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Duality, imprimitivity, reciprocity

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Abstract. Let G be a locally compact group, Γ a subgroup of G, U an irreducible projective representation (p.r.) of G; j a p.r. of Γ; U^j-representation of G induced by j. Let $A(U, j)$ be the space of (U, j)-automorphic forms. Then there exists a Hilbert space isomorphism $L_G(U, U_j) \approx A(U, j)$. Several reciprocity theorem of Frobenius type are corollaries. Any system of imprimitivity (E, U) based on G/Γ defines a (U, j)-automorphic form.

Introduction. The chief problem in the theory of group representations is to determine the space $L_G(U, U_1)$ of operators intertwining unitary representations (U, H) and (U_1, H_1) of a locally compact group G

$$L_G(U, U_1) = \{T \in L(H, H_1):\ TU(g) = U_1(g)T,\ g \in G\}.$$

The problem is most interesting when $U_1 = (U^j, H^j)$ is the representation induced by a unitary representation (j, V) of a subgroup $\Gamma \subset G$. In the case of a finite group G this problem was considered already by the founder of the theory of group representations — Ferdinand Georg Frobenius and was investigated by such masters of the theory as G. W. Mackey, F. Bruhat, I. M. Gelfand (and his school), R. I. Blattner (Lie-groups), Olšanski and many others.

K. Maurin and L. Maurin [6], [7] have given a complete characterization of the space $L_G(U, U^j)$ by means of s. c. (U, j)-automorphic forms.

As was recently remarked by Mackey [5], several problems in the theory of theta-functions and automorphic forms could be considered as a describing of the space $L_G(U, U^j)$, where U and U^j are projective representations and (U, H) is irreducible. Our chief result is the following:

Duality Theorem (Theorem 3.1). *Let (U, H) be an irreducible σ_1-representation and (j, V) a σ-representation of $\Gamma \subset G$. Then the Hilbert space $L_G(U, U^j)$ is unitarily isomorphic to the space $A(U, j)$ of (U, j)-automorphic forms.*

If we do not assume the irreducibility of (U, H), we obtain the following result, which can be considered as a generalisation (and even a sharpened form) of an important theorem of Blattner [1].

THEOREM 1.1. *Let (U, H) bean arbitrary σ_1-representation and (j, V) a σ-representation of $\Gamma \subset G$. Then there exists an injection $k: L_G(U, U^j) \to A(U, j)$.*

We obtain the Blattner Theorem (which was inspired by F. Bruhat [2]) if we take $U = U^M$, where (M, H^M) is the representation induced by a unitary (or σ-) representation M of another subgroup Γ_1 of G.

Since our notion of (U, j)-automorphic form is valid for arbitrary locally compact groups, it has been interesting to compare it with the famous notion of a "System of imprimitivity" introduced and investigated by G. W. Mackey. We prove in Section 3 (by a method which we owe to N. Skovhus-Poulsen) Theorem 3.1, which asserts that any system of imprimitivity defines a (U, j)-automorphic form.

In Section 4 we show that several reciprocity theorems proved in [7] are valid also for σ-representations.

We are happy to express here our gratitude to Dr Nils Skovhus-Poulsen for the manuscript of his very interesting talk on Mackey's Imprimitivity Theorem and to Dr A. Wawrzyńczyk for many invaluable conversations on induced representations.

1. PRELIMINARIES

A. Rho-functions. Let G be a locally compact group with the left invariant measure dg. Let $\Gamma \subset G$ be a subgroup of G provided with the left-invariant measure $d\gamma$. The modular functions are denoted by Δ_G and Δ_Γ. As is well known, on G there exists a strictly positive continuous function ϱ such that

$$\varrho(g\gamma) = \frac{\Delta_\Gamma(\gamma)}{\Delta_G(\gamma)}\varrho(g), \quad g \in G, \ \gamma \in \Gamma. \tag{1.1}$$

Such a function is called a *rho-function.*

We now fix a rho-function ϱ and let μ be a (positive) quasi-invariant measure on G/Γ associated with ϱ and defined by

$$\int_G \psi(g)\varrho(g)dg = \int_{G/\Gamma} d\mu(\dot{g}) \int_\Gamma \psi(g\gamma)d\gamma, \quad \psi \in C_0(G). \tag{1.2}$$

B. Induced representations. (For the following cf. Mackey [5], Blattner [1], Bruhat [2], Bourbaki, and the excellent monograph of G. Warner [8].) Let (j, V) be a continuous unitary representation of Γ in the Hilbert space $(V, \langle\cdot|\cdot\rangle)$. Denote by H^j the Hilbert space of (equivalence classes of) functions $f: G \to V$ which are

1° dg-measurable.

