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On a modification of a theorem of O. Olejnik and on its applications

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In investigations of the unicity of the solutions of boundary problems for elliptic partial differential equations of the second order, especially in the case of Neumann's problem, a theorem of O. Olejnik ([5]) plays an important part. The theorem gives information on the behaviour of a directed derivative of the solution of an elliptic equations at those points of the boundary of a domain under consideration at which the solution of the equation attains its non-negative l.u.b. or its non-positive g.l.b. For parabolic partial differential equations an analogous theorem is known ([3]).

The assumption that the quadratic form of the equation (elliptic or parabolic) is uniformly positively defined in the closure of the domain plays an important part in the proofs of the above theorems.

The purpose of this paper is to prove a modified version of Olejnik's theorems for the elliptic and parabolic equations and to apply it to the investigation of the unicity of the solutions. We replace the assumption mentioned above by another one.

§ 1. Let D be a bounded domain in the m-dimensional Euclidean space of points $X(x_1, \ldots, x_m)$. Let Σ be a subset of the boundary $F(D)$ of D given by the equation $G(X) = 0$, where the function G is of class C^2 in an m-dimensional neighbourhood V of Σ such that $\operatorname{grad}^2 G > 0$ in V. Let

$$(1)\qquad \mathcal{E}(u) \equiv \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X)\frac{\partial u}{\partial x_k} + c(X)u = f(X),$$

where the functions $a_{ij}(X) = a_{ji}(X)$, $b_k(X)$, $c(X)$, and $f(X)$ are defined and bounded in the closure $\bar{D}$ of D. Let $\sum_{i,j=1}^{m} a_{ij}(X)\lambda_i\lambda_j$ be positive in $\bar{D}$. Under all these assumptions we shall prove

THEOREM 1. *Suppose that* 1° $c(X) \leqslant -c_0$, $c_0 > 0$, $f(X) \geqslant 0$ $(f(X) \leqslant 0)$ *for* $X \in \bar{D}$; 2° X_0 *is a point of* Σ *such that there is a neighbourhood* N *of* X_0 *such that* $F(D) \cap N$ *belongs to* Σ; 3° $u(X)$ *is a solution of* (1) *regular in* $\bar{D}$ (*i.e.* $u(X)$ *is of class* C^2 *in* D *and continuous in* $\bar{D}$) *positive* (*negative*) *at* X_0 *and* $u(X) < u(X_0)$ $(u(X) > u(X_0))$ *for* $X \in D$; *then*

$$\overline{\lim_{X\to X_0}} \frac{u(X)-u(X_0)}{\overline{XX_0}} < 0 \qquad \Big(\underline{\lim_{X\to X_0}} \frac{u(X)-u(X_0)}{\overline{XX_0}} > 0\Big)$$

as $X \in D$ *tends to* X_0 *along a half-straightline which starts from* X_0, *is not tangent to* $F(D)$ *at* X_0 *and is contained in* D *in a neighbourhood of* X_0.

Proof. Since $F(D)$ is of class C^2 at X_0, according to [1] there is a closed sphere K whose boundary $F(K)$ is tangent to $F(D)$ at X_0 and whose other points belong to D. Let R be the radius of K and let $A(a_1, \ldots, a_m)$ be the centre of K. Let K_0 be a sphere of centre X_0 and of radius $\varrho_0 < R$. Let Ω denote the interior of $K \cap K_0$. Denote by S the part of the boundary $F(\Omega)$ contained in the interior of K and let σ denote the remaining part of $F(\Omega)$. Put

$$v(X) = e^{-hR^2} - e^{-hr^2},$$

where $r^2 = AX^2 = \sum_{i=1}^{m} (x_i - a_i)^2$ and $h > 0$ is a constant to be chosen later. We have $v(X) = 0$ for $X \in F(K)$ and

$$e^{hr^2}\mathcal{E}(v) = -4h^2 \sum_{i,j=1}^{m} a_{ij}(X)(x_i - a_i)(x_j - a_j) + \\ + 2h \sum_{i=1}^{m} [a_{ii}(X) + b_i(X)(x_i - a_i)] + c(X)[\exp h(r^2 - R^2) - 1],$$

whence

$$e^{hr^2}\mathcal{E}(v) \leqslant 2h \sum_{i=1}^{m} [a_{ii}(X) + b_i(X)(x_i - a_i)] - c(X).$$

