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The maximum principle for the systems of the difference equations

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In paper [1], p. 446, a discrete analogue of the strong maximum principle for a difference equation is presented. Here, it is generalized on a system of difference equations of the form (3).

Let Ω denote n-dimensional bounded domain of the Euclidean space E_n, consisting of a finite number of parallelepipeds with edges parallel to the hyperplanes of the coordinate system. Let us consider, in the domain Ω the system of equations of the form

$$(1) \qquad \sum_{s=1}^{n} a^s(x)\frac{\partial^2 u}{\partial x_s^2}+\sum_{s=1}^{n} B^s(x)\frac{\partial u}{\partial x_s}+C(x)u = F(x)\,,$$

where $x = (x_1, x_2, \ldots, x_n)$, $u = (u_1, u_2, \ldots, u_p)$, $F = (F_1, F_2, \ldots, F_p)$, B^s, C, $s = 1, 2, \ldots, n$, are matrices of the dimension p. Assume, that the functions $a^s(x) > 0$, matrices $B^s(x)$, $C(x)$ and vector $F(x)$ are bounded in $\Omega\cup\Gamma$, where Γ is the boundary of Ω.

Let $ih = (i_1h_1, i_2h_2, \ldots, i_nh_n)$, where $i = (i_1, i_2, \ldots, i_n)$ is a set of integers, $h = (h_1, h_2, \ldots, h_n)$, $h_s > 0$, $s = 1, 2, \ldots, n$.

Introduce $\bar{\Omega}_h = \{ih \in \bar{\Omega}\}$ ($\bar{\Omega}$ denotes the closure of the domain Ω). We shall call two points $jh, kh, j = (j_1, j_2, \ldots, j_n), k = (k_1, k_2, \ldots, k_n)$ *adjacent*, if $\sum_{s=1}^{n}|j_s-k_s| = 1$. Then, internal point of the net $\bar{\Omega}_h$ may be defined as a point $ih \in \Omega$, whose all adjacent points belong to $\bar{\Omega}$. The set of the points, which belong to $\bar{\Omega}$, but are not internal points of the net $\bar{\Omega}_h$, will be called the *boundary of* Ω_h and denoted by Γ_h. The set of internal points of the net $\bar{\Omega}_h$ will be denoted by Ω_h. The value of the function v, defined in the point ih will be denoted by v_i.

Left side of the system (1) may be approximated by the following scheme

$$(2) \qquad Lu_i \equiv \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s u_i+\sum_{s=1}^{n} B_i^s \bar{\Delta}_s u_i + C_i u_i\,,$$

where

$$ih \in \Omega_h\,, \qquad \nabla_s u_i = \frac{u_i - u_i^{s-1}}{h_s}\,, \qquad \Delta_s u_i = \frac{u_i^{s+1} - u_i}{h_s}\,,$$

$$\bar{\Delta}_s u_i = \tfrac{1}{2}(\nabla_s + \Delta_s) u_i\,, \qquad u_i^{s\pm 1} = u\big(i_1 h_1, i_2 h_2, \ldots, (i_s \pm 1) h_s, \ldots, i_n h_n\big)\,.$$

Let $v_i = (v_{1i}, v_{2i}, \ldots, v_{pi})$ be solution of the system equations

$$(3) \qquad Lv_i = F_i \quad \text{for} \quad ih \in \Omega_h\,, \qquad v_i = \psi_i \quad \text{for} \quad ih \in \Gamma_h\,,$$

where $\psi_i = (\psi_{1i}, \psi_{2i}, \ldots, \psi_{pi})$ is a given vector.

It is convenient to introduce following denotations

$$A_{i+}^{00} = A_{i-}^{00} = \tfrac{1}{2} C_i\,, \qquad A_{i+}^{l0} = A_{i-}^{l0} = \tfrac{1}{4} B_i^l\,,$$

$$A_{i+}^{0l} = A_{i-}^{0l} = \tfrac{1}{4}(B_i^l)^T\,, \qquad A_{i+}^{ls} = A_{i-}^{ls} = -\tfrac{1}{2} a_i^l E \delta_{ls}\,,$$

E is a unit matrix, $l, s = 1, 2, \ldots, n$, A^T is a transposed matrix A. $R_i = (v_i, v_i)^{1/2}$, $\overset{0}{\eta}_{i+} = \overset{0}{\eta}_{i-} = e_i$, where e_i is a unit vector, defined for $v_i \neq 0$, in direction of the vector v_i.

$$\Phi_i^+ = \sum_{l,s=0}^{n} (A_{i+}^{ls} \overset{l}{\eta}_{i+}, \overset{s}{\eta}_{i+})\,, \qquad \Phi_i^- = \sum_{l,s=0}^{n} (A_{i-}^{ls} \overset{l}{\eta}_{i-}, \overset{s}{\eta}_{i-})\,,$$

where $\overset{s}{\eta}_{i+} = \Delta_s e_i$, $\overset{s}{\eta}_{i-} = \nabla_s e_i$, $s = 1, 2, \ldots, n$. $D_{\max}$ is a set of the points $ih \in \Omega_h$ where the function R_i attains its maximum.

