ANNALES
POLONICI MATHEMATICI
XXV (1971)

On integro-differential equations of parabolic type

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Mlak [4] has shown that the first Fourier problem in a bounded domain for a semilinear equation of parabolic type has a maximum solution and a minimum solution. As a consequence he obtained a theorem on a weak differential inequalities.

In this paper we extend Mlak's results to the following system of parabolic integro-differential equations:

$$(0.1)\qquad L^k u^k \equiv \sum_{i,j=1}^{n} a_{ij}^k(x,t)u_{x_ix_j}^k + \sum_{i=1}^{n} b_i^k(x,t)u_{x_i}^k + c^k(x,t)u^k - u_t^k$$
$$= f^k\Big(x, t, u^1, \ldots, u^N, u_{x_1}^k, \ldots, u_{x_n}^k, \int_{G_t} u^1(y,t)\mu^1(x,t;dy), \ldots$$
$$\ldots, \int_{G_t} u^N(y,t)\mu^N(x,t;dy)\Big)\quad (k = 1, \ldots, N).$$

At first we prove a theorem on the existence of the maximum and minimum solutions for a more general system than (0.1) containing certain operators $B^k u$ on the right-hand side. The proof is based on a theorem (following from [7]) on the existence of a solution for the problem considered and on the assumption that operators $L^k u^k - B^k u$ fulfil certain strong inequalities. A theorem on weak inequalities follows from this proof.

The results mentioned above involve the system (0.1) as a particular case. Moreover, if functions f^k do not contain the derivatives $u_{x_i}^k$ $(i = 1, \ldots, n)$, then stronger versions of the theorems obtained for the general case (0.1) can be proved.

The author expresses his sincere gratitude to Professor P. Besala for suggesting the problem and for valuable advice.

1. Preliminaries. We shall use the notations of paper [7]. Here we recall only the definition of the domain G. Namely, by G we denote a bounded open domain of the Euclidean space of the variables $(x, t) = (x_1, \ldots, x_n, t)$ whose boundary consists of the domains R_0 and R_T of hyperplanes $t = 0$, $t = T = \text{const} > 0$, and of a surface S situated in the strip $0 \leqslant t \leqslant T$. The set $\Sigma = R_0 \cup S$ is called the *parabolic boundary* of the domain G.

The parabolic distance from P to $\Sigma^t = \Sigma \cap \{(x, \tau)\colon 0 \leqslant \tau \leqslant t\}$, where $P = (x, t)$ is any point in G, we denote by d_P, i.e.,

$$d_P = \inf_{Q \in \Sigma^t} d(P, Q).$$

For any points $P, P' \in G$ we put $d_{PP'} = \min(d_P, d_{P'})$.

Let us introduce the following norm:

$$\|u\|_{2+a}^G = \|u\|_a^G + \sum_{i=1}^n \|du_{x_i}\|_a^G + \sum_{i,j=1}^n \|d^2 u_{x_i x_j}\|_a^G + \|d^2 u_t\|_a^G \qquad (0 < a < 1),$$

where

$$\|d^m v\|_a^G = \sup_{P \in G} [(d_P)^m |v(P)|] + \sup_{P, P' \in G} \left\{ (d_{PP'})^{m+a} \frac{|v(P) - v(P')|}{[d(P, P')]^a} \right\}.$$

The set of all functions u for which $\|u\|_{2+a}^G < \infty$ will be denoted by $W_{2+a}(G)$.

Now we shall formulate two lemmas concerning the problem

$$(1.1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) u_{x_i x_j} + \sum_{i=1}^n b_i(x, t) u_{x_i} + c(x, t) u - u_t = f(x, t), \qquad (x, t) \in \bar{G} \setminus \Sigma,$$

$$(1.2) \qquad u(x, t) = \varphi(x, t), \qquad (x, t) \in \Sigma,$$

which follow from [1]. The following assumptions will be made:

(1.I) For any $(x, t) \in \bar{G}$ and $\xi \in E_n$ we have $a_{ij}(x, t) = a_{ji}(x, t)$,

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geqslant K_0 |\xi|^2 \qquad (K_0 > 0).$$

(1.II) There is such a constant $K_1 > 0$ that

$$|a_{ij}|_a^G, \ |b_i|_a^G, \ |c|_a^G, \ |a_{ij}|_{1-0}^S \leqslant K_1.$$

(1.III) The surface S belongs both to $\bar{\bar{C}}_{2+a}$ and to C_{2-0}.

(1.IV) The function $f(x, t)$ is of the class $C_a(G)$.

LEMMA 1 ([1], p. 69). *Let assumptions* (1.I)-(1.IV) *hold and let the function* $\varphi(x, t)$ *be continuous on* Σ. *Then there exists a unique solution* u *of problem* (1.1), (1.2) *and furthermore* $u \in W_{2+a}(G)$ (¹).

(¹) We have formulated Lemma 1 under stronger assumptions than those which follow from [1].

LEMMA 2. *Let assumptions* (1.I)-(1.IV) *be satisfied and suppose that* $u(x, t)$ *is a solution of the problem*

$$Lu = f(x, t) \text{ in } \bar{G}\setminus\Sigma, \quad u = 0 \text{ on } \Sigma.$$

Then for any β $(0 < \beta < 1)$ *there exists a constant* $K(\beta)$ *depending only on* β, K_0, K_1 *and* G *such that*

$$|u|_{1+\beta}^{G^\tau} \leqslant K(\beta)\tau^{(1-\beta)/2}|f|_0^{G^\tau},$$

where $G^\tau = G \cap \{(x, t)\colon 0 < t < \tau\}$, $0 < \tau \leqslant T$.