2° $f(g\gamma) = \left(\frac{\Delta_\Gamma(\gamma)}{\Delta_G(\gamma)}\right)^{1/2} j(\gamma^{-1})f(g), \; \gamma \in \Gamma, \; g \in G,$

3° $\|f(\cdot)\|_v^2$ is locally integrable on G,

4° $\int\limits_{G/\Gamma} \varrho(g)^{-1}\|f(g)\|_v^2 d\mu(\dot g) =: \|f\|_{H^j}^2 < \infty,$

5° $U^j(g)f(x) := f(g^{-1}x), \quad g, x \in G.$

Blattner [1] avoids the use of the quasi-invariant measure in the following way : Write

$$\psi^b(x) := \int\limits_\Gamma \psi(x\gamma)d\gamma, \qquad \psi \in C_0(G). \tag{1.3}$$

Then $\psi^b \in C_0(G/\Gamma)$, and as was proved by Bruhat [2], $C_0(G) \ni \psi \to \psi^b \in C_0(G/\Gamma)$ is a continuous surjection. For any f_1, f_2 satisfying 1°–3° we have on G/Γ a complex Radon measure μ_{f_1,f_2} defined by

$$\int\limits_G \langle f_1(g)|f_2(g)\rangle \psi(g)dg =: \int\limits_{G/\Gamma} \psi^b(\dot g)d\mu_{f_1,f_2}(\dot g). \tag{1.4}$$

Hence we have the important equations

$$(f_1|f_2)_{H^j} = \mu_{f_1,f_2}(G/\Gamma), \tag{1.5}$$

$$\|f_1\|_{H^j}^2 = \sup_{0 \leqslant \psi^b \leqslant 1} \int\limits_G \psi(g)\|f_1(g)\|_V^2 dg. \tag{1.6}$$

C. Intertwining operators. Let (U_i, H_i), $i = 1, 2$, be unitary representations of the group G. A map $T \in L(H_1, H_2)$ interwines (by definition) U_1 and U_2 if

$$TU_1(g) = U_2(g)T, \quad g \in G. \tag{1.7}$$

The space of intertwining operators is denoted by $L_G(U_1, U_2)$. If (U_1, H_1) is irreducible, then (as was remarked by Mackey) $L_G(U_1, U_2)$ has the natural Hilbert space structure

$$(T_1|T_2) := (T_2^*T_1\varphi|\varphi)_{H_1}\|\varphi\|^{-2}. \tag{1.8}$$

Since $T_2^*T_1$ commutes with $(U(g))$, $g \in G$, by the Schur Lemma $T_2^*T_1 = a\cdot \mathrm{id}_{H_1}$, where $a \in C$. Plainly $(T_1|T_2) = a$. Plainly

$$\|T_1\|^2 = (T_1|T_1).$$

2. DUALITY THEOREM (D.T.) FOR AUTOMORPHIC FORMS

This section is the core of the paper. We give here a much stronger version of our D.T. from K. Maurin–L. Maurin [6] and [7].

The space Φ. Let (U, H) be a unitary representation of G. Let Φ be the linear spas of $U(\psi)h := \int_G \psi(g) U(g)h$, where $\psi \in C_0(G)$, $h \in H$.

We equip Φ with the (natural) bornological topology in the following way: for each compact $K \subset G$, denote by $C_0(G, K) = \{\psi \in C_0(G) : \operatorname{spt} \psi \subset K\}$. $C_0(G, K)$ is a Banach space with norm $\|\psi\| = \|\psi\|_\infty = \sup|\psi(G)|$. $C_0(G) = \varinjlim C_0(G, K)$ is the inductive limit of Banach spaces $C_0(G, K)$. Denote by Φ_K = spas of $\{U(\psi)h, \psi \in C_0(G, K) h \in H\} = Q_K(C_0(G, K) \otimes H)$, where $\otimes$ denotes the projective tensor product and $Q_K(\psi \otimes h) = Q(\psi \otimes h) := U(\psi)h$.

DEFINITION. The space Φ is the *inductive limit* of Φ_K:

$$\Phi = \varinjlim Q_K(C_0(G, K) \otimes H) = \varinjlim \Phi_K,$$

where K runs through all compact subsets of G.

It is a bornological space as an inductive limit of normed spaces Φ_K (the topology of Φ is of course the inductive topology with respect to $(Q_K, C_0(G, K) \otimes H)$).

LEMMA 2.1. *$\Phi \subset H$ is dense in H and the imbedding is continuous.*

LEMMA 2.2. *The restriction $(U|\Phi, \Phi)$ of U to Φ is a continuous representation of G in Φ.*

LEMMA 2.3. *For each sequence $h_n \xrightarrow[n\to\infty]{} h$, in H*

$$U(\psi)h_n \to U(\psi)h \quad \text{in } \Phi.$$

LEMMA 2.4. *Let $(W, \|\cdot\|)$ be a normed space. Then the linear map $L : \Phi \to W$ is continuous if and only if for any compact set $K \subset G$ there is a positive constant $c(K)$ such that*

$$\|LU(\psi)h\| \leqslant c(K) \|\psi\|_\infty \|h\|_H \quad \text{for any } \psi \in C_0(G, K),\ h \in H.$$

These Lemmas were (more or less) proved in [1].

PROPOSITION 2.1. *Let $T \in L_G(U, U^j)$; then for each $\varphi = U(\psi)h \in \Phi$ the element $T\varphi \in H^j$ has a continuous representative*

$$G \ni x \to \int \psi(t)(Th)(t^{-1}x)\,dt \in V. \tag{2.1}$$

Proof in the Appendix!

