ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

Decompositions of tensor products of contractions

by IRENA BROSZKIEWICZ (Kraków)

Abstract. The present paper concerns canonical decompositions of some semigroups of contractions in Hilbert spaces. It is assumes that these semi-groups have unitary dilations. We apply advanced methods of prediction theory of stationary second order stochastic processes; see [2].

1. Let H be a complex Hilbert space. $L(H)$ stands for the algebra of all linear bounded operators in H. Suppose that G is an additive subgroup of the reals. We write $G_+ = \{t\colon t \in G,\ t \geq 0\}$. In all what follows, by a semi-group of contractions we mean a function $G_+ \ni t \to T(t) \in L(H)$ such that $T(0) = I$ (identity operator) and $T(t_1 + t_2) = T(t_1)T(t_2)$ for $t_1, t_2 \in G_+$ and $\|T(t)\| \leq 1$ for all $t \in G_+$.

A generalization of Sz.-Nagy theorem [3] proved in [1] asserts that then $T(t)$ has a unitary dilation. This means that there is a space K such that $H \subset K$ and a unitary representation $U(t) \in L(K)$ of G such that $T(t)f = PU(t)f$ for $f \in H$, $t \in G_+$, P being the orthogonal projection of K onto H. We then say that $U(\cdot)$ is a *unitary dilation* of $T(\cdot)$.

The space H can be written as $H = H^u \oplus H^c$ where H^u reduces all $T(t)$ to unitary operators and no non-zero part of H^c shares this property. The part $T^u(t)$ of $T(t)$ in H^u is called the *unitary part* of $T(t)$ and the part $T^c(t)$ of $T(t)$ in H^c a *completely non-unitary semigroup* of contractions. The representation $T(t) = T^u(t) \oplus T^c(t)$ is then unique; we call it the *canonical decomposition* of $T(t)$.

Suppose that we are given two semi-groups of contractions, $T_1(t) \in L(H_1)$ and $T_2(t) \in L(H_2)$. It is plain that the product $T(t) = T_1(t) \otimes T_2(t) \in L(H_1 \otimes H_2)$ is a semi-group of contractions. Our problem is to find out how the canonical decomposition of $T(t)$ depends on those of the factors $T_1(t)$ and $T_2(t)$.

2. Let $T(t)$ be a semi-group of contractions in H and $U(t)$ its unitary dilation, necessarily minimal.

We define

$$M_+ = \bigvee_{t \in G_+} U(t)H, \quad M_- = \bigvee_{-t \in G_+} U(t)H,$$

and

$$R_+ = \bigcap_{t \in G_+} U(t) M_+, \quad R_- = \bigcap_{t \in G_+} U(t) M_-.$$

It is proved in [2] that the unitary part H^u of the canonical decomposition of $T(t)$ satisfies the following conditions:

$$H^u = R_+ \cap R_- \tag{2.1}$$

and for $f \in H$

$$s-\lim_{t \to +\infty} U(t) T(t)^* f = P_+ f, \tag{2.2}$$

$$s-\lim_{t \to +\infty} U(t)^* T(t) f = P_- f \tag{2.3}$$

for $f \in H$, $P_\pm$ being the orthogonal projection onto $R_\pm$.

Suppose that we are given two semi-groups of contractions $T_1(t)$, $T_2(t)$ in H_1 and H_2, respectively. It is clear that if $U_1(t)$ and $U_2(t)$ are the corresponding unitary dilations, then $U(t) = U_1(t) \otimes U_2(t)$ is a unitary dilation of $T_1(t) \otimes T_2(t) \stackrel{\mathrm{df}}{=} T(t)$.

We define:

$$M_+^{(k)} = \bigvee_{t \in G_+} U_k(t) H_k \quad (k = 1, 2),$$

$$M_-^{(k)} = \bigvee_{-t \in G_+} U_k(t) H_k \quad (k = 1, 2)$$

and

$$R_\pm^{(k)} = \bigcap_{t \in G_+} U_k(t) M_\pm^{(k)} \quad (k = 1, 2),$$

$P_\pm^{(k)} =$ the orthogonal projection onto $R_\pm^{(k)}$ $(k = 1, 2)$.