This lemma follows from the proof of Theorem 4 of [1] (p. 191).

2. Differential equations containing operators. For every $\tau \in (0, T]$ let B^k $(k = 1, \dots, N)$ be an operator defined on the set $C_{1,0}^N(G^\tau)$ of all vector-functions $u(x, t) = (u^1(x, t), \dots, u^N(x, t))$ continuous in $\bar{G}^\tau$ and possessing in $\bar{G}^\tau\setminus\Sigma^\tau$ continuous derivatives $u_{x_i} = (u_{x_i}^1, \dots, u_{x_i}^N)$ $(i = 1, \dots, n)$ with values belonging to the set of all functions defined in $\bar{G}^\tau\setminus\Sigma^\tau$.

In this section we shall prove the existence of the maximum and minimum solutions of the problem

$$L^k w^k = B^k w, \quad (x, t) \in \bar{G}^\tau\setminus\Sigma^\tau, \tag{2.1}$$

$$w^k(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma^\tau \ (k = 1, \dots, N) \tag{2.2}$$ ([2]).

The method employed in the proof requires the existence of a solution of this problem without assuming a certain consistency condition for the functions φ^k appearing in the theorems obtained in paper [7]. Therefore at first we state a suitable existence theorem for the above problem which will be applied in our proof.

The following assumptions are introduced for $k = 1, \dots, N$; $0 < \tau \leqslant T$ (comp. section 2 of [7]):

(2.I) The coefficients of L^k satisfy assumptions (1.I) and (1.II).

(2.II) The functions φ^k are of class $C_{1+\beta}(G)$ $(\alpha < \beta < 1)$.

(2.III) Operators B^k map the space $C_{1+\alpha}^N(G^\tau)$ into the set

$$\bigcup_{0<\varepsilon<1} C_\varepsilon(G^\tau)$$

and there are constants $A_1, A_2, A_3 \geqslant 0$, $0 \leqslant \lambda < 1$ (independent of τ) such that for any $u \in C_{1+\alpha}^N(G^\tau)$ the following inequality holds:

$$|B^k u|_0^{G^\tau} \leqslant A_1 + A_2(|u|_1^{G^\tau})^\lambda + A_3|u|_1^{G^\tau}$$

([2]) The solution $u(x, t)$ of (2.1), (2.2) is called a *maximum* (*minimum*) *solution* if for every solution $w(x, t)$ of (2.1), (2.2) the inequalities $w^k(x, t) \leqslant u^k(x, t)$ $(w^k(x, t) \geqslant u^k(x, t))$ hold in $\bar{G}^\tau$ $(k = 1, \dots, N)$.

(2.IV) Operators $B^k u$ are continuous in the space $C^N_{1+\alpha}(G^\tau)$ in the following sense: if $u, u_m \in C^N_{1+\alpha}(G^\tau)$ and $\lim_{m\to\infty} |u_m - u|^{G^\tau}_{1+\alpha} = 0$, then $\lim_{m\to\infty} |B^k u_m - B^k u|^{G^\tau}_0 = 0$.

THEOREM 1. *If assumptions* (1.III), (2.I)-(2.IV) *are satisfied and*

$$K(\alpha) N A_3 \tau^{(1-\alpha)/2} < 1 \quad (^3), \tag{2.3}$$

then there exists a solution $w(x, t) = \{w^k(x, t)\}$ *of the problem* (2.1), (2.2); *furthermore* $w \in C^N_{1+\beta}(G^\tau) \cap W^N_{2+\varepsilon}(G^\tau)$ (4), *where* $0 < \varepsilon < 1$ *is a certain constant.*

This theorem can be proved by the same considerations as those for Theorem 1 of [7], making use of Lemmas 1 and 2. As a consequence we obtain the following remark:

Remark. Under the assumptions of Theorem 1 there exists a solution w of problem (2.1), (2.2) which belongs to $C^N_{1+\beta}(G^\tau) \cap W^N_{2+\varepsilon}(G^\tau)$ and fulfils the inequality

$$|w|^{G^\tau}_{1+\beta} \leqslant \Big(NA_1 + NA_2 M_0^\lambda + NA_3 M_0 + \sum_{k=1}^{N} |L^k \Phi^k|^{G^\tau}_0\Big) K(\beta) \tau^{(1-\beta)/2} + |\Phi|^{G^\tau}_{1+\beta}, \tag{2.4}$$

where M_0 is the greater number of the following ones:

$$\left[\frac{2K(\alpha) N A_2 \tau^{(1-\alpha)/2}}{1 - K(\alpha) N A_3 \tau^{(1-\alpha)/2}}\right]^{1/(1-\lambda)},$$

$$\frac{2\big[K(\alpha)\tau^{(1-\alpha)/2}\big(NA_1 + \sum_{k=1}^{N} |L^k \Phi^k|^{G^\tau}_0\big) + |\Phi|^{G^\tau}_{1+\alpha}\big]}{1 - K(\alpha) N A_3 \tau^{(1-\alpha)/2}}.$$

Before stating a theorem on the existence of the maximum and minimum solutions we make the following assumption:

(2.V) If functions $u = (u^1, \ldots, u^N)$ and $v = (v^1, \ldots, v^N)$, regular in $\overline{G^\tau}$, satisfy the inequalities $L^k u^k - B^k u > L^k v^k - B^k v$, $(x, t) \in \overline{G^\tau} \setminus \Sigma^\tau$ $(k = 1, \ldots, N)$, $u(x, t) < v(x, t)$ (5), $(x, t) \in \Sigma^\tau$, then $u(x, t) < v(x, t)$ in $\overline{G^\tau}$.

