0) \cdot \overline{\mathrm{conv}}\, \big(f[I_0 \times \bigcup \{v[J] \colon v \in S_n\}]\big)\Big) \\ \leqslant |t-t_0| \cdot \mathscr{L}\big(f[I_0 \times \bigcup \{v[J] \colon v \in S_n\}]\big) \\ \leqslant h \cdot k \cdot \mathscr{L}\big(\bigcup \{v[J] \colon v \in S_n\}\big) = h \cdot k \cdot \mathscr{L}(S_n).$$

Finally, we conclude that

$$\mathscr{L}(S_n) = \sup \{\mathscr{L}(S_n(t)) \colon t \in I\} \leqslant n^{-1} h + h \cdot k \cdot \mathscr{L}(S_n),$$

which completes the proof.

3. Let $r_0 \geqslant 0$, $0 \leqslant \varepsilon_0 \leqslant b$. Let us denote:

by Y — a conditionally compact set consisting of some $y \in E$ such that $\|y\| \leqslant r_0$;

by Γ — a conditionally compact set consisting of some $\psi \in C([m, t_0], E)$ such that $\sup \{\|\psi(t) - \varphi(t)\| \colon m \leqslant t \leqslant t_0\} \leqslant \varepsilon_0$.

By $(\mathrm{PC}_{y,\psi})$ we shall denote the problem of finding the solution of the equation

$$x'(t) = y + f\big(t, x(g(t))\big)$$

satisfying the initial condition

$$x(t) = \psi(t) \quad \text{for } m \leqslant t \leqslant t_0,$$

where $y \in Y$ and $\psi \in \Gamma$.

Let us denote by $\mathscr{X}$ the set of all solutions of the problem ($\mathrm{PC}_{y,\psi}$) on J with y ranging over Y and ψ ranging over Γ. We prove

3.1. THEOREM. *Let assumption* (A) *be satisfied, let* $h \leqslant \min\left(a, \dfrac{K+b-\varepsilon_0}{M+r_0}\right)$ *and* $0 \leqslant h \cdot k < 1$. *Then there exists an E-solution of problem* ($\mathrm{PC}_{y,\psi}$) *on the interval J. Moreover,* $\mathscr{X}$ *is a compact subset of* $C(J, E)$.

Proof. First, we prove our theorem for the case $\varepsilon_0 = r_0 = 0$. By a slight modification of the argument used in [5] (see [14]) we prove that there exists a compact set H satisfying condition (*) of Theorem 2.1: Indeed, let us put

$$R(X) = \varphi[P] + \overline{\bigcup_{0 \leqslant \lambda \leqslant h} \lambda \cdot \operatorname{conv}(f[I \times X])} \quad \text{for } X \subset B, \text{ where } P = [m, t_0],$$

$$\Omega = \{X \subset B\colon R(X) \subset X\} \quad \text{and} \quad H = \bigcap \{X\colon X \in \Omega\}.$$

As $\varphi[P]$ is a compact set, $R(X)$ is closed. We can verify that $R(B) \subset B$, $\varphi[P] \subset H$ and $R(H) = H$. Hence, by the properties of the measure of non-compactness, we conclude that H is compact. The set H satisfies the assumptions of Theorem 2.1 and therefore $E_n \neq \emptyset$ for $n \geqslant 1$. By 2.3 $\mathscr{L}(E_n) \to 0$ as $n \to \infty$. Consequently, the existence of an E-solution for (PC) follows from 2.2 (c).

Since the function $F_y(t, x) = y + f(t, x)$ satisfies on $I_0 \times B$ condition (A), it follows by the above reasoning that the E-solution for ($\mathrm{PC}_{y,\psi}$) exists on J.

