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On some class of midconvex functions

by KAZIMIERZ NIKODEM (Bielsko-Biała)

Abstract. The class of functions having the representation $f = F+A$, where F is convex and A is additive, is considered. Numerous properties and conditions characterizing such functions are given.

1. Let $\boldsymbol{R}$ denote the real line and assume that D is a convex subset of the space $\boldsymbol{R}^n$. A function $f\colon D \to \boldsymbol{R}$ is said to be *convex* iff $f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$ for all $x, y \in D$ and $t \in [0, 1]$; it is said to be *midconvex* (or *convex in the sense of Jensen*) iff $f\left(\frac{1}{2}(x+y)\right) \leqslant \frac{1}{2}\left(f(x)+f(y)\right)$ for all $x, y \in D$. We say that a function $A\colon \boldsymbol{R}^n \to \boldsymbol{R}$ is *additive* iff it satisfies the Cauchy functional equation $A(x+y) = A(x)+A(y)$ for all $x, y \in \boldsymbol{R}^n$.

Assume that D is a fixed open convex subset of $\boldsymbol{R}^n$ and consider the following class of functions:

$$\mathscr{F} := \{f\colon D \to \boldsymbol{R}\colon f \text{ has the representation } f = F+A, \text{ where } F\colon D \to \boldsymbol{R} \text{ is convex and } A\colon \boldsymbol{R}^n \to \boldsymbol{R} \text{ is additive}\}.$$

In this paper, we give many properties and conditions characterizing functions belonging to this class.

First notice that all convex functions defined on D belong to $\mathscr{F}$ and all functions belonging to $\mathscr{F}$ are midconvex. Moreover, $f+g \in \mathscr{F}$ and $cf \in \mathscr{F}$ whenever $f, g \in \mathscr{F}$ and $c \geqslant 0$. So we have the following.

PROPOSITION 1. *The class $\mathscr{F}$ is a cone in the space of all functions from D into $\boldsymbol{R}$. It contains the class of all convex functions defined on D and is contained in the class of all midconvex functions defined on D.*

PROPOSITION 2. *If a function $f\colon D \to \boldsymbol{R}$ belongs to the class $\mathscr{F}$, then it is either continuous on D or its graph is dense in $D \times \boldsymbol{R}$.*

Proof. This follows immediately from the fact that discontinuous additive functions $A\colon \boldsymbol{R}^n \to \boldsymbol{R}$ have their graphs dense in $\boldsymbol{R}^{n+1}$ (cf. [4], p. 277).

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Example 1. Assume that a: $\boldsymbol{R}^n \to \boldsymbol{R}$ is a discontinuous additive function and g: $\boldsymbol{R} \to \boldsymbol{R}$ is a convex, non-constant, bounded below function. Then the function $f := g \circ a$ is midconvex, discontinuous and bounded below. So, in view of Proposition 2, it does not belong to the class $\mathscr{F}$.

The next example shows that there exist midconvex functions f: $D \to \boldsymbol{R}$, with graphs dense in $D \times \boldsymbol{R}$, which do not belong to $\mathscr{F}$.

Example 2. Assume that a: $\boldsymbol{R} \to \boldsymbol{R}$ is a discontinuous additive function and put $f(x) := a(x) + \exp(a(x))$, $x \in \boldsymbol{R}$. Of course, f is midconvex and $f \notin \mathscr{F}$ (cf. Example 1 and Proposition 1). Fix a point $(x_0, y_0) \in \boldsymbol{R}^2$ and a number $\varepsilon > 0$. Since the function $t \to t + \exp t$ maps $\boldsymbol{R}$ onto $\boldsymbol{R}$, there exists a point $t_0 \in \boldsymbol{R}$ such that $t_0 + \exp t_0 = y_0$. By continuity of the function exp there exists a $\delta \in (0, \frac{1}{2}\varepsilon)$ such that $|\exp t - \exp t_0| < \frac{1}{2}\varepsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Since the graph of a is dense in $\boldsymbol{R}^2$, we infer that $a(x) \in (t_0 - \delta, t_0 + \delta)$ for some $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Then

$$\begin{aligned} |y_0 - a(x) - \exp(a(x))| &\leqslant |y_0 - t_0 - \exp t_0| + |t_0 - a(x)| + |\exp t_0 - \exp(a(x))| \\ &\leqslant 0 + \delta + \tfrac{1}{2}\varepsilon < \varepsilon. \end{aligned}$$

Thus, $(x, f(x)) \in (x_0 - \varepsilon, x_0 + \varepsilon) \times (y_0 - \varepsilon, y_0 + \varepsilon)$, which proves that the graph of f is dense in $\boldsymbol{R}^2$.

