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Generalized Fredholm eigenvalues of a Jordan curve

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Abstract. The aim of this paper is to give a definition of Fredholm eigenvalues of a Jordan curve Γ without any reference to the Neumann–Poincaré kernel and without any regularity assumptions on Γ. To this end we introduce conjugate holomorphic eigenfunctions of Γ (abbreviated: CHE), i.e. functions holomorphic in complementary domains of Γ whose boundary values on Γ satisfy relation (2.6). The real parameter λ in (2.6) has most of the familiar properties of classical Fredholm eigenvalues of Γ.

1. Introduction. Let Γ be a Jordan curve in the finite plane C and let D, $D^* \ni \infty$, be its complementary domains. Many important problems in conformal mapping and the potential theory (e.g. the solution of the interior and the exterior Dirichlet, or Neumann problem, conformal mapping of D, or D^* onto the unit disk Δ) can be reduced to the solution of a linear integral equation of Fredholm type with the Neumann–Poincaré kernel:

$$k(\xi, t) = -\frac{1}{\pi}\frac{\partial}{\partial n_\xi}\log|\xi - t|, \qquad \xi, t \in \Gamma, \tag{1.1}$$

or its transposition. Here $\partial/\partial n_\xi$ denotes the derivative along the interior normal of Γ at ξ. For details see [2], [5], [12], [13]. If $\Gamma \in C^3$ and $\varkappa(t)$ denotes the curvature of Γ at $t \in \Gamma$, then putting $2\pi k(t, t) = \varkappa(t)$ we obtain a kernel continuously differentiable w.r.t. the arc length s on Γ. The eigenvalues of k, i.e., the real numbers λ such that the homogeneous integral equation

$$\mu(t) = \lambda \int_\Gamma k(\xi, t)\mu(\xi)\, ds, \qquad t \in \Gamma, \tag{1.2}$$

has a non-trivial real-valued solution μ, are called *Fredholm eigenvalues of* Γ. The smallest positive Fredholm eigenvalue λ_0 of Γ is of particular interest since it determines the rate of convergence of the Neumann series for the kernel k.

If Γ is a quasicircle, i.e., if Γ admits a K-quasiconformal (abbreviated: K-qc) reflection (for the definition of K-qc reflection and properties of quasicircles cf. [10], [3]), then $\lambda_0 \geqslant (K+1)/(K-1)$, as pointed out by Ahlfors [1]. Some recent papers exhibit another interesting connection between Fredholm eigenvalues and qc mappings: In some extremal problems involving conformal mappings with qc extension it is rather the smallest positive Fredholm eigenvalue λ_0 than the maximal dilatation that appears in the extremal case, cf. [9], [15].

In his seminal paper on Fredholm eigenvalues [14] Schiffer was able to prove many interesting properties of Fredholm eigenvalues of a Jordan curve Γ under the assumption $\Gamma \in C^3$. This paper aims at giving a definition of Fredholm eigenvalues without any reference to the kernel (1.1) and without any regularity assumptions on Γ. To this end we introduce the notion of conjugate holomorphic eigenfunctions of Γ (abbreviated: CHE), i.e., a pair f, F of functions holomorphic in D and $D^* \cup \{\infty\}$, resp., whose boundary values on Γ satisfy relation (2.6) and we call the corresponding real number λ in (2.6) a Fredholm eigenvalue of Γ. Under some additional assumptions on f and F we are able to show that the spectrum $\Lambda(\Gamma)$, i.e., the set of all Fredholm eigenvalues of Γ has all the familiar properties of the spectrum in the classical sense. Moreover, if $\Gamma \in C^3$ and $F(\infty) = 0$, a corresponding eigenvalue is also a Fredholm eigenvalue in the classical sense.

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2. Conjugate holomorphic eigenfunctions. In this section we shall derive some characteristic properties of classical Fredholm eigenvalues which will serve as a basis for their generalization. Let Γ be a Jordan curve of the class C^3. If $\tau(\xi)$: $\Gamma \to \boldsymbol{R}$ is real-valued and continuously differentiable w.r.t the arc length s on Γ then the integral

$$I(z) = \frac{1}{\pi i}\int_\Gamma \frac{\tau(\xi)\,d\xi}{\xi - z} = \frac{1}{\pi i}\int_\Gamma \tau(\xi)\,d_\xi \log(\xi - z) \tag{2.1}$$

represents in the complementary sets D, $D^* \ni \infty$ of Γ holomorphic functions f, F which may be called *conjugate holomorphic potentials* of double layer with density τ. In fact, $\operatorname{Re} I(z)$ is the logarithmic potential of double layer with density τ as generated by the kernel (1.1), i.e.,

$$\operatorname{Re} I(z) = -\frac{1}{\pi}\int_\Gamma \tau(\xi)\frac{\partial}{\partial n_\xi}\log|\xi - z|\,ds, \qquad z \in \boldsymbol{C}\setminus\Gamma. \tag{2.2}$$