THEOREM 2.1. *Let $T \in L_G(U, U^j)$. Define $\eta = \eta_T$ by*

$$\eta(\varphi) := (T\varphi)(e) = \int \psi^*(t) Th(t)\,dt, \qquad \varphi = U(\psi)h \in \Phi \tag{2.2}$$

(where the right-hand side is understood as the value of the continuous representative of $T\varphi$ at e, and $\psi^*(t) := \psi(t^{-1}) \Delta_G(t)^{-1}$). Then $\Phi \ni \varphi \to \eta(\varphi) \in V$ has the following properties:

2(a) $\eta \in L(\Phi, V)$, *i.e.* η *is continuous,*
2(b) $\eta(U(\gamma)\varphi) = \varrho(\gamma)^{-1/2} j(\gamma)\eta(\varphi)$, $\gamma \in \Gamma$,
2(c) $f_{\eta,\varphi}(\cdot) \in H^j$, *where* $f_{\eta,\varphi}(g) := \eta(U(g)\varphi)$.

Proof. Since

$$(T\varphi)(g) = U^j(g^{-1})(T\varphi)(e) = TU(g^{-1})\varphi(e) = \eta(U(g^{-1})\varphi),$$

points (b) and (c) follow from the definition of induced representation.

It suffices to prove (a). Let $a \in C_0(G)$, $0 \leqslant a \leqslant 1$ and $a = 1$ on $\operatorname{spt}\psi^*$ — support of ψ^*

$$\begin{aligned}\|\eta(\varphi)\|_V &\leqslant \int |\psi^*(g)|\,\|Th(g)\|_V dg = \int |\psi^*(g)|\,a(g)\,\|Th(g)\|_V\,dg\\ &\leqslant \|\psi^*\|_{L^2(G)}\Big(\int a(g)^2\,\|Th(g)\|_V^2 dg\Big)^{1/2}\\ &\leqslant \|\theta^b\|_\infty \|\psi^*\|_{L^2}\|Th\|_{H^j} \leqslant c(K)\|\psi\|_\infty \|h\|_H \qquad \text{by (1.6)}\end{aligned}$$

$(\theta := a^2)$ and Lemma 2.4. □

COROLLARY 2.1. *For G unimodular*

$$\|\eta(\varphi)\|_V \leqslant \|\psi\|_{L^2}\,\|\theta^b\|_\infty\,\|Th\|_{H^j}.$$

COROLLARY 2.2. *For G unimodular, Γ compact, η is continuous even in the relative $L^2(G)$-topology on $C_0(G)$.*

Proof. Since $0 \leqslant \theta^b \leqslant 1$

$$\|\eta(\varphi)\|_V \leqslant \|\psi\|_{L^2(G)}\|Th\|_{H^j} \leqslant \|T\|\,\|\psi\|_{L^2}\,\|h\|.$$

DEFINITION 2.3. If (U, H) is irreducible, then the map $\eta : \Phi \to V$ which satisfies 2(a)–2(c) is called a *(U, j)-automorphic form.*

COROLLARY 2.4. *If G/Γ admits a finite invariant measure μ and* $\dim H < \infty$, *then* 2(a) *and* 2(b) *imply* 2(c).

Proof. Since $\dim H < \infty$, $\Phi = H$, we have

$$\begin{aligned}\big\|\eta(U^{-1}(\cdot)\varphi)\big\|_{H^j}^2 &= \int\limits_{G/\Gamma} \big\|\eta(U^{-1}(g))\varphi\big\|_V^2 d\mu \leqslant c \int\limits_{G/\Gamma} \|U^{-1}(g)\varphi\|_H^2 d\mu\\ &\leqslant c\mu(G/\Gamma)\,\|\varphi\|^2 < \infty.\end{aligned}$$

Since in the theory of automorphic functions the space G/Γ is compact (Gelfand–Piateckij, Šapiro, Olšanskij) or G/Γ admits a finite invariant measure ("modular functions"), the following two Corollaries are of some interest:

COROLLARY 2.3. *If G/Γ is compact, then from* 2(b) *follows* 2(c).

Proof. We have to prove that, for each $\varphi \in \Phi$, $\eta(U^{-1}(\cdot)\varphi) \in H^j$. But it is obvious since

$$\varrho(\cdot)^{-1}\big\|\eta(U^{-1}(\cdot)\varphi)\big\|_V^2 \in C(G/\Gamma).$$

PROPOSITION 2.2. *If G/Γ is compact or G/Γ possesses a finite invariant measure and* $\dim H < \infty$, *then each $\eta \in L(\Phi, V)$ which is $\varrho^{-1/2} j$-covariant (i.e. satisfies* 2(b)) *satisfies* 2(c). *Hence in these cases an automorphic form is characterised by* 2(a) *and* 2(b) *only.*

We have proved that any $T \in L_G(U, U^j)$ defines an automorphic form (Theorem 2.1). The duality theorem asserts that a $(U, \dot{J})$-automorphic from defines an intertwining operator $T_\eta \in L(U, U^j)$; more precisely:

THEOREM 2.2. *Let η be a (U, j)-automorphic form; then the map*

$$\overset{0}{T}_\eta\colon \Phi \to H^j, \quad \text{where} \quad \overset{0}{T}_\eta \varphi := f_{\eta,\varphi}(\cdot)$$

can be extended (uniquely) to $T_\eta \in L(U, U^j)$.