2. In the next theorem we shall give a few conditions characterizing midconvex functions belonging to the class $\mathscr{F}$. Let us begin with some definitions. A function A: $\boldsymbol{R}^n \to \boldsymbol{R}$ is *affine* iff it is the sum of an additive function and a constant. A function f: $D \to \boldsymbol{R}$ is said to be *concave* (*midconcave*) iff the function $-f$ is convex (midconvex). It is said to be *locally approximately midconcave* iff for every $x_0 \in D$ and for every $\varepsilon > 0$ there exists an $r > 0$ such that the open ball $B(x_0, r)$ is included in D and $f(\frac{1}{2}(x+y)) \geqslant \frac{1}{2}(f(x) + f(y)) - \varepsilon$ for all $x, y \in B(x_0, r)$. We say that the *Jensen difference* of a function f: $D \to \boldsymbol{R}$ is *bounded on a ball* $B \subset D$ iff there exists a constant $M > 0$ such that $|f(\frac{1}{2}(x+y)) - \frac{1}{2}(f(x) + f(y))| \leqslant M$ for all $x, y \in B$. Finally, let us recall the following definition of the class $\mathscr{A}$ of sets introduced by Ger and Kuczma [1]: a *set* $T \subset \boldsymbol{R}^n$ *belongs to* $\mathscr{A}$ iff every midconvex function defined on a convex open set D containing T and bounded above on T is continuous in D. It is worth to remind that sets having positive inner Lebesgue measure as well as second category sets with the Baire property belong to the class $\mathscr{A}$. However, there exist nowhere dense sets of the Lebesgue measure zero which also belong to $\mathscr{A}$ (cf. [4], p. 210).

Theorem 1. *Let* f: $D \to \boldsymbol{R}$ *be a midconvex function. Then the following conditions are equivalent*:

(1) f *is locally upper bounded at each point* $x \in D$ *by an affine function*;

(2) f *is locally upper bounded at each point* $x \in D \setminus \{0\}$ *by an additive function*;

(3) *f is upper bounded on a set $T \subset D$ belonging to the class $\mathscr{A}$ by a midconvex function g: $D \to \mathbf{R}$;*

(4) *f is the sum of a continuous function and a midconcave function;*

(5) *f is locally approximately midconcave on D;*

(6) *the Jensen difference of f is bounded on an open ball $B \subset D$;*

(7) $f \in \mathscr{F}$.

Proof. (1)$\Rightarrow$(2). Fix a point $y = (y_1, \ldots, y_n) \in D \setminus \{0\}$. By the assumption there exist a neighbourhood $U \subset D$ of y, an additive function a: $\mathbf{R}^n \to \mathbf{R}$ and a constant $c \in \mathbf{R}$ such that $f(x) \leqslant a(x)+c$ for all $x \in U$. Since $y \neq 0$, we have $y_i \neq 0$ for some $i \in \{1, \ldots, n\}$. Assume, for example, that $y_i > 0$ and take an $\varepsilon \in (0, y_i)$ such that the ball $B(y, \varepsilon) \subset U$. Consider the function A: $\mathbf{R}^n \to \mathbf{R}$ defined by

$$A(x) := \frac{|c|}{y_i - \varepsilon} x_i, \qquad x = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

This function is additive and $A(x) > |c|$ for all $x \in B(y, \varepsilon)$ because for such x we have $x_i > y_i - \varepsilon$. Hence $f(x) \leqslant a(x) + A(x)$, $x \in B(y, \varepsilon)$, and the function $a + A$ is additive. In the case where $y_i < 0$ the proof is analogous.

The implication (2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(4). Put $\varphi(x) := f(x) - g(x)$, $x \in D$. Clearly, φ is midconvex and non-negative on T. Since $T \in \mathscr{A}$, we infer that φ is continuous. So $f = \varphi + g$ is of the claimed form.