Let us prove now the following

THEOREM 1. *If the set $D_{\max}$ is non-empty and $\Phi_i^+ + \Phi_i^- \leqslant 0$, $(v_i, F_i) \geqslant 0$ for $ih \in D_{\max}$, then $D_{\max} = \Omega_h$.*

Proof. Let us perform following substitutions into the system (3)

$$\bar{\Delta}_s v_i = \bar{\Delta}_s(e_i R_i) = \tfrac{1}{2}(e_i^{s+1} \Delta_s R_i + R_i \Delta_s e_i + e_i^{s-1} \nabla_s R_i + R_i \nabla_s e_i)\,,$$

$$\nabla_s \Delta_s v_i = \nabla_s \Delta_s (e_i R_i) = e_i \nabla_s \Delta_s R_i + \nabla_s e_i \nabla_s R_i + \Delta_s e_i \Delta_s R_i + R_i \nabla_s \Delta_s e_i\,.$$

$$(4) \qquad \sum_{s=1}^{n} a_i^s e_i \nabla_s \Delta_s R_i + \sum_{s=1}^{n} \big[a_i^s (\nabla_s e_i \nabla_s R_i + \Delta_s e_i \Delta_s R_i) + \tfrac{1}{2} B_i^s (e_i^{s+1} \Delta_s R_i + e_i^{s-1} \Delta_s R_i)\big] + R_i \Big[C_i e_i + \tfrac{1}{2} \sum_{s=1}^{n} B_i^s (\Delta_s e_i + \nabla_s e_i) + \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s e_i\Big] = F_i\,.$$

Multiplying both sides of the lth equation, $l = 1, 2, \ldots, p$, of the system (4) by the lth component of the vector e_i and taking a sum of all equations, the following equations may be obtained

$$(5)\quad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s R_i + \sum_{s=1}^{n} b_i^s \Delta_s R_i + \sum_{s=1}^{n} \bar{b}_i^s \nabla_s R_i + \\ + R_i\Big[(C_i e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \Delta_s e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \nabla_s e_i, e_i) + \\ + \sum_{s=1}^{n} a_i^s (e_i, \nabla_s \Delta_s e_i)\Big] = (e_i, F_i),$$

where

$$b_i^s = a_i^s (e_i, \Delta_s e_i) + \tfrac{1}{2} (B_i^s e_i^{s+1}, e_i),$$
$$\bar{b}_i^s = a_i^s (e_i, \nabla_s e_i) + \tfrac{1}{2} (B_i^s e_i^{s-1}, e_i),$$

for $ih \in \Omega_h$, $s = 1, 2, \ldots, n$.

Identity $(e_i, e_i) \equiv 1$ implies

$$(6)\quad (e_i, \nabla_s \Delta_s e_i) = -\tfrac{1}{2}[(\Delta_s e_i, \Delta_s e_i) + (\nabla_s e_i, \nabla_s e_i)].$$

Let us replace the expression $(e_i, \nabla_s \Delta_s e_i)$ in equation (5) by the right-hand side of equation (6). Thus, we obtain

$$(7)\quad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s R_i + \sum_{s=1}^{n} b_i^s \Delta_s R_i + \sum_{s=1}^{n} \bar{b}_i^s \nabla_s R_i + \\ + R_i\Big[(C_i e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \Delta_s e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \nabla_s e_i, e_i) - \\ - \tfrac{1}{2} \sum_{s=1}^{n} a_i^s (\Delta_s e_i, \Delta_s e_i) - \tfrac{1}{2} \sum_{s=1}^{n} a_i^s (\nabla_s e_i, \nabla_s e_i)\Big] = (e_i, F_i).$$

Using the notation described above, equation (7) may be rewritten in the form

$$(8)\quad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s R_i + \sum_{s=1}^{n} b_i^s \Delta_s R_i + \sum_{s=1}^{n} \bar{b}_i \nabla_s R_i + (\Phi_i^+ + \Phi_i^-) R_i = (e_i, F_i), \\ ih \in \Omega_h.$$