Theorems 2.1 and 2.2 give:

DUALITY THEOREM FOR AUTOMORPHIC FORMS (D.T.). *If we provide the space $A(U, j)$ — of (U, j)-automorphic forms with the natural Hilbert structure*

$$(\eta_1|\eta_2) := \big(\eta_1(U(\cdot)\varphi)\,|\,\eta_2(U(\cdot)\varphi)\big)_{H^j}\|\varphi\|^{-2} = (f_{\eta_1,\varphi}|f_{\eta_2,\varphi})_{H^j}\|\varphi\|^{-2},$$

then the $L_G(U, U^j)$ and $A(U, j)$ are isomorphic as Hilbert spaces:

$$(T_{\eta_1}|T_{\eta_2}) = (\eta_1|\eta_2) \quad \text{for any } \eta_i \in A(U, j),\ i = 1, 2,$$

$$(\eta_{T_1}|\eta_{T_2}) = (T_1|T_2) \quad \text{for any } T_i \in L_G(U, U^j).$$

Remark. In the next section we shall extend these results to projective σ-representations.

Such extension is necessary — as was remarked by Mackey [5] — to embrace the classical automorphic forms and θ-functions.

We precede the proof of the Theorem 2.2 by several lemmas

LEMMA 2.5. *Let η satisfy* 2(a)–2(c); *then $T_0 = \overset{0}{T}_\eta\colon \Phi \to H^j$ is sequentially closed.*

Proof. Let $\varphi_n \to 0$ in Φ and $T_0\varphi_n \to h$ in H^j, $n \to \infty$. Since $(T_0\varphi_n)$ is a Cauchy sequence in H^j, there exists a subsequence $n_k \to \infty$ such that $(T_0\varphi_{n_k})(g)$ is convergent for almost all $g \in G$ (it is the Riesz–Fischer theorem for H^j proved e.g. in Blattner [1]). But

$$T_0\varphi_{n_k}(g) = \eta\big(U(g)\varphi_{n_k}\big) \to 0 \quad \text{for all } g \text{ for } n_k \to \infty$$

(by the continuity of η). Hence $h(g) = 0$ for almost all $g \in G$: thus $h = 0$. □

LEMMA 2.6. *Let $\psi \in C_0(G, K)$; then for any compact $M \subset G$ the map $M \ni t \to U(t)U(\psi)h \in \Phi_{MK}$ is continuous.*

Proof. $U(t)U(\psi)h = U(L_t\psi)h = Q(L_t\psi \otimes h) \in \Phi_{MK}$. Since $G \ni t \mapsto L_t\psi \in C_0(G)$ is continuous, $G \ni t \to Q(L_t\psi \otimes h) \in \Phi$ is continuous.

We recall a classical result as

LEMMA 2.7. *Let B_1, B_2 be Banach spaces and let $T: B_1 \to B_2$ be closed and defined on a dense linear subset of B_1. If $f: G \to B_1$ is B_1-(Bochner) μ-integrable in B_1 and Tf is B_2-Bochner μ-integrable, then*

$$T\Big(\int_G f d\mu\Big) = \int_G (Tf)\,d\mu.$$

LEMMA 2.8. *Let $T_0\varphi := \eta\big(U(\cdot)^{-1}\varphi\big)$. Then $T_0 U(\psi)\varphi = U^j(\psi)T_0\varphi$ for any $\varphi \in \Phi$, $\psi \in C_0(G)$.*

Proof. Let $\varphi \in \Phi_K$ and $\psi \in C_0(G, M)$.

Since $T_0: \Phi \to H^j$ is sequentially closed, $T|\Phi_B: \Phi_B \to H^j$ is sequentially closed for any compact $B \subset G$. By Lemma 2.6, we can consider $M \ni t \mapsto U(t)\varphi \in \overline{\Phi}_{MK}$ (Banach space: completion of the normed space Φ_{MK}). Denote by $\overline{T}_0$ the closure of $T_0|\Phi_{MK}$. Since $\psi(\cdot)U(\cdot)\varphi$ is continuous, it is Bochner integrable and $\psi(\cdot)\overline{T}_0 U(\cdot)\varphi = \psi(\cdot)U^j(\cdot)T_0\varphi$ is a Bochner-integrable H^j-valued function. Thus in virtue of Lemma 2.7 we have

$$\begin{aligned} T_0 U(\psi)\varphi &= \overline{T}_0 \int \psi(t)\,U(t)\varphi\,dt = \int \overline{T}_0\big(\psi(t)\,U(t)\varphi\big)\,dt \\ &= \int \psi(t)\overline{T}_0\,U(t)\varphi\,dt = \int \psi(t)\,U^j(t)T_0\varphi\,dt \\ &= U^j(\psi)T_0\varphi. \ \square \end{aligned}$$

LEMMA 2.9. *T_0 considered as a map from a dense subset Φ of the Hilbert space H into H^j is closable.*

Proof. Let $\varphi_n \to 0$ in H and $T_0\varphi_n \to v$ in H^j.