In order to prove that (4) implies (5) fix a point $x_0 \in D$ and a positive number ε. By the continuity of φ there exists a ball $B(x_0, \delta) \subset D$ such that $|\varphi(x) - \varphi(x_0)| < \frac{1}{2}\varepsilon$ for each $x \in B(x_0, \delta)$. Then for all $x, y \in B(x_0, \delta)$ we have

$$\begin{aligned} f\left(\tfrac{1}{2}(x+y)\right) - \frac{f(x)+f(y)}{2} &= \varphi\left(\tfrac{1}{2}(x+y)\right) - \frac{\varphi(x)+\varphi(y)}{2} + g\left(\tfrac{1}{2}(x+y)\right) - \frac{g(x)+g(y)}{2} \\ &\geqslant \varphi\left(\tfrac{1}{2}(x+y)\right) - \frac{\varphi(x)+\varphi(y)}{2} \\ &> \varphi(x_0) - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\left(\varphi(x_0) + \tfrac{1}{2}\varepsilon + \varphi(x_0) + \tfrac{1}{2}\varepsilon\right) = -\varepsilon, \end{aligned}$$

which was to be proved.

The implication (5)$\Rightarrow$(6) is obvious.

(6)$\Rightarrow$(7). Assume that $\left|f\left(\frac{1}{2}(x+y)\right) - \frac{1}{2}(f(x)+f(y))\right| \leqslant M$ for all $x, y \in B$ and some $M > 0$. By a theorem of Z. Kominek on the local stability of the Jensen functional equation (cf. [3]) there exist a function g: $B \to \mathbf{R}$ satisfying the Jensen equation and a constant $M_1 > 0$ such that $|f(x) - g(x)| \leqslant M_1$ for all $x \in B$. It is well known that g must be of the form $g = A + c$, where A: $\mathbf{R}^n \to \mathbf{R}$ is an additive function and c is a real constant. Let $F(x) := f(x) - A(x)$, $x \in D$. Then F is midconvex and $F(x) = f(x) - g(x) + c \leqslant M_1 + c$, $x \in B$, i.e., F is upper bounded on B. Hence, in view of a theorem of

Bernstein–Doetsch (cf. [4], [9]), F is continuous and so it is convex. Thus $f = F+A$ belongs to the class $\mathscr{F}$.

The implication (7) $\Rightarrow$ (1) is evident. In such a way the proof is finished. $\square$

Remark. The implication (3) $\Rightarrow$ (7) of Theorem 1 is connected with some problem posed by the author and has been proved independently by C. T. Ng [6], Kominek [3] and the author [8]. The proof presented here bases on the method used in [3].

3. Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \boldsymbol{R}^m$ $(m \geqslant 2)$. We say that x is *majorized by* y, and write $x \prec y$, if $(x_1, \ldots, x_m) = (y_1, \ldots, y_m) \cdot P$ for some double stochastic $m \times m$ matric P. A function φ: $S \to \boldsymbol{R}$ defined on a subset S of $\boldsymbol{R}^m$ is said to be *Schur-convex* on S iff $\varphi(x) \leqslant \varphi(y)$ for all $x, y \in S$, $x \prec y$ (cf. [5], p. 62). It is known that if f: $D \to \boldsymbol{R}$ is a convex function, then for every $k \geqslant 2$ the function φ: $D^k \to \boldsymbol{R}$ defined by

$$\varphi(x_1, \ldots, x_k) := f(x_1) + \ldots + f(x_k)$$

is Schur-convex; however, the sum may be Schur-convex also for non-convex functions. Recently C. T. Ng [7] has proved that a function f generates Schur-convex sums if and only if it belongs to the class $\mathscr{F}$. The main part of the proof of this theorem consisted in showing (by use of a theorem of De Bruijn and Kemperman [2]) that functions satisfying the functional inequality

$$f(tx+(1-t)y) + f((1-t)x+ty) \leqslant f(x) + f(y)$$

belong to the class $\mathscr{F}$. In the next theorem we shall give another proof of this fact basing on the implication (3) $\Rightarrow$ (7) of Theorem 1. We shall also give some connection with quasi-convex functions. Recall that a function f: $D \to \boldsymbol{R}$, where D is a convex subset of $\boldsymbol{R}^n$, is said to be *quasi-convex* iff

$$f(tx+(1-t)y) \leqslant \max\{f(x), f(y)\}$$

for all $x, y \in D$ and $t \in [0, 1]$ (cf. [9], p. 228). Of course every convex function is quasi-convex but quasi-convex functions need not be even midconvex. However, we have the following