By the boundedness of the coefficients of equation (1) there are positive constants M_1 and M_2 such that

(2) $$\mathcal{E}(v) \leqslant (M_1 h + M_2) e^{-hr^2},$$

whence

(3) $$\mathcal{E}(v) \leqslant 1$$

for $h \geqslant h_0$, where h_0 is sufficiently large. Let

$$w(X) = u(X) - u(X_0) - \lambda v(X);$$

then

(4) $$\mathcal{E}(w) = f(X) - c(X)u(X_0) - \lambda \mathcal{E}(v).$$

If $\lambda < \lambda_0 = c_0 u(X_0)$ and $h \geqslant h_0$ then $\mathcal{E}(w) > 0$ for $X \in \Omega$.

By 3° we have

$$w(X) = u(X) - u(X_0) \leqslant 0\,, \quad X \in \sigma\,, \tag{5}$$

$$u(X) - u(X_0) \leqslant -a_0\,, \quad X \in S\,, \tag{6}$$

where $a_0 = \text{const} > 0$. Therefore there is a $\lambda_1 > 0$ such that $w(X) \leqslant 0$ for $X \in S$ and $0 < \lambda < \lambda_1$. Hence according to (4) and (5) we have

$$\mathcal{E}(w) > 0\,, \quad X \in \Omega \quad \text{and} \quad w(X) \leqslant 0\,, \quad X \in F(\Omega)\,, \tag{7}$$

if $0 < \lambda < \min(\lambda_1, \lambda_0)$ and $h \geqslant h_0$. By the extremum property of the solutions of elliptic equations ([4], p. 155), $w(X) \leqslant 0$ for $X \in \Omega$. Therefore

$$u(X) - u(X_0) \leqslant \lambda[v(X) - v(X_0)]\,, \quad X \in \Omega\,, \tag{8}$$

because $v(X_0) = 0$.

Let l be an open half-straightline starting from X_0 such that l is not tangent to $F(D)$ at X_0 and l belongs to D in a neighbourhood of X_0. By (8)

$$\varlimsup_{\substack{X \to X_0 \\ X \in l}} \frac{u(X) - u(X_0)}{\overline{XX_0}} \leqslant \lambda \varlimsup_{\substack{X \to X_0 \\ X \in l}} \frac{v(X) - v(X_0)}{\overline{XX_0}} = \lambda \left.\frac{dv}{dl}\right|_{X = X_0}. \tag{9}$$

But

$$\left.\frac{dv}{dl}\right|_{X = X_0} = 2he^{-hR^2}\boldsymbol{r}_0 \boldsymbol{l}\,,$$

where $\boldsymbol{r}_0 = \boldsymbol{AX}_0$ and $\boldsymbol{l}$ is a unit vector on l with its origin at X_0. Thus

$$\left.\frac{dv}{dl}\right|_{X = X_0} < 0$$

and the proof is completed.

Remark 1. Because of (4) the assertion of Theorem 1 remains true if the assumption 1° is replaced by 1°a $c(X) \leqslant 0$ and $f(X) \geqslant f_0$ $(f(X) \leqslant -f_0)$, where $f_0 > 0$. If 1°a is satisfied Theorem 1 remains true also if $u(X_0) = 0$.

Let $F(D)$ be of class C^2. Let $l(X)$ denote a fixed half-straightline starting from $X \in F(D)$, not tangent to $F(D)$ at X and contained in D in a neighbourhood of X.

THEOREM 2. *Suppose that $c(X) \leqslant -c_0 < 0$ and $f(X) \geqslant 0$ $(f(X) \leqslant 0)$ for $X \in \bar{D}$ and $\alpha(X), \beta(X)$ are real functions defined on $F(D)$ such that $\alpha(X) \geqslant 0$, $\beta(X) \leqslant 0$, $\alpha^2(X) + \beta^2(X) > 0$, $X \in F(D)$, and $u(X)$ is a solution of* (1) *regular in $\bar{D}$, such that the derivative du/dl exists for $X \in F(D)$ and*

$$L(u) \equiv \alpha(X)\frac{du}{dl} + \beta(X)u \geqslant 0 \quad (L(u) \leqslant 0)\,, \quad X \in F(D); \tag{10}$$

then $u(X) \leqslant 0$ $(u(X) \geqslant 0)$ for $X \in \bar{D}$.