THEOREM 2. *If the assumptions of Theorem* 1 *and* (2.V) *are fulfilled, then there exist a maximum solution* $v = \{v^k\}$ *and a minimum solution* $u = \{u^k\}$ *of problem* (2.1), (2.2); *moreover,* $v, u \in C^N_{1+\beta}(G^\tau) \cap W^N_{2+\varepsilon}(G^\tau)$ *for some* $0 < \varepsilon < 1$.

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(3) $K(\alpha)$ is the constant appearing in Lemma 2.

(4) $W^N_{2+\alpha}(G)$ denotes the set of all functions $w(x, t)$ with a finite norm $\|w\|^G_{2+\alpha} = \sum_{k=1}^{N} \|w^k\|^G_{2+\alpha}$.

(5) I.e. $u^k < v^k$ $(k = 1, \ldots, N)$.

Proof. We apply a method similar to that of Mlak [4]. Namely, in order to obtain the maximum solution v, let us consider for $m = 1, 2,$ the problem

$$L^k v_m^k = B^k v_m - \frac{1}{m}, \quad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau, \tag{2.5}$$

$$v_m^k(x, t) = \varphi^k(x, t) + \frac{1}{m}, \quad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N). \tag{2.6}$$

This problem possesses, by the remark to Theorem 1, a solution $v_m = \{v_m^k\}$ such that

$$|v_m|_{1+\beta}^{G^\tau} \leqslant M_1, \tag{2.7}$$

where M_1 denotes the expression on the right-hand side of (2.4) with

$$A_1, \ |L^k \Phi^k|_0^{G^\tau}, \ |\Phi|_{1+\alpha}^{G^\tau}, \ |\Phi|_{1+\beta}^{G^\tau}$$

replaced by

$$A_1 + 1, \ |L^k \Phi^k|_0^{G^\tau} + |c^k|_0^{G^\tau}, \ |\Phi|_{1+\alpha}^{G^\tau} + N, \ |\Phi|_{1+\beta}^{G^\tau} + N$$

respectively.

It follows from (2.7) and Theorem 4 of [1] (p. 188) that there exist a subsequence $\{v_{m'}\}$ of the sequence $\{v_m\}$ and a function $v \in C_{1+\beta}^N(G^\tau)$ such that

$$\lim_{m' \to \infty} |v_{m'} - v|_{1+\alpha}^{G^\tau} = 0. \tag{2.8}$$

We will show (in a similar way to that followed by Kusano [2]) that v is a solution of problem (2.1), (2.2). According to Lemma 1 the problem

$$L^k \bar{v}^k = B^k v, \quad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau, \tag{2.9}$$

$$\bar{v}^k(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N) \tag{2.10}$$

has a unique solution $\bar{v}$ which belongs to $W_{2+\varepsilon}^N(G^\tau)$ for some $0 < \varepsilon < 1$. Using relations (2.5), (2.6), (2.9), (2.10), we obtain

$$L^k\left(v_{m'}^k - \bar{v}^k - \frac{1}{m'}\right) = B^k v_{m'} - B^k v - \frac{1}{m'} c^k(x, t), \quad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau,$$

$$v_{m'}^k(x, t) - \bar{v}^k(x, t) - \frac{1}{m'} = 0, \quad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N)$$

and hence, by Lemma 2,

$$|v_{m'}^k - \bar{v}^k|_{1+\beta}^{G^\tau} \leqslant K(\beta)\left[|B^k v_{m'} - B^k v|_0^{G^\tau} + \frac{1}{m'}|c^k|_0^{G^\tau}\right] + \frac{1}{m'}. \tag{2.11}$$

By virtue of (2.8), (2.11) and (2.IV)

$$\lim_{m'\to\infty} |v_{m'} - \bar{v}|_{1+\beta}^{G^\tau} = 0$$

and thus $\bar{v} \equiv v$. This means, by (2.9), (2.10), that the vector function v is a solution of the problem in question (⁰).

It remains to prove that v is the maximum solution. Indeed, assuming that a function $w = \{w^k\}$ is a solution of the problem (2.1), (2.2) and taking into considerations relations (2.5), (2.6) and assumption (2.V), we have

$$w(x, t) < v_m(x, t), \qquad (x, t) \in \overline{G^\tau} \ (m = 1, 2, \ldots).$$

Thus $w(x, t) \leqslant v(x, t)$ in $\overline{G^\tau}$, which completes the proof in the case of the maximum solution.

To receive the minimum solution we consider, for $m = 1, 2, \ldots$, the problem

$$L^k u_m^k = B^k u_m + \frac{1}{m}, \qquad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau,$$

$$u_m^k(x, t) = \varphi^k(x, t) - \frac{1}{m}, \qquad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N).$$

The further proceeding is the same as in the proof of the existence of the maximum solution.