PROPOSITION 3. *A function f: $D \to \boldsymbol{R}$ defined on a convex open subset D of $\boldsymbol{R}^n$ is convex if and only if it is midconvex and quasi-convex.*

Proof. Assume that f is midconvex and quasi-convex and consider the level sets $L_n := \{x \in D: f(x) \leqslant n\}$, $n \in N$. Since $\bigcup_{n \in N} L_n = D$, there exists an $m \in N$ such that L_m is of the second category. By the quasi-convexity the set L_m is convex and so it has the Baire property. Hence L_m belongs to the class $\mathscr{A}$. Thus f being midconvex and upper bounded on L_m is convex. The converse implication is trivial. $\square$

If a function $f: D \to \mathbf{R}$ is midconvex and instead of the quasi-convexity it satisfies a somewhat weaker condition, then it belongs to the class $\mathcal{F}$. Namely, we have the following

THEOREM 2. *Let $f: D \to \mathbf{R}$ be a function defined on a non-empty open convex subset of $\mathbf{R}^n$. Then the following statements are equivalent:*

(1) *f satisfies the functional inequality*

$$f(tx+(1-t)y)+f((1-t)x+ty) \leqslant f(x)+f(y)$$

for all $x, y \in D$ and $t \in [0, 1]$;

(2) *f is midconvex and satisfies the functional inequality*

$$f(tx+(1-t)y)+f((1-t)x+ty) \leqslant 2\max\{f(x), f(y)\}$$

for all $x, y \in D$ and $t \in [0, 1]$;

(3) *$f \in \mathcal{F}$.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$. Fix a point $p \in D$ and take an $\varepsilon > 0$ such that the closed ball $B(p, \varepsilon)$ is included in D. Assume that $\{e_1, \ldots, e_n\}$ is the standard ortonormal base in $\mathbf{R}^n$. Denote by L_i the line segment joining the points $a_i := p+\varepsilon e_i$ and $b_i := p-\varepsilon e_i$, $i = 1, \ldots, n$. For every $x \in L_i$ there exists a $t \in [0, 1]$ such that $x = ta_i+(1-t)b_i$. Then $2p-x = (1-t)a_i+tb_i$. Hence, using the assumed inequality, we obtain

$$f(x)+f(2p-x) \leqslant 2\max\{f(a_i), f(b_i)\} \quad \text{for all } x \in L_i.$$

Put $M := \max\{f(a_1), \ldots, f(a_n), f(b_1), \ldots, f(b_n)\}$ and consider the function $g: B(p, \varepsilon) \to \mathbf{R}$ defined by $g(x) := -f(2p-x)$, $x \in B(p, \varepsilon)$. In view of the midconvexity of f we infer that g is midconcave. Moreover, on account of the above inequality, we have

$$f(x) \leqslant g(x)+2M \quad \text{for all } x \in \bigcup_{i=1}^{n} L_i.$$

Since f is midconvex and g is midconcave, we have the same estimation on the set

$$\left\{\frac{x_1+\ldots+x_n}{n}: x_1, \ldots, x_n \in \bigcup_{i=1}^{n} L_i\right\}.$$

This set belongs to the class $\mathcal{A}$ because its interior is non-empty. Therefore, by Theorem 1, there exist an additive function $A: \mathbf{R}^n \to \mathbf{R}$ and a convex function $F_1: B(p, \varepsilon) \to \mathbf{R}$ such that $f = A+F_1$ on $B(p, \varepsilon)$. Put $F(x) := f(x) - A(x)$, $x \in D$. Then F is midconvex and $F = F_1$ on $B(p, \varepsilon)$, so by the theorem of Bernstein–Doetsch F is convex. Thus $f = F+A \in \mathcal{F}$.