If we put

$$\text{p.v.}\, I(t) = \alpha(t) + i\beta(t), \tag{2.3}$$

where p.v. denotes the principal value of integral (2.1) at the point $t \in \Gamma$, then

$\alpha+i\beta$ is continuous on Γ and by Privalov's theorem on Cauchy type integrals [4] both f and F have continuous extensions on $\bar{D}$ and $\overline{D^*}$, resp., which satisfy

$$f(\xi) = \alpha(\xi)+\tau(\xi)+i\beta(\xi), \quad F(\xi) = \alpha(\xi)-\tau(\xi)+i\beta(\xi), \quad \xi\in\Gamma. \tag{2.4}$$

Following Schiffer [14] we may consider the double layer potentials with density $\tau = \lambda\mu$, where μ is an eigenfunction associated with the eigenvalue λ. For any non-trivial solution μ of (1.2) $\mu(\xi(s))$ is a finite linear combination of $D(s, s_k; \lambda)$, where $D(s, s'; \lambda)$ is Fredholm's first minor; cf. [11], p. 43 and [5], p. 235. Therefore $\mu(\xi(s))$ is continuously differentiable w.r.t. s. Moreover,

$$\operatorname{Re}\text{p.v.}\, I(t) = \int_\Gamma \tau(\xi)k(\xi, t)\,ds = \lambda\int_\Gamma \mu(\xi)k(\xi, t)\,ds = \mu(t)$$

so that

$$\text{p.v.}\, I(t) = \mu(\xi)+i\nu(\xi), \quad \xi\in\Gamma. \tag{2.5}$$

In view of (2.3) and (2.5) equations (2.4) take the form

$$f(\xi) = (1+\lambda)\mu(\xi)+i\nu(\xi), \quad F(\xi) = (1-\lambda)\mu(\xi)+i\nu(\xi), \tag{2.6}$$

where μ, ν are real-valued and continuous on Γ. Relations (2.6) can be written in the following concise form

$$f(\xi) = L\circ F(\xi), \quad F(\xi) = l\circ f(\xi), \tag{2.7}$$

where

$$L(w) = (1-\lambda)^{-1}(w+\lambda\bar{w}), \quad l(w) = (1+\lambda)^{-1}(w-\lambda\bar{w}) = L^{-1}(w). \tag{2.8}$$

From (2.6) we obtain

$$[f(\xi)-F(\xi)]/[f(\xi)+\bar{F}(\xi)] = \lambda = \text{const}, \quad \xi\in\Gamma. \tag{2.9}$$

Conversely, if f, F are holomorphic and non-constant in D, D^*, resp., and have continuous extensions to the closure of respective domains such that (2.6) holds for some real λ and $F(\infty) = 0$, then λ is a Fredholm eigenvalue of a sufficiently smooth Γ. In fact, both μ and λ can be recovered from the boundary values $f(\xi)$, $F(\xi)$ by means of (2.6) and (2.9) and then formula (2.1) with $\tau = \lambda\mu$ determines f and F. Now, it is easily verified that (1.2) holds so that λ is a Fredholm eigenvalue of Γ. This characterization of Fredholm eigenvalues is also implicitly contained in Theorem 5 in [7].

Therefore holomorphic functions f, F whose boundary values satisfy (2.6) may be used in defining Fredholm eigenvalues of a Jordan curve Γ in the following way.

DEFINITION 1. We call (f, F) a pair of *conjugate holomorphic eigenfunctions* (abbreviated: CHE) of a Jordan curve Γ in C if f and F are non-constant functions holomorphic in the domains D and $D^* \cup (\infty)$, resp., that satisfy the following conditions.

(i) Both f and F have extensions to the closures of respective domains which are continuous in the spherical metric and satisfy the boundary relation (2.7) on Γ.

(ii) If h and h^* map conformally the unit disk Δ onto D and D^*, resp., then both $f{\circ}h$ and $F{\circ}h^*$ belong to $H^1(\Delta)$.

(iii) For any $w \in \partial\Delta$ there exists a neighbourhood N_w of w such that both $f{\circ}h$ and $F{\circ}h^*$ are univalent in $N_w \cap \Delta$.

Definition 2. If the functions f, F satisfy the assumptions of Definition 1, then the real constant λ in (2.9) and the real-valued function

$$\mu(\xi) = \tfrac{1}{2}[f(\xi) + \bar{F}(\xi)], \quad \xi \in \Gamma, \tag{2.10}$$

are called *Fredholm eigenvalue* and a *Fredholm eigenfunction* of Γ associated with λ, resp.

Note that no regularity conditions are imposed on Γ in the above stated definitions. In what follows we shall use the term classical Fredholm eigenvalues, whenever Fredholm eigenvalues in the usual sense appear. If $F(\infty) = 0$ and Γ is sufficiently smooth, then λ satisfying the conditions of Definition 2 is a classical Fredholm eigenvalue of Γ, as pointed out above.