By Lemma 2.3, $U(\psi) \in L(H, \Phi)$: $U(\psi)\varphi_n \to 0$ in Φ. But by Lemma 2.8

$$T_0 U(\psi)\varphi_n = U^j(\psi)T_0\varphi_n \to U^j(\psi)v \quad \text{in } H^j, \quad n \to \infty.$$

Hence, by the sequential closedness of $T_0: \Phi \to H^j$ (Lemma 2.5), $U^j(\psi)v = 0$ for each $\psi \in C_0(G)$. Thus $v = 0$ (for $\psi \to \delta_0$, $U^j(\psi)v \to v$).

LEMMA 2.10. *Let $T = T_\eta$ be the H-space closure of T_0. Then*

$$(*) \qquad TU(g) = U^j(g)T, \quad g \in G.$$

Thus by the theorem of v. Neumann and Najmark T is bounded and defined on the whole H. Hence $T \in L_G(U, U^j)$.

Proof of $(*)$. Let $h \in D(T)$, $D(T)$ domain of T.

Thus there exists a sequence $\varphi_n \in \Phi$ such that $\varphi_n \to h$ and $T\varphi_n = T_0\varphi_n \to Th$, $n \to \infty$.

Since $U(g)\varphi_n \to U(g)h$ and $TU(g)\varphi_n = T_0 U(g)\varphi_n = U^j(g)T_0\varphi_n \to U^j(g)Th$, by the closedness of T (the closure of T_0) $TU(g)\varphi_n \to TU(g)h$ we have

$$U^j(g)Th = \lim U^j(g)T_0\varphi_n = \lim T_0 U(g)\varphi_n = \lim TU(g)\varphi_n = TU(g)h. \ \square$$

Proof of D. T. It suffices to check that the linear maps $l : L_G(U, U^j) \to A(U, j)$, where $lT := \eta_T$, $\eta_T(\varphi) := (T\varphi)(e)$ and $k : A(U, j) \to L_G(U, U^j)$, $k\eta = T_\eta$, where $T_\eta\varphi := \eta(U^{-1}(\cdot)\varphi)$ (given in the preceding Theorems), satisfy $(l \circ k)\eta \equiv \eta$, $\eta \in A(U, j)$, and $(k \circ l)\, T \equiv T$, $T \in L_G(U, U^j)$.

But this is immediate:

$$lk\eta(\varphi) = \eta_{k\eta}(\varphi) = k\eta\varphi(e) = (T_\eta\varphi)(e) = \eta(\varphi) \quad \text{for every } \varphi \in \Phi,$$

$$\begin{aligned}(klT\varphi)(g) &= T_{lT}(\varphi)(g) = lT\big(U(g^{-1})\varphi\big) = T\big(U(g^{-1})\varphi\big)(e) \\ &= (T\varphi)(g) \quad \text{for all } g \in G \text{ and } \varphi \in \Phi.\end{aligned}$$

The isometry of k is almost obvious:

$$\begin{aligned}\|\varphi\|^2(T_{\eta_1}|T_{\eta_2}) &= a(\varphi|\varphi) = (T^*_{\eta_2}T_{\eta_1}\varphi|\varphi) = (T_{\eta_1}\varphi|T_{\eta_2}\varphi) \\ &= \eta_1\big(U^{-1}(\cdot)\varphi\big)|\eta_2\big(U^{-1}(\cdot)\varphi\big)_{H^j} = (\eta_1|\eta_2)\|\varphi\|^2 \\ &= (f_{\eta_1,\varphi}|f_{\eta_2,\varphi})_{H^j}. \ \square\end{aligned}$$

Remark. Our definition of $(\eta_1|\eta_2)$ is a far-reaching generalization of the scalar product introduced by Petersson in the early thirties.

3. EXTENSION OF D.T. FOR PROJECTIVE REPRESENTATIONS

Now we shall extend the results of Section 2 to projective σ-representations. Such a generalization is necessary for several applications (cf. Mackey [5])

DEFINITION 3.1. A continuous mapping $G \ni g \mapsto U(g) \in L(H)$, where $U(g)$ are unitary $U(e) = \mathrm{id}_H$ is a *projective representation* with multiplier σ, or a σ-representation if

$$U(g_1)U(g_2) = \sigma(g_1, g_2)U(g_1 \cdot g_2), \tag{3.1}$$

where $\sigma : G \times G \to S$ ($S = \{z \in C : |z| = 1\}$) such that

$$\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_3)\sigma(g_2, g_3), \tag{3.2}$$

$$\sigma(g, e) = \sigma(e, g) = 1. \tag{3.3}$$

Induced σ-representations. Let (j, V) be a σ-representation of the subgroup $\Gamma \subset G$, where σ satisfies (3.1)–(3.3). The space H^j of Section 1 is modified as follows:

DEFINITION 3.2.

1° $f : G \to V$ are dg-measurable,

2° $f(g\gamma) = \varrho(\gamma)^{1/2}j(\gamma)^{-1}\sigma(g, \gamma)f(g)$, $g \in G$, $\gamma \in \Gamma$,

3° $\|f(\cdot)\|_v^2$ is locally integrable on G,

4° $\int_G \varrho(g)^{-1}\|f(g)\|_v^2 d\mu(g) =: \|f\|_{H^j}^2 < \infty$,

5° $U_g^jf(x) = \sigma(g, g^{-1}x)f(g^{-1}x)$.