Therefrom

$$(9)\quad \bar{L} R_i \equiv \sum_{s=1}^{n} \left(\frac{a_i^s}{h_s^2} + \frac{b_i^s}{h_s}\right) R_i^{s+1} + \sum_{s=1}^{n} \left(\frac{a_i^s}{h_s^2} - \frac{\bar{b}_i^s}{h_s}\right) R_i^{s-1} + \\ + \sum_{s=1}^{n} \left(\frac{\bar{b}_i^s}{h_s} - \frac{b_i^s}{h_s} - \frac{2a_i^s}{h_s^2}\right) R_i + (\Phi_i^+ + \Phi_i^-) R_i = (e_i, F_i).$$

Let $D_{\max} \neq \Omega_h$. From the assumption it follows that the set $D_{\max}$ is non-empty. Thus, there exists such a point $jh \in D_{\max}$ that at least in one point $kh \in \bar{\Omega}_h$ adjacent to point jh, $R_k < R_j$.

Let

$$S = \{1, 2, \ldots, n\},$$

$$S^+ = \left\{ s \in S\colon \frac{a_i^s}{h_s^2} + \frac{b_i^s}{h_s} > 0, \; ih \in \Omega_h \right\},$$

$$S^- = \left\{ s \in S\colon \frac{a_i^s}{h_s^2} - \frac{\bar{b}_i^s}{h_s} > 0, \; ih \in \Omega_h \right\}.$$

Recalling the assumptions we see that equation (9) implies the following inequality

$$
\begin{aligned}
(10) \quad LR_j < R_j\Bigg[& \sum_{s\in S^+} \left(\frac{a_j^s}{h_s^2} + \frac{b_j^s}{h_s}\right) + \sum_{s \in S^-} \left(\frac{a_j^s}{h_s^2} - \frac{\bar{b}_j^s}{h_s}\right) + \\
& + \sum_{s\in S} \left(\frac{\bar{b}_j^s}{h_s} - \frac{b_j^s}{h_s} - \frac{2a_j^s}{h_s^2}\right) + \Phi_j^+ + \Phi_j^- \Bigg] \\
= R_j\Bigg[& \sum_{s\in S-S^-} \left(\frac{\bar{b}_j^s}{h_s} - \frac{a_j^s}{h_s^2}\right) - \sum_{s\in S-S^+} \left(\frac{b_j^s}{h_s} + \frac{a_j^s}{h_s^2}\right) + \\
& + (C_j e_j, e_j) + \frac{1}{2} \sum_{s\in S} (B_j^s \Delta_s e_j, e_j) + \frac{1}{2} \sum_{s \in S} (B_j^s \nabla_s e_j, e_j) - \\
& - \frac{1}{2} \sum_{s\in S} a_j^s (\Delta_s e_j, \Delta_s e_j) - \frac{1}{2} \sum_{s\in S} a_j^s (\nabla_s e_j, \nabla_s e_j) \Bigg] \\
= R_j\Bigg\{ & \sum_{s\in S-S^-} \left[\frac{a_j^s}{h_s} (e_j, \nabla_s e_j) + \frac{1}{2h_s} (B_j^s e_j^{s-1}, e_j) - \frac{a_j^s}{h_s^2} + \right. \\
& \left. + \frac{1}{2} (B_j^s \nabla_s e_j, e_j) - \frac{a_j^s}{2} (\nabla_s e_j, \nabla_s e_j) \right] + \sum_{s\in S-S^+} \left[-\frac{a_j^s}{h_s} (e_j, \Delta_s e_j) - \right. \\
& \left. - \frac{1}{2h_s} (B_j^s e_j^{s+1}, e_j) - \frac{a_j^s}{h_s^2} + \frac{1}{2} (B_j^s \Delta_s e_j, e_j) - \frac{a_j^s}{2} (\Delta_s e_j, \Delta_s e_j) \right] + \\
& + (C_j e_j, e_j) + \frac{1}{2} \sum_{s\in S^+} [(B_j^s \Delta_s e_j, e_j) - a_j^s (\Delta_s e_j, \Delta_s e_j)] + \\
& + \frac{1}{2} \sum_{s \in S^-} [(B_j^s \Delta_s e_j, e_j) - a_j^s (\nabla_s e_j, \nabla_s e_j)] \Bigg\} \\
\leqslant R_j\Bigg\{ & \sum_{s\in S-S^-} \left[\frac{a_j^s}{h_s} (\nabla_s e_j, \nabla_s e_j)^{1/2} + \frac{K_s}{h_s} - \frac{a_j^s}{h_s^2} + \right.
\end{aligned}
$$