The implication $(3) \Rightarrow (1)$ is evident. This completes the proof. □

4. In this section we shall give another application of Theorem 1. Let us begin with some preliminaries. For a given set $B \subset \boldsymbol{R}^{n+1}$ $(= \boldsymbol{R}^n \times \boldsymbol{R})$ we denote by $\pi(B)$ its projection on the space $\boldsymbol{R}^n$ and by B_x, $x \in \boldsymbol{R}^n$, its x-section (i.e., $B_x := \{y \in \boldsymbol{R}: (x, y) \in B\}$). Assume that B is a convex subset of $\boldsymbol{R}^{n+1}$ and A: $\boldsymbol{R}^n \to \boldsymbol{R}$ is an additive function and consider the set

$$C := \bigcup_{x \in \pi(B)} \big(\{x\} \times (B_x + A(x))\big).$$

One can easily check that C is midconvex (i.e., $\frac{1}{2}(c_1 + c_2) \in C$ whenever $c_1, c_2 \in C$), $\pi(C)$ is convex and C_x is convex for each $x \in \pi(C)$. It seems interesting to ask if for every set $C \subset \boldsymbol{R}^{n+1}$ having the above properties there exists an additive function A: $\boldsymbol{R}^n \to \boldsymbol{R}$ such that the corresponding set

$$B := \bigcup_{x \in \pi(C)} \big(\{x\} \times (C_x - A(x))\big)$$

is convex. Without any further assumptions such a question has negative answer (see Example 3). However, if $\pi(C)$ is open and all the sections C_x, $x \in \pi(C)$, are compact, the answer is positive (the convexity of C_x need not be assumed then). Namely, we have the following

THEOREM 3. *Assume that C is a midconvex subset of $\boldsymbol{R}^{n+1}$ $(n \geqslant 1)$. If its projection $\pi(C)$ on the space $\boldsymbol{R}^n$ is convex and open and for every $x \in \pi(C)$ the section C_x is compact, then there exists an additive function A: $\boldsymbol{R}^n \to \boldsymbol{R}$ such that the set $B = \bigcup_{x \in \pi(C)} \big(\{x\} \times (C_x - A(x))\big)$ is convex.*

Proof. Let us notice first that sections C_x are convex because they are closed and, in view of the midconvexity of C, $\frac{1}{2}(C_x + C_x) = C_x$. Put $D := \pi(C)$ and consider the functions f, g: $D \to \boldsymbol{R}$ defined by

$$f(x) := \inf C_x \quad \text{and} \quad g(x) := \sup C_x$$

for all $x \in D$. By the midconvexity of C it follows that f is midconvex and g is midconcave. Moreover, $f \leqslant g$ on D. Therefore, in virtue of Theorem 1, there exist an additive function A: $\boldsymbol{R}^n \to \boldsymbol{R}$ and a convex function F: $D \to \boldsymbol{R}$ such that $f = A + F$ on D. The function $G := g - A$ defined on D is then concave because it is midconcave and locally lower bounded ($G \geqslant F$ on D). Now, using the fact that $C_x = [f(x), g(x)]$, $x \in D$, we obtain

$$B = \bigcup_{x \in D} \big(\{x\} \times (C_x - A(x))\big) = \bigcup_{x \in D} \big(\{x\} \times [F(x), G(x)]\big).$$

This shows that the set B is convex. □

EXAMPLE 3. Consider the set $C := \{(x, y) \in \boldsymbol{R}^2: y \geqslant |a(x)|\}$, where a: $\boldsymbol{R} \to \boldsymbol{R}$ is a discontinuous additive function. This set is midconvex, its projection on the first axis $\pi(C) = \boldsymbol{R}$ is convex and open and all the sections C_x, $x \in \boldsymbol{R}$, are convex. However, there is no additive function A: $\boldsymbol{R} \to \boldsymbol{R}$ such that the set $B = \bigcup_{x \in R} (\{x\} \times (C_x - A(x)))$ is convex (cf. Example 1). Analogous properties has the set $C := \boldsymbol{R} \times (0, 1) \cup \boldsymbol{Q} \times \{0\}$, where $\boldsymbol{Q}$ denotes the set of all rationals.

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
BIELSKO-BIAŁA, POLAND

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