Definition 3.3. Let (U, H) be a σ_1-irreducible representation of G and let (j, V) be a σ-representation of Γ. A mapping $\eta: \Phi \to V$ is called a $(U, j; \sigma, \sigma_1)$-*automorphic form* if

3(a) $\eta \in L(\Phi, V)$,

3(b) $\eta(U(\gamma)\varphi) = \varrho(\gamma)^{-1/2} j(\gamma)\eta(\varphi)$,

3(c) $(\eta(U(g^{-1})\varphi)) \in H^j$.

Theorem 3.1. *The Duality Theorem extends without any changes to projective representations*:

The spaces $L_G(U, U^j)$ *and* $A(U, j; \sigma, \sigma_1)$ *are isomorphic as Hilbert spaces.*

In order to see this we shall make some simple

Remarks. Let G^{σ_1} be a σ_1-extension of G: i.e. $G^{\sigma_1} = S \times G$ and the group operation is defined as follows:

$$(z_1, g_1)(z_2, g_2) := (z_1 z_2 \sigma_1(g_1, g_2), g_1 g_2). \tag{3.4}$$

Plainly

$$(1, e) = 1, \qquad (z, g)^{-1} = \left(\frac{1}{z\sigma_1(g, g^{-1})}, g^{-1}\right). \tag{3.5}$$

Following Mackey [5] we define (U^0, H) as a unitary representation of G^{σ_1} in H by

$$U^0(z, g) := zU(g). \tag{3.6}$$

The subsequent lemma shows that our space Φ is good for σ_1-representations:

Lemma 3.1. *Let* $\psi_1(z, g) := \psi(g)$, *where* $\psi \in C_0(G)$ (*thus* $\psi_1 \in C_0(G^{\sigma_1})$). *Put* $c = \int_S z\,dz$; *then*

$$U^0(\psi_1)h = cU(\psi)h, \tag{3.7}$$

$$U(t)U(\psi) = U(L_t\psi), \tag{3.8}$$

$$G \ni t \to U(t)\varphi \in \Phi \quad \textit{is continuous for } \varphi \in \Phi. \tag{3.9}$$

$$\textit{For each } \psi \in C_0(G), \quad U(\psi) \in L(H, \Phi). \tag{3.10}$$

Proof in Appendix.

Lemma 3.2. (σ-*representation* (U, H) *is irreducible*) $\Leftrightarrow$ ((U^0, H) *is irreducible*).

Lemma 3.3. *If* (U, H) *is an irreducible projective representation of* G *and* (U_1, H_1) *any* σ-*representation, then every closed densely defined intertwining map* $T: H \to H_1$ *such that* $TU(g) = U_1(g)T$ *is bounded.*

Proof. Since T intertwines the irreducible unitary representation U^0 of G^{σ_1} with U_1^0, then by the v. Neumann–Najmark Theorem T is bounded.

Since all lemmas and propositions are valid for projective representations (cf. Appendix), the Duality Theorem for $(U, j; \sigma, \sigma_1)$-forms is proved. $\square$

4. AUTOMORPHIC FORMS AND SYSTEMS OF IMPRIMITIVITY (S.I.)

We recall

DEFINITION 4.1. Let $P: C_0(G/\Gamma) \to L(H)$ be such a $*$-homomorphism that

$$U(g)P(\psi^b)U(g)^{-1} = P(L_g\psi^b).$$

Then the pair $\{P, U\}$ is called a *system of imprimitivity* with base G/Γ.

In this section we prove the following

THEOREM 4.1. 1° *Each system of imprimitivity defines an automorphic form* η *by the formula*

$$\eta(\varphi) := [\varphi], \quad \textit{where } [\varphi] := \varphi + \operatorname{Ker}\beta,$$

$\beta(\cdot, \cdot)$ *being a semi-scalar product on* Φ *given by*

$$\beta(\varphi, \varphi) := \lim_{\psi \to \delta_e} \big(P(\psi^b)\varphi | \varphi\big), \qquad \operatorname{Ker}\beta = \{\varphi \in \Phi : \beta(\varphi, \varphi) = 0\}.$$

Proof 1°. (This construction we owe to Niels Skovhus-Poulsen; he used it in his elegant proof of the imprimitivity theorem communicated on the conference in Aarhus, May 1972: "Open House for Functional Analysis".)

Let $\beta(\varphi, \varphi) := \lim_{\psi \to \delta_e} \big(P(\psi^b)\varphi | \varphi\big)_H$.

Plainly $\beta(\varphi, \varphi) \geqslant 0$ and β is hermitian on $\Phi \times \Phi$. Let $\Phi \ni \varphi \mapsto [\varphi] = \varphi + \operatorname{Ker}\beta \in V_0 := \Phi/\operatorname{Ker}\beta$ be the natural projection and $\langle [\varphi_1] | [\varphi_2] \rangle := \beta(\varphi_1, \varphi_2)$. Then $(V_0, \langle \cdot | \cdot \rangle)$ is a pre-Hilbert space.