Obviously, $\mathscr{X}$ is closed, bounded and all the functions belonging to $\mathscr{X}$ are equicontinuous. We have

$$\mathscr{X}(t) \subset \bigcup \{\psi[P]\colon \psi \in \Gamma\} \quad \text{for } t \in P$$

and

$$\mathscr{X}(t) \subset \{\psi(t_0)\colon \psi \in \Gamma\} + \\ + \bigcup_{0 \leqslant \lambda \leqslant h} \lambda \cdot Y + \bigcup_{0 \leqslant \lambda \leqslant h} \lambda \cdot \overline{\operatorname{conv}}\left(f\left[I \times \bigcup \{x[J]\colon x \in \mathscr{X}\}\right]\right)$$

for $t \in I$. From this and from the assumptions it follows (cf. the proof of 2.3) that $\mathscr{L}(\mathscr{X}) = 0$, and therefore $\mathscr{X}$ is compact.

4. Let S be a cone in E. Denote by $\leqslant$ the partial order in E generated by S. Assume, moreover, that S is a cone with a non-void interior and $f\colon I \times B \to E$ is a function such that $x_1 \leqslant x_2$ implies $f(t, x_1) \leqslant f(t, x_2)$.

This assumption remains valid throughout all this section and will not be repeated in formulations of particular theorems.

Modifying the proof from [13], Theorem 70.1, we obtain the following result on "strong" differential inequalities:

4.1. *Let the functions x, y from $[m, t_0+h)$ into B be continuous. Moreover, let*

$$x(t) \leqslant y(t) \quad (or,\ x(t) < y(t)\,(^1)) \quad for\ m \leqslant t \leqslant t_0,$$

$$D^+x(t) \leqslant f\big(t, x(g(t))\big) \quad for\ t_0 \leqslant t < t_0+h,$$

$$f\big(t, y(g(t))\big) < D^+y(t) \quad (or,\ f\big(t, y(g(t))\big) \leqslant D^+y(t)) \quad for\ t_0 \leqslant t < t_0+h.$$

Under our assumptions the inequality $x(t) < y(t)$ holds for every $t_0 < t < t_0+h$.

We introduce the definition: A solution $x^0(\cdot)$ of the problem $(PC_{y,\psi})$ on the interval J is called the *maximal integral* of $(PC_{y,\psi})$ on J, if for every solution $x(\cdot)$ of $(PC_{y,\psi})$ on J we have the inequality $x(t) \leqslant x^0(t)$ for $t_0 \leqslant t < t_0+h$.

In a similar way one can introduce the notion of the *minimal integral* of $(PC_{y,\psi})$ on J.

4.2. THEOREM. *Let assumption* (A) *be satisfied,* $r_0 > 0$, *and let* $h \leqslant \min\left(a, \dfrac{K+b}{r_0+M}\right)$, $0 \leqslant h \cdot k < 1$. *Then there exists a unique maximal (minimal) integral of* (PC) *on* J, *and this integral is the limit of a sequence of* E-*solutions of* (PC) *on* J.

Proof. Let (y_k) be a sequence in E converging to zero and such that $y_k > \theta$, $\|y_k\| \leqslant r_0$ $(k = 1, 2, \ldots)$. By Theorem 3.1, the problem

$$x'(t) = y_k + f\big(t, x(g(t))\big), \qquad x(t) = \varphi(t) \quad \text{for } m \leqslant t \leqslant t_0$$

possesses an E-solution $x_k(\cdot)$ in the interval J and $\{x_k(\cdot)\colon k \geqslant 1\}$ is a compact subset of $C(J, E)$.

Denote by $x(\cdot)$ a solution of (PC) on J, and let $(x_{k_n}(\cdot))$ be a subsequence of $(x_k(\cdot))$ such that $x_{k_n}(\cdot) \to x^0(\cdot)$ as $n \to \infty$. We have

$$x(t) = x_k(t) \quad (k = 1, 2, \ldots) \quad \text{for } m \leqslant t \leqslant t_0+h,$$

$$D^+x(t) = f\big(t, x(g(t))\big) \quad \text{for } t_0 \leqslant t < t_0+h$$

and it is easy to check that

$$f\big(t, x_k(g(t))\big) < D^+x_k(t) \quad (k = 1, 2, \ldots) \quad \text{for } t_0 \leqslant t < t_0+h.$$

Therefore, by Theorem 4.1, $x(t) < x_k(t)$ for every $k \geqslant 1$ and $t_0 < t < t_0+h$. Consequently, $x^0(\cdot)$ is a solution of (PC) on J such that $x(t) \leqslant x^0(t)$ for $t_0 \leqslant t < t_0+h$. By an analogous argument we obtain the assertion concerning the minimal integral of (PC) on J.