From the proof of Theorem 2 the following theorem on weak inequalities easily results.

THEOREM 3. *Let all the assumptions of Theorem 2 hold true. Suppose that a vector-function $w(x, t) = \{w^k(x, t)\}$, regular in $\overline{G^\tau}$, fulfils the following inequalities*:

$$L^k w^k \underset{(\leqslant)}{\geqslant} B^k w, \qquad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau \ (k = 1, \ldots, N),$$

$$w(x, t) \leqslant v(x, t) \qquad (w(x, t) \geqslant u(x, t)), \qquad (x, t) \in \Sigma^\tau.$$

Then $w(x, t) \leqslant v(x, t)$ $(w(x, t) \geqslant u(x, t))$ in $\overline{G^\tau}$, where $v = \{v^k\}$ $(u = \{u^k\})$ is the maximum (minimum) solution of problem (2.1), (2.2).

3. Differential equations containing functionals. For every $\tau \in (0, T]$ let $\Psi(x, t; u(\cdot, t))$ $((x, t) \in \overline{G^\tau} \setminus \Sigma^\tau,\ u = (u^1, \ldots, u^N) \in C_{1,0}^N(G^\tau))$ be a system of functionals

$$\Psi^1(x, t; u^1(\cdot, t)), \ldots, \Psi^N(x, t; u^N(\cdot, t)).$$

(⁰) It is easy to observe, in view of the obvious inequalities (see (2.5), (2.6))

$$L^k v_{m+1}^k - B^k v_{m+1} > L^k v_m^k - B^k v_m, \qquad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau,$$

$$v_{m+1}^k(x, t) < v_m^k(x, t), \qquad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N)$$

and assumption (2.V), that the sequence $\{v_m\}$ is decreasing. Therefore, it follows from the uniform convergence of the subsequence $\{v_{m'}\}$ that the sequence $\{v_m\}$ is also uniformly convergent to the solution v.

We shall derive corollaries from Theorems 2 and 3 for the operators B^k given by the formulas

(3.1) $$B^k u = f^k\big(x, t, u, u_x^k, \Psi\big(x, t; u(\cdot, t)\big)\big),$$

where $u_x^k = (u_{x_1}^k, \ldots, u_{x_n}^k)$ and functions $f^k(x, t, p, q, r)$ are defined on $\bar{G} \times E_{N+n+N}$.

We make the following assumptions ($k = 1, \ldots, N$):

(3.I) The functions $f^k(x, t, p, q, r)$ satisfy a uniform Hölder condition in every bounded set $\bar{G} \times H$ ($H \subset E_{N+n+N}$). Moreover, there exist constants $A_4, A_5, A_6 \geqslant 0$, $0 \leqslant \lambda < 1$ such that for any $(x, t, p, q, r) \in \bar{G} \times E_{N+n+N}$

$$|f^k(x, t, p, q, r)| \leqslant A_4 + A_5 |(p, q, r)|^\lambda + A_6 |(p, q, r)|,$$

where $|(p, q, r)| = \sum_{i=1}^{N} |p_i| + \sum_{j=1}^{n} |q_j| + \sum_{i=1}^{N} |r_i|$.

(3.II) The functions $f^k(x, t, p, q, r)$ are non-increasing with respect to the variables $p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_N, r_1, \ldots, r_N$.

(3.III) For any $z, \bar{z} \in C_{1+a}(G^\tau)$ we have

$$|\Psi^k\big(x, t; z(\cdot, t)\big) - \Psi^k\big(x, t; \bar{z}(\cdot, t)\big)|_0^{G^\tau} \leqslant A_7 |z - \bar{z}|_0^{G^\tau},$$

$A_7 \geqslant 0$ being a constant independent of τ.

(3.IV) If $z \in C_{1+a}(G^\tau)$, then the functions $\Psi^k\big(x, t; z(\cdot, t)\big)$ are uniformly Hölder continuous in G^τ.

(3.V) The functionals $\Psi^k\big(x, t; z(\cdot, t)\big)$ are non-decreasing with respect to the functions $z(x, t)$, regular in $\overline{G^\tau}$ (7).

THEOREM 4. *Let assumptions* (1.III), (2.I), (2.II), (3.I)-(3.V) *be satisfied and let*

(3.2) $$K(a) N A_6 (A_7 + 1) \tau^{(1-a)/2} < 1.$$

Then Theorem 2 holds true in case (3.1).

For the proof we need the following

LEMMA 3. *Let assumptions* (1.III), (3.II), (3.V) *and* (1.I) (*with* a_{ij} *replaced by* a_{ij}^k) (8) *be satisfied. Suppose that vector-functions* $u = (u^1, \ldots, u^N)$ *and* $v = (v^1, \ldots, v^N)$ *are regular in* $\overline{G^\tau}$ *and fulfil the inequalities*

$$L^k u^k - f^k\big(x, t, u, u_x^k, \Psi\big(x, t; u(\cdot, t)\big)\big) > L^k v^k - f^k\big(x, t, v, v_x^k, \Psi\big(x, t; v(\cdot, t)\big)\big),$$
$$(x, t) \in G^\tau \setminus \Sigma^\tau \ (k = 1, \ldots, N), \qquad u(x, t) < v(x, t), \qquad (x, t) \in \Sigma^\tau.$$

(7) I.e. if the functions $z(x, t), \bar{z}(x, t)$ are regular in $\overline{G^\tau}$ and $z(x, t) \geqslant \bar{z}(x, t)$, then $\Psi^k(x, t; z(\cdot, t)) \geqslant \Psi^k(x, t; \bar{z}(\cdot, t))$ ($k = 1, \ldots, N$).