Proof. If $u(X) = \text{const}$, then according to (1) and because of the assumption on $c(X)$ and $f(X)$, we have $u(X) \leqslant 0$ $(u(X) \geqslant 0)$. Consider the case where $u(X) \neq \text{const}$ for $X \in \bar{D}$. If Theorem 2 where not true, then l.u.b. of $u(X)$ (g.l.b. $u(X)$) would be positive (negative) in $\bar{D}$. According to E. Hopf ([2]) there exists $X_0 \in F(D)$ such that $u(X) < u(X_0)$, $u(X_0) > 0$ $(u(X) > u(X_0)$, $u(X_0) < 0)$ for $X \in D$. But in virtue of Theorem 1 we would have $du/dl < 0$ $(du/dl > 0)$ at X_0. This, however, contradicts (10). The proof is completed.

COROLLARY 1. *Under the assumptions of Theorem 2 the function $u \equiv 0$ is the only solution of the equation $\mathcal{E}(u) = 0$ regular in $\bar{D}$ satisfying the condition $L(u) = 0$ on $F(D)$. In particular, if $\beta(X) \equiv 0$, we get the unicity of the solution of Neumann's problem for equation* (1).

§ 2. We shall now consider parabolic equations.

Let D be a bounded domain in the $(m+1)$-dimensional Euclidean space of points $(t, X) = (t, x_1, \ldots, x_m)$. Let D be contained between the hyperplanes $t = 0$ and $t = T$. Assume that the part D_0 of $\bar{D}$ common with $t = 0$ is not empty. Let $\bar{S}$ denote the closure of the subset of $F(D)$ of those points for which $t \neq 0$ and $t \neq T$. The subset of $\bar{S}$ of the points for which $t \neq 0$ will be denoted by S. Let $\Gamma = S \cup D_0$. We shall consider the parabolic equation

$$(11)\quad \mathfrak{F}(u) \equiv \sum_{i,j=1}^{m} a_{ij}(t, X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(t, X)\frac{\partial u}{\partial x_k} + c(t, X)u - \frac{\partial u}{\partial t} = f(t, X)$$

where $a_{ij}(t, X) = a_{ji}(t, X)$, $b_k(t, X)$, $c(t, X)$, and $f(t, X)$ are defined and bounded in $\bar{D}$ and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(t, X)\lambda_i\lambda_j$ is positively defined in $\bar{D}$.

The following theorem gives the strong extremum property for the solution of (11).

THEOREM 3. *If* 1° *$u(t, X)$ is a solution of* (11) *regular in $\bar{D}$,* 2° *$c(t, X) \leqslant -c_0 < 0$ and $f(t, X) \geqslant 0$ $(f(t, X) \leqslant 0)$,* 3° *$u(t, X)$ attains positive maximum (negative minimum) at $(t_0, X_0) \in \bar{D}\setminus\Gamma$, then $u(t, X) = u(t_0, X_0)$ at each point $(t, X) \in \bar{D}$ such that $t < t_0$ and there is a curve of the form $X = X(t)$ connecting (t_0, X_0) with (t, X) and contained in $\bar{D}\setminus\Gamma$.*

The proof is analogous to the proof of a theorem in [3], where assumption 2° has the form 2°a $\sum_{i,j=1}^{m} a_{ij}(t, X)\lambda_i\lambda_j \geqslant \mu \sum_{i=1}^{m} \lambda_i^2$, $\mu > 0$ and $f(t, X) \equiv 0$.

Theorem 3 remains true also if 2° is replaced by 2°b $c(t, X) \leqslant 0$ and $f(t, X) \geqslant f_0$ $(f(t, X) \leqslant -f_0)$ where $f_0 > 0$.