Plainly $\Phi \ni \varphi \to \eta(\varphi) := [\varphi] \in V_0$ is linear and continuous. Since $\varrho(\gamma)\beta(\varphi_1, \varphi_2) = \beta\big(U(\gamma)\varphi_1, U(\gamma)\varphi_2\big)$,

$$j_0(\gamma)[\varphi] := [\varrho(\gamma)^{-1/2} U(\gamma)\varphi]$$

is a well-defined isometry on V_0. But $j_0(\gamma_1\gamma_2) = j_0(\gamma_1) \cdot j_0(\gamma_2)$, $\gamma_1, \gamma_2 \in \Gamma$; hence by closure we obtain a unitary representation $\gamma \to \overline{j_0}(\gamma) =: j(\gamma)$ (j, V) of the subgroup Γ in the Hilbert space $(V, \langle \cdot | \cdot \rangle)$.

Thus $j(\gamma)\eta(\varphi) = \varrho(\gamma)^{-1/2}\eta\big(U(\gamma)^{-1}\varphi\big)$

$$\infty > \big(P(\psi^b)\varphi|\varphi\big) = \int_G \beta\big(U(g^{-1})\varphi,\ U(g^{-1})\varphi\big)\psi(g)\,dg$$

$$= \int_G \|\eta\big(U(g^{-1})\varphi\big)\|_v^2\psi(g)\,dg,$$

and

$$\|f_{\eta,\varphi}\|_{H^j}^2 := \sup_{0\leqslant\psi^b\leqslant 1} \big(P(\psi^b)\varphi|\varphi\big) = \|\varphi\|_H^2.$$

Thus $\varphi\to[\varphi]$ is a (U, j)-automorphic form. □

5. RECIPROCITY THEOREMS

In this section we formulate several reciprocity theorem of the Frobenius–Bruhat type. They are immediate generalizations of the theorems proved in Part II of G.D.T., for unitary representations (U, H) and (j, V). Since these proofs use only the Duality Theorem, its corollaries from Section 3 and Proposition 2.2, they go over without any change to projective σ-representations and we formulate the results only.

THEOREM 5.1. *If G/Γ is compact or if G/Γ has a finite invariant measure and* $\dim H < \infty$ *and (U, H) and (j, V) are projective representations, then there holds the Frobenius reciprocity*

$$L_G(U, U^j) \cong L_\Gamma(j_\varrho, U|\Gamma), \tag{F.R}$$

where $j_\varrho(\gamma) := \varrho(\gamma)^{-1/2}j(\gamma)$.

THEOREM 5.2. *Let G be unimodular and σ-compact and let the projective representation U be irreducible and square-integrable, i.e.* $\big(U(\cdot)h|k\big) \in L^2(G)$ *for any h, k. Then if Γ is compact we have the isometric isomorphism*

$$L_G(U, U^j) \cong (H-S)_\Gamma(j, U|\Gamma), \tag{5.1}$$

where the right-hand side of (5.1) *denotes the space of Hilbert–Schmidt maps: $H\to V$ interwining for $U|\Gamma$ and j. This space is provided with the Hilbert–Schmidt scalar product.*

APPENDIX

Proofs of lemmas. We shall now give the proof of a generalized version of Proposition 2.1 (for σ-representations). Let (U, H) be a projective representation of G, and (U^j, H^j) – a representation induced by the σ-representation (j, V) of the subgroup $\Gamma\subset G$. If intertwines U and U^j, then the continuous function

$$f(x) := \int_G \psi(t)\sigma(t, t^{-1}x)Th(t^{-1}x)\,dt \tag{A.1}$$

is a representative of $TU(\psi)h$, where $\psi\in C_0(G)$, $h\in H$.

The proof consists of two points:

1° $f(\cdot)$ – defined by (A.1) is an element of H^j,

2° $f = U^j(\psi)Th$.

Ad 1°: Plainly f is locally integrable. Take $0 \leqslant \theta \in C_0(G)$, $0 \leqslant \theta^b \leqslant 1$,

$$\begin{aligned} I := &\left|\int \theta(g)\langle f(g)|f(g)\rangle_V dg\right| \\ = &\left|\int \theta(g)\left\langle \int \psi(t)\sigma(t, t^{-1}g)Th(t^{-1}g)\,dt | f(g)\right\rangle dg\right| \\ \leqslant &\int |\psi(t)| \left|\int \theta(g)\sigma(t, t^{-1}g)\langle Th(t^{-1}g)|f(g)\rangle dg\right| dt. \end{aligned}$$