(8) Assumptions (1.I), (1.III) can be replaced by weaker ones (see, for instance, [6], p. 191).

Under these assumptions $u(x,t) < v(x,t)$ *in* $\overline{G^\tau}$.

The method of proving this lemma is the same as that used to prove the theorem on strong differential inequalities (see [3] or [6], p. 191).

Now, by the above lemma, Theorem 4 results from Theorem 2.

As a consequence of Theorem 3 and Lemma 3 we obtain the following

THEOREM 5. *Let all the assumptions of Theorem 4 be fulfilled. Suppose that for a function* $w(x,t) = \{w^k(x,t)\}$, *regular in* $\overline{G^\tau}$, *we have the inequalities*

$$L^k w^k \underset{(\leqslant)}{\geqslant} f^k\big(x, t, w, w_x^k, \Psi(x, t; w(\cdot, t))\big), \quad (x,t)\in\overline{G^\tau}\setminus\Sigma^\tau \ (k = 1, \ldots, N),$$
$$w(x,t) \leqslant v(x,t) \quad (w(x,t) \geqslant u(x,t)) \ \text{on } \Sigma^\tau.$$

Under these assumptions $w(x,t) \leqslant v(x,t)$ $(w(x,t) \geqslant u(x,t))$ *in* $\overline{G^\tau}$, $v = \{v^k\}$ $(u = \{u^k\})$ *being the maximum (minimum) solution of problem* (2.1), (2.2) *in case* (3.1).

Now let $\Psi^k\big(x, t; z(\cdot, t)\big)$ $((x,t)\in\overline{G}\setminus\Sigma,\ 1 \leqslant k \leqslant N)$ be a functional defined on the set of all functions $z(x,t)$, continuous in $\overline{G}$ and let $f^k(x, t, p, r)$ $(1 \leqslant k \leqslant N)$ be a function defined on $\overline{G}\times E_{N+N}$.

At present, applying Chaplygin's method, we shall prove the existence of the maximum and minimum solutions of the problem

$$L^k w^k = f^k\big(x, t, w, \Psi(x, t; w(\cdot, t))\big), \quad (x,t)\in\overline{G}\setminus\Sigma, \tag{3.3}$$

$$w^k(x,t) = \varphi^k(x,t), \quad (x,t)\in\Sigma \ (k = 1, \ldots, N) \tag{3.4}$$

under weaker assumptions than those of Theorem 4.

The following assumptions are introduced $(k = 1, \ldots, N)$:

(3.VI) The functions $f^k(x, t, p, r)$ are non-increasing with respect to the variables $p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_N, r_1, \ldots, r_N$ and satisfy a uniform Hölder conditions in every bouded set $\overline{G}\times H$ $(H \subset E_{N+N})$.

(3.VII) If a function $z(x,t)$ is uniformly Hölder continuous in G, then also functions $\Psi^k\big(x, t; z(\cdot, t)\big)$ have this property (with exponents of Hölder continuity which may be different from that of the function $z(x,t)$).

(3.VIII) For any functions $z(x,t)$ and $\bar{z}(x,t)$, uniformly Hölder continuous in G, we have

$$\big|\Psi^k\big(x, t; z(\cdot, t)\big) - \Psi^k\big(x, t; \bar{z}(\cdot, t)\big)\big|_0^G \leqslant A_8 |z - \bar{z}|_0^G,$$

$A_8 \geqslant 0$ being a constant.

(3.IX) The functionals $\Psi^k\big(x, t; z(\cdot, t)\big)$ are non-decreasing in the set of all functions $z(x,t)$, regular in $\overline{G}$.

(3.X) There exist functions $u_0(x,t) = \{u_0^k(x,t)\}$ and $v_0(x,t) = \{v_0^k(x,t)\}$, regular and uniformly Hölder continuous in $\bar{G}$, such that

(3.5) $$L^k u_0^k > f^k\big(x, t, u_0, \Psi(x, t; u_0(\cdot, t))\big), \quad (x,t)\in\bar{G}\setminus\Sigma,$$

(3.6) $$L^k v_0^k < f^k\big(x, t, v_0, \Psi(x, t; v_0(\cdot, t))\big), \quad (x,t)\in\bar{G}\setminus\Sigma \ (k = 1, \ldots, N)$$

(3.7) $$u_0^k(x,t) < \varphi^k(x,t) < v_0^k(x,t), \quad (x,t)\in\Sigma.$$

THEOREM 6. *If assumptions* (1.III), (2.I), (2.II), (3.VI)-(3.X) *are satisfied, then problem* (3.3), (3.4) *has a minimum solution* $u = \{u^k\}$ *and a maximum solution* $v = \{v^k\}$. *Moreover*, $u_0 \leqslant u \leqslant v \leqslant v_0$ *and* $u, v \in C_{1+\beta}^N(G) \cap W_{2+\varepsilon}^N(G)$ *for some* ε, $0 < \varepsilon < 1$.