$$+K_s(\nabla_s e_j, \nabla_s e_j)^{1/2} - \frac{a_j^s}{2}(\nabla_s e_j, \nabla_s e_j)\Big] +$$

$$+\sum_{s\in S-S^+}\Big[\frac{K_s}{h_s} - \frac{a_j^s}{h_s^2} + \frac{a_j^s}{h_s}(\Delta_s e_j, \Delta_s e_j)^{1/2} +$$

$$+K_s(\Delta_s e_j, \Delta_s e_j)^{1/2} - \frac{a_j^s}{2}(\Delta_s e_j, \Delta_s e_j)\Big] +$$

$$+K_0 + \sum_{s\in S^+}\Big[K_s(\Delta_s e_j, \Delta_s e_j)^{1/2} - \frac{a_j^s}{2}(\Delta_s e_j, \Delta_s e_j)\Big] +$$

$$+\sum_{s\in S^-}\Big[K_s(\nabla_s e_j, \nabla_s e_j)^{1/2} - \frac{a_j^s}{2}(\nabla_s e_j, \nabla_s e_j)\Big]\Big\},$$

where

$$K_0 = n \max_{l,s} \max_{ih\in\bar{\Omega}_h} |C_i^{ls}|, \qquad K_s = \frac{n}{2} \max_{l,r} \max_{ih\in\bar{\Omega}_h} |B_i^{slr}|.$$

Let us consider, the following set of auxiliary functions

$$G_i^s(x) = \left(K_s + \frac{a_i^s}{h_s}\right)x - \frac{a_i^s}{2}x^2 + \frac{K_s}{h_s} - \frac{a_i^s}{h_s^2}, \qquad H_i^s(x) = K_s x - \frac{a_i^s}{2}x^2,$$

$s = 1, 2, \ldots, n$.

From (10) it may be concluded, that

$$\text{(11)} \quad \bar{L}R_j \leqslant R_j\Big(\sum_{s\in S-S^+} G_{\max}^s + \sum_{s\in S-S^-} G_{\max}^s + \sum_{s\in S^+} H_{\max}^s + \sum_{s\in S^-} H_{\max}^s + K_0\Big),$$

where

$$G_{\max}^s = \max_{-\infty<x<+\infty} G_j^s(x) = -\frac{a_j^s}{2h_s^2} + \frac{K_s^2}{2a_j^s} + \frac{2K_s}{h_s},$$

$$H_{\max}^s = \max_{-\infty<x<+\infty} H_j^s(x) = \frac{K_s^2}{2a_j^s}.$$

It is easy to see that the right-hand side expression of the inequality (11) is negative for sufficiently small h_s.

Let $S^+ = S^- = S$. Then equation (9) (cf. [1], p. 446) implies the following inequality

$$\text{(12)} \qquad \bar{L}R_j < (\Phi_j^+ + \Phi_j^-)R_j \leqslant 0.$$

If $S^+ \neq S$ or $S^- \neq S$, then from inequality (11) immediately follows

$$\text{(13)} \qquad \bar{L}R_j < 0.$$

On the other hand, we have $\bar{L}R_j \geqslant 0$, which contradicts (12) and (13). Thus, we conclude that the assumption $D_{\max} \neq \Omega_h$ was false and this terminates proof of Theorem 1.

We shall prove, that if the matrix C satisfies the condition

$$(C_i v_i, v_i) \leqslant -\mu(v_i, v_i), \quad ih \in \bar{\Omega}_h,$$

where $\mu \geqslant \frac{\bar{K}n}{4a}$, $\bar{K} = \max_s K_s^2$, $a = \min_s \min_{x \in \bar{\Omega}} a^s(x)$, then

$$\Phi_i^+ + \Phi_i^- \leqslant 0 \quad \text{for} \quad ih \in \bar{\Omega}_h.$$

Namely

$$(14) \quad \Phi_i^+ + \Phi_i^- = (C_i e_i, e_i) + \tfrac{1}{2}\sum_{s=1}^{n} [(B_i^s \Delta_s e_i, e_i) - a_i^s(\Delta_s e_i, \Delta_s e_i)] + $$
$$+ \tfrac{1}{2}\sum_{s=1}^{n} [B_i^s \nabla_s e_i, e_i) - a_i^s(\nabla_s e_i, \nabla_s e_i)]$$
$$\leqslant -\mu + \tfrac{1}{2}\sum_{s=1}^{n} [K_s(\Delta_s e_i, \Delta_s e_i)^{1/2} - a(\Delta_s e_i, \Delta_s e_i)] +$$
$$+ \tfrac{1}{2}\sum_{s=1}^{n} [K_s(\nabla_s e_i, \nabla_s e_i)^{1/2} - a(\nabla_s e_i, \nabla_s e_i)].$$