(We can interchange the integrations (Tonelli!) since the integrand has a compact support.) But

$$\begin{aligned} &\left|\int \theta(g)\sigma(t, t^{-1}g)\langle Th(t^{-1}g)|f(g)\rangle_V dg\right| \\ &= \left|\int \theta(g)\int \sigma(t, t^{-1}g)\langle Th(t^{-1}g)|\psi(s)\sigma(s, s^{-1}g)Th(s^{-1}g)\rangle ds\, dg\right| \\ &\leqslant \int |\psi(s)| \int \theta(g)\left|\sigma(t, t^{-1}g)\right| |\sigma(s, s^{-1}g)| |\langle Th(t^{-1}g)|Th(s^{-1}g)\rangle| dg\, ds \\ &= \int |\psi(s)| \int \theta(g) \left|\langle U^j(t)Th(g)|\, U^j(s)Th(g)\rangle\right| dg\, ds \\ &\leqslant \int |\psi(s)| \int \theta(g) \|U^j(t)Th(g)\|_V \|U^j(s)Th(g)\|_V dg\, ds \\ &\leqslant \int |\psi(s)| \Big(\int \theta(g)\|U^j(t)Th(g)\|_V^2 dg\Big)^{1/2} \Big(\int \theta(g)\|U^j(s)T\varphi(g)\|_V^2 dg\Big)^{1/2} ds \\ &\leqslant \int |\psi(s)|\, \|\theta^b\|_\infty \|U^j(t)Th\|_{H^j} \|U^j(s)Th\|_{H^j} ds \leqslant \|\psi\|_1 \|\theta^b\|_\infty \|Th\|_{H^j}^2. \end{aligned}$$

Hence finally

$$I \leqslant \int |\psi(t)|\,dt\, \|\psi\|_1 \|\theta^b\|_\infty \|Th\|_{H^j}^2 = \|\psi\|_1^2 \|Th\|_{H^j}^2 \|\theta^b\|_\infty .$$

But

$$\begin{aligned} \|f\|_{H^j}^2 &= \sup_{0 \leqslant \theta^b \leqslant 1} \int_G \theta(g)\|f(g)\|_V^2 dg \\ &\leqslant \sup_{0 \leqslant \theta^b \leqslant 1} (\|\psi\|_1^2 \|Th\|_{H^j}^2 \|\theta^b\|_\infty) = \|\psi\|_1^2 \|Th\|_{H^j}^2 < \infty. \end{aligned}$$

Hence $f \in H^j$.

Ad 2°. It suffices to prove the identity

$$(f|l)_{H^j} \underset{l}{=} (U^j(\psi)Th|l)_{H^j} \quad \text{for } l \text{ from a total set in } H^j.$$

But we have a classical

SUBLEMMA (Mackey, cf. Bruhat [2] or G. Warner [8]).

$$\Big\{l : l(x) := \int_\Gamma \varrho(\gamma)^{-1/2}\theta(x\gamma)\sigma(x, \gamma)j(\gamma)v\,d\gamma,\ \theta \in C_0(G),\ v \in V\Big\},$$

is total in H^j and each l has a compact support modulo Γ (i.e. if $S = \operatorname{spt} l$, then $\pi(S) \subset K$ is a compact subset of G/Γ, where $\pi(x) := x\Gamma$).

Proof of 2°. Write $F_t(\pi(x)) = \varrho(x)^{-1}\langle U^j(t)Th(x)\,|\,l(x)\rangle_V$. But $\mathrm{spt}F_t \subset K$ since $\mathrm{spt}F_t \subset \mathrm{spt}\,l$. Thus

$$(t, \dot{x}) \mapsto \psi(t)\,\varrho(x)^{-1}\langle U^j(t)Th(x)\,|\,l(x)\rangle_V$$

is a measurable function with support $\mathrm{spt}\,\psi\times K$ (a compact subset in $G\times(G/\Gamma)$). Hence we can apply the Tonelli Theorem:

$$\begin{aligned}(f\,|\,l)_{H^j} &= \int_{G/\Gamma} \varrho(x)^{-1}\langle f(x)\,|\,l(x)\rangle\, d\mu(\dot{x})\\ &= \int_{G/\Gamma} \varrho(x)^{-1} \int_G \psi(t)\langle\sigma(t, t^{-1}x)Th(t^{-1}x)\,|\,l(x)\rangle\, dt\, d\mu(\dot{x})\\ &= \int_G \psi(t)\Big(\int_{G/\Gamma} \varrho(x)^{-1}\sigma(t, t^{-1}x)\langle Th(t^{-1}x)\,|\,l(x)\rangle\, d\mu(\dot{x})\Big)\, dt\\ &= \int_G \psi(t)\big(U^j(t)Th\,|\,l\big)_{H^j} dt = \big(U^j(\psi)Th\,|\,l\big)_{H^j}. \ \square\end{aligned}$$

LEMMA A.1 (Skovhus–Poulsen). *Let $\{P, U\}$ be an S.I. based on G/Γ and let for $h, k\in H$, $\nu_{h,k}$ be Radon masure on G such that*

$$\big(P(\psi^b)\,h\,|\,k\big) = \int_G \psi(g)\,d\nu_{h,k}(g).$$

If $h, k\in\Phi$, then there exists a continuous function $m_{h,k}(\cdot)$ on G such that

$$d\nu_{h,k}(g) = m_{h,k}(g)\,dg. \tag{A.2}$$

If $h = U(\psi)x$, $k = U(\theta)y$, $\psi, \theta\in C_0(G)$, $x, y\in H$, then there exists a measure μ (depending on x, y) on $G\times G$ such that

$$m_{h,k}(g) = \int_{G\times G} \psi(lg)\,\bar{\theta}(cg)\,d\mu(b, c).$$

Remark. The sesquilinear form $\beta(\cdot,\cdot)$ in Theorem 4.1 is given by: $\beta(h, k) := m_{h,k}(e)$, $h, k\in\Phi$.

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