Proof. In view of assumptions (3.VII), (3.VIII) there is a constant $N_2 > 0$ such that if a function $z(x,t)$ is uniformly Hölder continuous in G and

$$|z|_0^G \leqslant N_1 = |u_0|_0^G + |v_0|_0^G,$$

then

$$\big|\Psi^k(x, t; z(\cdot, t))\big|_0^G \leqslant \big|\Psi^k(x, t; u_0^k(\cdot, t))\big|_0^G + A_8|z|_0^G + A_8|u_0^k|_0^G \leqslant N_2.$$

Let us put

$$H_1 = \{(p, r) : |p_k| \leqslant N_1, |r_k| \leqslant N_2 \ (k = 1, \ldots, N)\}.$$

By hypothesis, the functions $f^k(x, t, p, r)$ are uniformly Hölder continuous (with some exponent α') in $\bar{G}\times H_1$. Denote their Hölder coefficients by N_{3k}. Hence, and from the monotonicity of f^k it easily follows that for any

$$(x,t)\in\bar{G}, \quad (p, r), (\bar{p}, \bar{r})\in H_1, \quad (p, r) > (\bar{p}, \bar{r})$$

we have the inequalities

(3.8) $$f^k(x, t, p, r) - f^k(x, t, \bar{p}, \bar{r}) < \zeta_k(p, \bar{p}) = N_3|p_k - \bar{p}_k|^{\alpha'} \quad (k = 1, \ldots, N),$$

where N_3 is a constant greater than $\max_k N_{3k}$.

Now, using Theorem 1, we construct a sequence of vector-functions $u_m(x,t) = \{u_m^k(x,t)\}\in C_{1+\beta}^N(G)$ $(k = 1, \ldots, N;\ m = 1, 2, \ldots)$ by solving successively the problems

(3.9) $$L^k u_m^k = f^k\big(x, t, u_{m-1}, \Psi(x, t; u_{m-1}(\cdot, t))\big) + \zeta_k(u_m, u_{m-1}), \quad (x,t)\in\bar{G}\setminus\Sigma,$$

(3.10) $$u_m^k(x,t) = \varphi^k(x,t) - \eta/m, \quad (x,t)\in\Sigma,$$

where $\eta > 0$ is such a constant (existing by (3.7)) that

(3.11) $$u_0^k(x,t) + \eta < \varphi^k(x,t) < v_0^k(x,t) - \eta, \quad (x,t)\in\Sigma \ (k = 1, \ldots, N).$$

Applying the method of induction, one can show that for $k = 1, \ldots, N$; $m = 1, 2, \ldots$ the following inequalities hold:

$$u_{m-1}^k(x, t) < u_m^k(x, t), \quad (x, t) \in \bar{G}, \tag{3.12}$$

$$u_m^k(x, t) < v_0^k(x, t), \quad (x, t) \in \bar{G}, \tag{3.13}$$

$$L^k u_m^k > f^k\big(x, t, u_m, \Psi(x, t; u_m(\cdot, t))\big), \quad (x, t) \in \bar{G} \setminus \Sigma. \tag{3.14}$$

Indeed, let us take $m = 1$. Relation (3.12) follows, by Lemma 3, from (3.9)-(3.11) and from the obvious inequality

$$L^k u_0^k > f^k\big(x, t, u_0, \Psi(x, t; u_0(\cdot, t))\big) + \zeta_k(u_0, u_0), \quad (x, t) \in \bar{G} \setminus \Sigma$$

(resulting from (3.5)). Similarly, combining relations (3.7), (3.9)-(3.11) with the inequality

$$L^k v_0^k < f^k\big(x, t, u_0, \Psi(x, t; u_0(\cdot, t))\big) + \zeta_k(v_0, u_0), \quad (x, t) \in \bar{G} \setminus \Sigma$$

(which is a consequence of (3.6) and (3.8)) we get (3.13). Finally, taking advantage of (3.9), (3.12), (3.13) (for $m = 1$) and (3.8), we obtain (3.14). The reasoning in the second step of induction is the same as that for $m = 1$.

It follows from inequalities (3.12), (3.13) that $|u_m|_0^G \leqslant N_1$. Hence, recalling the definition of the constant N_2 and the uniform continuity of f^k in $\bar{G} \times H_1$, we obtain

$$\Big|f^k\big(x, t, u_{m-1}, \Psi(x, t; u_{m-1}(\cdot, t))\big)\Big|_0^G + |\zeta_k(u_m, u_{m-1})|_0^G \leqslant N_4$$
$$(k = 1, \ldots, N;\ m = 1, 2, \ldots).$$

These inequalities and Lemma 2 applied to (3.9), (3.10) imply the estimate

$$|u_m|_{1+\beta}^G \leqslant N_5 \quad (m = 1, 2, \ldots).$$

By the above estimate and by (3.12), (3.13) the sequence $\{u_m\}$ is uniformly convergent to a function $u \in C_{1+\beta}^N(G)$.