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(1) $x < y$ means $y - x \in \operatorname{Int}(S)$; the interior of S is denoted by $\operatorname{Int}(S)$.

The proof is finished.

For E-solutions the theorem on the continuous dependence of the maximal (minimal) integral of (PC) on the initial data is true. More precisely, we have the following theorem:

Let us put

$$\mathscr{V} = \{\psi \in C([m, t_0], E)\colon \varphi(t) < \psi(t) \text{ and } \|\psi(t) - \varphi(t)\| \leqslant \varepsilon_0 \text{ for } m \leqslant t \leqslant t_0\},$$

and for any $\psi \in \mathscr{V}$ denote the maximal integral of the problem

$$(+) \qquad x'(t) = f\big(t, x(g(t))\big), \quad x(t) = \psi(t) \quad \text{for } m \leqslant t \leqslant t_0$$

on the interval J by $x^0_\psi(\cdot)$.

4.3. THEOREM. *Let assumption* (A) *be satisfied*, $r_0 > 0$, $0 < \varepsilon_0 \leqslant b$, *and let* $h \leqslant \min\left(a, \dfrac{K+b-\varepsilon_0}{M+r_0}\right)$, $0 \leqslant h \cdot k < 1$. *Then the function* $\psi \to x^0_\psi$ *maps continuously* $\mathscr{V}$ *into* $C(J, E)$.

Proof. Let $(\psi_k(\cdot))$ be a sequence in $\mathscr{V}$ converging to $\varphi(\cdot)$. By Theorem 3.1, the set of all solutions of $(+)$ on J with $\psi(\cdot)$ ranging over $\{\psi_k(\cdot)\colon k \geqslant 1\}$, is compact in $C(J, E)$.

On account of Theorem 4.2, there exists a maximal integral $x^0(\cdot)$ of (PC) on J and a maximal integral $x^0_{\psi_k}(\cdot)$ of $(+)$ on J if $\psi = \psi_k$ $(k = 1, 2, \ldots)$. Let $(y^0_k(\cdot))$ be a subsequence of $(x^0_{\psi_k}(\cdot))$ and let $(y^0_{n_k}(\cdot))$ be a subsequence of $(y^0_k(\cdot))$ such that $y^0_{n_k}(\cdot) \to y_0(\cdot)$ as $k \to \infty$. To prove our theorem it is sufficient to show that $y_0(t) = x^0(t)$ for $m \leqslant t \leqslant t_0 + h$.

Obviously, $y_0(\cdot)$ is a solution of (PC) on J. Hence, $x^0(t) = y_0(t)$ for $m \leqslant t \leqslant t_0$, and $y_0(t) \leqslant x^0(t)$ for $t_0 \leqslant t < t_0 + h$. Since

$$\begin{aligned} &x^0(t) < y^0_{n_k}(t) && \text{for } m \leqslant t \leqslant t_0,\\ &D^+x^0(t) = f\big(t, x^0(g(t))\big) && \text{for } t_0 \leqslant t < t_0 + h,\\ &D^+y^0_{n_k}(t) = f\big(t, y^0_{n_k}(g(t))\big) && \text{for } t_0 \leqslant t < t_0 + h, \end{aligned}$$

we infer from Theorem 4.1 that $x^0(t) < y^0_{n_k}(t)$ for $t_0 < t < t_0 + h$. Hence it follows that $x^0(t) \leqslant y_0(t)$ for $t_0 < t < t_0 + h$, and thus the proof is completed.

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