Applying (2.2) to $U(t) = U_1(t) \otimes U_2(t)$, $T(t) = T_1(t) \otimes T_2(t)$ we get for $f = f_1 \otimes f_2$ $(f_k \in H_k)$,

$$\begin{aligned} s-\lim_{t \to +\infty} U(t) T(t)^* f &= s-\lim_{t \to +\infty} \big(U_1(t) T_1(t)^* f \otimes U_2(t) T_2(t)^* f_2\big) \\ &= P_+^{(1)} f_1 \otimes P_+^{(2)} f_2 \\ &= P_+^{(1)} \otimes P_+^{(2)} (f_1 \otimes f_2) \\ &= P_+ (f_1 \otimes f_2), \end{aligned} \tag{2.4}$$

where $P_+ =$ the orthogonal projection on R_+, corresponding to $U(t) = U_1(t) \otimes U_2(t)$. Similarly, starting from (2.3) we prove that $(P_-^{(1)} \otimes P_-^{(2)})(f_1 \otimes f_2) = P_-(f_1 \otimes f_2)$, where P_- is the orthogonal projection on R_- corresponding to $U(t) = U_1(t) \otimes U_2(t)$. Since $f_1 \otimes f_2$ span $H_1 \otimes H_2$, we get that

$$P_{\pm} = P_{\pm}^{(1)} \otimes P_{\pm}^{(2)}. \tag{2.5}$$

It follows that for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned}(P_+ P_-)^n &= \big((P_+^{(1)} \otimes P_+^{(2)})(P_-^{(1)} \otimes P_-^{(2)})\big)^n \\ &= (P_+^{(1)} P_-^{(1)} \otimes P_+^{(2)} P_-^{(2)})^n \\ &= \big((P_+^{(1)} P_-^{(1)})^n \otimes (P_+^{(2)} P_-^{(2)})^n\big).\end{aligned} \tag{2.6}$$

By von Neumann–Wiener lemma and (2.1)–(2.3), $(P_+ P_-)^n \to$ projection onto the unitary part for $T(t) = T_1(t) \otimes T_2(t)$.

Also $(P_+^{(1)} P_-^{(1)})^n \xrightarrow{s}$ projection onto the unitary part for $T_1(t)$ and $(P_+^{(2)} P_-^{(2)})^n \xrightarrow{s}$ projection onto the unitary part of $T_2(t)$.

The following theorem is now deduced from (2.1) and (2.6).

THEOREM. *The unitary part of the canonical decomposition of* $T(t) = T_1(t) \otimes T_2(t)$ *equals to the tensor product of the unitary parts of* $T_1(t)$ *and* $T_2(t)$, *respectively.*

Let T_1, T_2 $(k = 1, 2)$ be two contractions in H_1 and H_2, respectively, and let $H_k = H_k^u \oplus H_k^c$ be the canonical decomposition of T_k. By our theorem the unitary part corresponding to $T_1 \otimes T_2$ equals $H_1^u \otimes H_2^u$ and the completely non-unitary part equals to $(H_1^u \otimes H_2^c) \oplus (H_1^c \otimes H_2^c) \oplus (H_1^c \otimes H_2^u)$.

References

[1] W. Mlak, *Unitary dilations in space of ordered groups*, Ann. Polon. Math. 17 (1965), 321–328.

[2] –, *Dissipative random processes*, Lecture Notes in Math. No. 656, 139–169.

[3] B. Sz.-Nagy, C. Foias, *Harmonic analysis of operators on Hilbert space*, Acad. Kiado, Budapest 1970.

AKADEMIA ROLNICZA IM. HUGONA KOŁŁĄTAJA
KATEDRA ZASTOSOWAŃ MATEMATYKI, KRAKÓW

Reçu par la Rédaction le 1985.12.16