We shall show, as in the proof of Theorem 2, that the function u is a solution of problem (3.3), (3.4). Indeed, the problem

$$L^k w^k = f^k\big(x, t, u, \Psi(x, t; u(\cdot, t))\big), \quad (x, t) \in \bar{G} \setminus \Sigma, \tag{3.15}$$

$$w^k(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma \ (k = 1, \ldots, N) \tag{3.16}$$

possesses, in virtue of Lemma 1, a unique solution $w \in W_{2+\varepsilon}^N(G)$, $0 < \varepsilon < 1$ being a constant. Subtracting (3.15), (3.16) from (3.9), (3.10) respectively, we receive

$$L^k\left(u_m^k+\frac{\eta}{m}-w^k\right)=f^k\big(x,t,u_{m-1},\Psi(x,t;u_{m-1}(\cdot,t))\big)+\zeta_k(u_m,u_{m-1})+$$

$$+\frac{\eta}{m}\sigma^k(x,t)-f^k\big(x,t,u,\Psi(x,t;u(\cdot,t))\big),\qquad (x,t)\in\bar{G}\setminus\Sigma,$$

$$u_m^k(x,t)+\frac{\eta}{m}-w^k(x,t)=0,\qquad (x,t)\in\Sigma\ \ (k=1,\ldots,N;\ m=1,2,\ldots).$$

Applying Lemma 2 to these relations and taking into considerations assumptions (3.VI), (3.VIII), we conclude, by a limit passage, that $w\equiv u$.

It is easy to check that the function u is a minimum solution of the problem in question. Indeed, suppose a function $\bar{w}$ is a solution of this problem. Hence, and by (3.14), we get $\bar{w}>u_m$ in virtue of Lemma 3. Thus $\bar{w}\geqslant u$.

The maximum solution v of problem (3.3), (3.4) can be obtained as a limit of the sequence $\{v_m\}$, where

$$L^kv_m^k=f^k\big(x,t,v_{m-1},\Psi(x,t;v_{m-1}(\cdot,t))\big)-\zeta_k(v_m,v_{m-1}),\qquad (x,t)\in\bar{G}\setminus\Sigma,$$

$$v_m^k(x,t)=\varphi^k(x,t)+\frac{\eta}{m},\qquad (x,t)\in\Sigma\quad (k=1,\ldots,N;\ m=1,2,\ldots).$$

The further proceeding is the same as that for the minimum solution.

The inequality $u_0\leqslant u\leqslant v\leqslant v_0$ is an immediate consequence of the above considerations. This completes the proof of our theorem.

From the proof of Theorem 6 one can derive the following theorem on weak inequalities.

THEOREM 7. *Let the assumptions of Theorem* 6 *be fulfilled and suppose that a function* $w(x,t)=\{w^k(x,t)\}$, *regular in* $\bar{G}$, *satisfies the inequalities*

$$L^kw^k\underset{(\leqslant)}{\geqslant}f^k\big(x,t,w,\Psi(x,t;w(\cdot,t))\big),\qquad (x,t)\in\bar{G}\setminus\Sigma\ \ (k=1,\ldots,N),$$
$$w(x,t)\leqslant v(x,t)\qquad (w(x,t)\geqslant u(x,t)),\ (x,t)\in\Sigma.$$

Then $w(x,t)\leqslant v(x,t)$ $(w(x,t)\geqslant u(x,t))$ *in* $\bar{G}$, *where* $v=\{v^k\}$ $(u=\{u^k\})$ *is the maximum (minimum) solution of problem* (3.3), (3.4).

Note, that Theorems 6 and 7 constitute a generalization of Mlak's results [4].

4. Integro-differential equations. In this section we shall formulate corollaries to the theorems of the previous section for integro-differential equations.

Denote by $\mu^k(x,t;D)$ $(k=1,\ldots,N)$ a non-negative measure (depending on $(x,t)\in\bar{G}$) defined on the σ-field $\mathfrak{M}$ of all Lebesgue measurable subsets of the domain

$$D_0 = \overline{\bigcup_{0 \leq t \leq T} G_t}, \quad \text{where } G_t = \{x\colon (x, t) \in \bar{G} \setminus S\}.$$

We make the following assumptions ($k = 1, \ldots, N$):

(4.I) For any $(x, t) \in \bar{G}$ the measures $\mu^k(x, t; D_0)$ are finite.

(4.II) There exists a finite non-negative measure $\bar{\mu}$ (defined on $\mathfrak{M}$) such that for any $D \in \mathfrak{M}$ and for any points $P = (x, t)$, $P' = (x', t')$ of the domain $\bar{G}$ we have

$$|\mu^k(x, t; D) - \mu^k(x', t'; D)| \leq \bar{\mu}(D)[d(P, P')]^\gamma,$$

$0 < \gamma < 1$ being a constant.

(4.III) There is a positive constant A_9 such that for any $D \in \mathfrak{M}$

$$\mu^k(x, t; D) \leq A_9 \mathfrak{m}(D),$$

where $\mathfrak{m}(D)$ is the Lebesgue measure of D (9).

From assumptions (4.I), (4.II) it follows that for any $(x, t) \in \bar{G}$

$$\mu^k(x, t; D_0) \leq A_{10} = R^\gamma \bar{\mu}(D_0) + \max_{1 \leq i \leq N} \inf_{(x', t') \in G} \mu^i(x', t'; D_0),$$

where R denotes the diametr in the parabolic distance of the domain G.