The function $H_i^s(x) = K_s x - ax^2$ attains its maximum equal $K_s^2/4a$. Hence, inequality (14) implies

$$(15) \quad \Phi_i^+ + \Phi_i^- \leqslant -\mu + \frac{\bar{K}n}{4a} \leqslant 0 \quad \text{for} \quad \mu \geqslant \frac{\bar{K}n}{4a}.$$

THEOREM 2. *If v_i is a solution of the system* (3) *and* $\Phi_i^+ + \Phi_i^- \leqslant 0$ *for $ih \in \Omega_h$, then v_i satisfies the inequality*

$$(16) \quad (v_i, v_i)^{1/2} \leqslant K[\max_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} + \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2}], \quad ih \in \bar{\Omega},$$

where the constant $K > 0$ does not depend on h.

This theorem will be proved with the aid of the following

LEMMA (cf. [2], p. 328). *If v_i is a solution of the system* (3) *and the function g_i conforms to the assumptions*

$$\text{(i)} \quad -\bar{L}g_i \geqslant \max_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} \quad \text{for} \quad ih \in \Omega_h,$$

$$\text{(ii)} \quad g_i \geqslant \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2} \quad \text{for} \quad ih \in \Gamma_h,$$

then

$$(v_i, v_i)^{1/2} \leqslant g_i \quad \text{for} \quad ih \in \bar{\Omega}_h.$$

Proof of lemma. Let $z_i = R_i - g_i$, $R_i = (v_i, v_i)^{1/2}$. From (i) and equation (9), it follows that

$$\bar{L}z_i = \bar{L}R_i - \bar{L}g_i = (e_i, F_i) - \bar{L}g_i \geqslant 0 \quad \text{for} \quad ih \in \Omega \ .$$

Thus, the function z_i conforms to the maximum principle (cf. Theorem 1). From (ii) it follows that

$$R_i - g_i \leqslant 0 \quad \text{for} \quad ih \in \Gamma_h \, .$$

Thus

$$R_i \leqslant g_i \quad \text{for} \quad ih \in \bar{\Omega}_h \, ,$$

q.e.d.

Proof of Theorem 2. Assume, that the domain Ω belongs to the half-space $x_1 \geqslant 0$ it could be always achieved by means of suitable choice of the coordinate system. Let

$$g_i = [\exp(a\bar{x}_1) - \exp(ax_1)] \max_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} + \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2} \, ,$$

where $\bar{x}_1 \geqslant x_1 + 1$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a fixed point in the space E_n. Now, we shall show that the function g_i satisfies the assumptions of the lemma. From the definition of the function g_i it follows

$$\frac{g_i^{s+1} - g_i^{s-1}}{2h_s} = \begin{cases} -a\exp(ax_1) \max\limits_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} + O(h_1^2) & \text{for} \quad s = 1 \, , \\ 0 & \text{for} \quad s = 2, 3, \ldots, n; \end{cases}$$

$$\frac{g_i^{s+1} - 2g_i + g_i^{s-1}}{h_s^2} = \begin{cases} -a^2\exp(ax_1) \max\limits_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} + O(h_1^2) & \text{for} \quad s = 1 \, , \\ 0 & \text{for} \quad s = 2, 3, \ldots, n \, . \end{cases}$$

Ad (i):

$$\begin{aligned} -\bar{L}g_i = & \left[\exp(ax_1)\left(a_i^1 a^2 + (b_i^s + \bar{b}_i^s) a\right) - (\Phi_i^+ + \Phi_i^-)\left(\exp(a\bar{x}_1) - \exp(ax_1)\right)\right] \times \\ & \times \max_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} - (\Phi_i^+ + \Phi_i^-) \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2} + O(h_1^2) \\ & \geqslant \max_{ih \in \Omega_h} (F_i, F_i)^{1/2} \, , \end{aligned}$$

where a sufficiently large number.

Ad (ii):

$$g_i \geqslant \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2} \quad \text{for} \quad ih \in \Gamma_h \, .$$

Thus, lemma implies

$$(v_i, v_i)^{1/2} \leqslant [\exp(a\bar{x}_1) - 1] \max_{ih \in \bar{\Omega}_h} (F_i, F_i)^{1/2} + \max_{ih \in \Gamma_h} (v_i, v_i)^{1/2}. \tag{17}$$

The inequality (16) follows immediately from (17), c.q.d.

References

[1] И. С. Березин, Н. П. Жидков, *Методы вычислений*, т. II, Москва 1962.

[2] Р. Курант, *Уравнения с частными производными*, Москва 1964.

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