A counterpart of Theorem 1 for the parabolic equation (11) is the following

THEOREM 4. *Suppose that* 1° $c(t, X) \leqslant -c_0 < 0$ *and* $f(t, X) \geqslant 0$ ($f(t, X) \leqslant 0$) *in* $\bar{D}$, 2° *the surface* S *is of class* C^2 *in a neighbourhood of* $P_0 \in S$, 3° *the normal* n *of* S *at* P_0 *is not parallel to the axis* t, 4° $u(t, X)$ *is a solution of* (11) *regular in* $\bar{D}$ *and attaining its positive l.u.b.* (*negative g.l.b.*) *at* P_0; *then either*

$$\varlimsup_{P\to P_0} \frac{u(P)-u(P_0)}{\overline{PP_0}} < 0 \qquad \Big(\varliminf_{P\to P_0} \frac{u(P)-u(P_0)}{\overline{PP_0}} > 0\Big)$$

as $P \in D$ *tends to* P_0 *along any half-straightline* l *starting from* P_0 *and forming an acute angle with the interior normal* n *of* S *at* P_0, *or there exists a neighbourhood* $O(P_0)$ *of* P_0 *such that* $u(t, X) = \text{const}$ *for* $(t, X) \in O(P_0) \cap \bar{D}$ *and* $t \leqslant t_0$.

Proof. If $u(t, X) \neq \text{const}$ for $(t, X) \in O(P_0) \cap \bar{D}$ and $t \leqslant t_0$, then according to Theorem 3 there exists a neighbourhood $O_1(P_0) \subset O(P_0)$ such that $u(P) < u(P_0)$ for $P \in O_1(P_0) \cap D$. The remaining part of the proof is analogous to the proof of Theorem 1 (cf. [3]).

Remark 2. Assumption 1° of Theorem 4 may be replaced by 1°a $c(t, X) \leqslant 0$ and $f(t, X) \geqslant f_0$ ($f(t, X) \leqslant -f_0$) where $f_0 > 0$.

The proof is almost the same (cf. Remark 1).

Remark 3. If the half-straightline l of Theorem 4 is perpendicular to the axis t, the assumption $c(t, X) \leqslant -c_0$ may be omitted. If fact, the function $v(t, X) = u(t, X)e^{-(M+c_0)t}$ (where $c(t, X) \leqslant M$, $c_0 > 0$) satisfies an equation of the form (11) in which $\tilde{c}(t, X) \leqslant -c_0$.

But

$$\frac{u(P)-u(P_0)}{\overline{PP_0}} = e^{(M+c_0)t_0} \frac{v(P)-v(P_0)}{\overline{PP_0}}.$$

Hence and by Theorem 4 we get our assertion.

Theorem 4 implies (cf. [3]) the following

THEOREM 5. *If* 1° *the surface* S *is of class* C^2 *and at no point* $P \in S$ *the normal* n *of* S *at* P *is parallel to the axis* t, 2° $c(t, X) \leqslant -c_0 < 0$ *and* $f(t, X) \equiv 0$, *then the function* $u(t, X) \equiv 0$ *is the only solution of* (11) *regular in* $\bar{D}$ *such that there exists a directed derivative* du/dl (*where* l *forms an acute angle with the interior normal of* S *at* P) *for all* $P \in S$ *and*

$$u(0, X) = 0\,, \qquad \frac{du}{dl} + a(t, X)u = 0 \qquad (a(t, X) \leqslant 0) \text{ on } S\,.$$

Remark 4. Theorem 5 implies the unicity of the solution of the second boundary problem (Neumann's problem) for equation (11). In the case where the half-staightline l is perpendicular to the axis t, according to Remark 3, the assumption $c(t, X) \leqslant -c_0 < 0$ of Theorem 5 may be omitted. In this case Theorem 5 and its corollaries give an extension of a theorem of [3], where the quadratic form of (11) is uniformly positive in $\bar{D}$,

$$\sum_{i,j=1}^{m} a_{ij}(t, X)\lambda_i\lambda_j \geqslant \mu \sum_{i=1}^{m}{}' \lambda_i^2, \qquad \mu > 0 .$$

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