THEOREM 8. *If assumptions* (1.III), (2.I), (2.II), (3.I), 3(.II), (4.I)-(4.III) *are satisfied and*

$$K(\alpha) N A_6 (A_{10} + 1) \tau^{(1-\alpha)/2} < 1,$$

then Theorem 2 remains true for the problem

$$(4.1) \qquad L^k w^k = f^k\Big(x, t, w, w_x^k, \int_{G_t} w(y, t)\mu(x, t; dy)\Big), \quad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau,$$

$$(4.2) \qquad w^k(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma^\tau \ (k = 1, \ldots, N),$$

where $\int_{G_t} w(y, t)\mu(x, t; dy)$ *is the system of integrals*

$$\int_{G_t} w^1(y, t)\mu^1(x, t; dy), \ldots, \int_{G_t} w^N(y, t)\mu^N(x, t; dy).$$

The proof consists in applying Lemma 4 of paper [7] and Theorem 4. Next, by Lemma 4 and by Theorem 6, we obtain

THEOREM 9. *Let assumptions* (1.III), (2.I), (2.II), (3.VI), (4.I)-(4.III) *and* (3.X) *(with functionals replaced by integrals) be satisfied. Then Theorem* 6

(9) If S is a cylindrical surface, then condition (4.III) is superfluous in all the theorems of this section.

remains valid for the problem

$$L^k w^k = f^k\Big(x, t, w, \int_{G_t} w(y, t)\mu(x, t; dy)\Big), \quad (x, t) \in \bar{G}\setminus\Sigma, \tag{4.3}$$

$$w^k(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma \ (k = 1, \ldots, N). \tag{4.4}$$

Using Lemma 4 of [7] and Theorems 5 and 7 one can easily obtain the following theorems:

THEOREM 10. *Under the assumptions of Theorem* 8, *if a function* $w(x, t) = \{w^k(x, t)\}$, *regular in* $\bar{G}^\tau$, *satisfies the inequalities*

$$L^k w^k \underset{(\leqslant)}{\geqslant} f^k\Big(x, t, w, w_x^k, \int_{G_t} w(y, t)\mu(x, t; dy)\Big), \quad (x, t) \in \bar{G}^\tau\setminus\Sigma^\tau \ (k = 1, \ldots, N),$$

$$w(x, t) \leqslant v(x, t) \quad (w(x, t) \geqslant u(x, t)), \quad (x, t) \in \Sigma^\tau,$$

then $w(x, t) \leqslant v(x, t)$ $(w(x, t) \geqslant u(x, t))$ *in* $\bar{G}^\tau$, *where* $v(x, t)$ $(u(x, t))$ *is the maximum (minimum) solution of problem* (4.1), (4.2).

THEOREM 11. *We preserve the assumptions of Theorem* 9. *Suppose that a function* $w(x, t) = \{w^k(x, t)\}$, *regular in* $\bar{G}$, *fulfils the inequalities*

$$L^k w^k \underset{(\leqslant)}{\geqslant} f^k\Big(x, t, w, \int_{G_t} w(y, t)\mu(x, t; dy)\Big), \quad (x, t) \in \bar{G}\setminus\Sigma \ (k = 1, \ldots, N),$$

$$w(x, t) \leqslant v(x, t) \quad (w(x, t) \geqslant u(x, t)), \ (x, t) \in \Sigma.$$

Then $w(x, t) \leqslant v(x, t)$ $(w(x, t) \geqslant u(x, t))$ *in* $\bar{G}$, $v(x, t)$ $(u(x, t))$ *being the maximum (minimum) solution of problem* (4.3), (4.4).

We conclude this section by giving an example showing that the assumptions of Theorem 9 do not imply the uniqueness of problem (4.3), (4.4). Moreover, the example shows that it can really happen that $u(x, t) \not\equiv v(x, t)$.

EXAMPLE. Let us write

$$G = \Big\{(x, t): \ -\frac{\pi}{2} < x < \frac{\pi}{2}, \ 0 < t < T = \frac{\pi}{4}\Big\}.$$

Following Mlak [5] (comp. [6], p. 215) we consider the problem

$$w_{xx} + \frac{2}{\pi} e^{-\pi}(\sin x) w_x - w_t = g(x, w) - \frac{1}{\pi} e^{-\pi} \sin^2 x \int_{-\pi/2}^{\pi/2} w(y, t)\, dy, \tag{4.5}$$

$$(x, t) \in \bar{G}\setminus\Sigma,$$

$$w(x, t) = \cos x, \quad (x, t) \in \Sigma, \tag{4.6}$$

where

$$g(x, z) = \begin{cases} \sqrt{\cos^2 x - z^2} - z & \text{if } |z| \leqslant \cos x, \\ -z & \text{if } |z| \geqslant \cos x. \end{cases}$$

Observe that all the assumptions of Theorem 9 are satisfied (in particular, we can put $u_0(x, t) = -e^{t+1}+1$ and $v_0(x, t) = e^{t+1}-1$), but there are two different solutions $w_1(x, t) = \cos x$ and $w_2(x, t) = \cos x \cos t$ of problem (4.5), (4.6). Hence and by Theorem 9, this problem has a maximum solution $v(x, t)$ and a minimum solution $u(x, t)$ which are different and, moreover,

$$-e^{t+1}+1 \leqslant u(x, t) \leqslant \cos x \cos t \leqslant \cos x \leqslant v(x, t) \leqslant e^{t+1}-1.$$

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Reçu par la Rédaction le 20. 1. 1970