Assumption (i) was suggested by the classical case. Assumption (ii) enables us to determine f and F in a unique manner from their boundary values. Assumption (iii) was suggested by the fact that it holds in all cases where the eigenvalues are known. Moreover, it is useful in proving various properties of spectrum $\Lambda(\Gamma)$ as known from the classical case. In particular it implies finite valence of CHE. The rejection of $\lambda = 1$ from $\Lambda(\Gamma)$, as well as the absence of the condition $F(\infty) = 0$ in our definition may be considered even as an advantage since Schiffer achieved in [14] the same by investigating the derivatives of CHE rather than CHE themselves. This approach results in the symmetry of $\Lambda(\Gamma)$ and its invariance under Moebius transformations.

3. Properties of generalized Fredholm eigenvalues. In what follows we denote by $\Lambda(\Gamma)$ the spectrum of a Jordan curve Γ, i.e., the set of its all Fredholm eigenvalues. The references that follow a property discussed below are related to the classical case.

I. There is no Jordan curve with Fredholm eigenvalues ± 1.

If $\lambda = 1$, then $\operatorname{Re} F{\circ}h^* = 0$ on $\partial\Delta$ and by (ii) this also holds in Δ. Hence $\operatorname{Re} F = 0$ in D^* and consequently $F = \text{const}$ contrary to Definition 1. The case $\lambda = -1$ can be treated analogously. Thus our definition eliminates the eigenvalue $\lambda = 1$ known from the classical theory.

II. If $\lambda \in \Lambda(\Gamma)$, then $-\lambda \in \Lambda(\Gamma)$.

Suppose f, F are CHE associated with the eigenvalue λ. It follows from (2.6) that $f_1 = -i(1-\lambda)f$, $F_1 = -i(1+\lambda)F$ are CHE associated with the eigenvalue $-\lambda$.

III. All $\lambda \in \Lambda(\Gamma)$ satisfy $|\lambda| > 1$.

Suppose that λ, $-1 < \lambda < 1$, belongs to $\Lambda(\Gamma)$ and f, F are CHE associated with λ. By assumption (iii) in Definition 1 we may apply the argument principle due to its topological character. For any $w \in f(D)$ not on $f(\Gamma)$ the index $n(f(\Gamma), w)$ is positive. Since $F(\xi) = l \circ f(\xi)$ and the Jacobian of l is positive for $|\lambda| < 1$, the index $n(F(-\Gamma), w)$ is negative at $w \in l \circ f(D)$ not on $F(\Gamma)$ which is impossible in absence of poles.

IV. If T is a Moebius transformation and the Jordan curves Γ, $\Gamma_1 = T(\Gamma)$ are both situated in the finite plane, then $\Lambda(\Gamma) = \Lambda(\Gamma_1)$; cf. [14], p. 1195.

Let f, F be CHE associated with $\lambda \in \Lambda(\Gamma)$. If the inside domains of Γ and Γ_1 correspond under T, then obviously $f_1 = f \circ T^{-1}$, $F_1 = F \circ T^{-1}$ are CHE of Γ_1 associated with λ. If T maps the inside of Γ onto the outside of Γ_1, then $f_1 = F \circ T^{-1}$, $F_1 = f \circ T^{-1}$ are CHE of Γ_1 associated with $-\lambda$. Hence $\Lambda(\Gamma) = \Lambda(\Gamma_1)$, in view of II.

V. If $\lambda > 1$ and one of the Dirichlet integrals $\iint_D |f'|^2$, $\iint_{D^*} |F'|^2$ is finite, so is the other one and we have

$$(3.1) \qquad (\lambda-1)\iint_D |f'|^2 = (\lambda+1)\iint_{D^*} |F'|^2;$$

cf. [14], p. 1193.

The functions f, F map simply connected domains D, D^* onto the Riemann surfaces S, S^* over the w-plane. Relation (2.7) induces a homeomorphic correspondence between their boundary curves. The affine mapping l carries S onto a homeomorphic Riemann surface $\tilde{S}$ whose projection on the w-plane, as well as the boundary curve, are the same as those of S^*, in view of (2.7). By the argument principle each point w of the projection has the same number of preimages in both $\tilde{S}$ and S^*, if possible branch points are counted with due multiplicity. Thus the areas of $\tilde{S}$ and S^* are equal. Since $\tilde{S}$ arises from S by the affine mapping l with the Jacobian $J_l = (1-\lambda)/(1+\lambda)$, we have

$$|\tilde{S}| = (\lambda-1)/(\lambda+1)|S| = |S^*|, \qquad \text{i.e.,} \qquad (\lambda-1)|S| = (\lambda+1)|S^*|$$

which proves (3.1).

The proof of equality (3.1) given in [14] for the classical case was based on the isometry property of Hilbert transform in L^2.

VI. If Γ admits a K-qc reflection and CHE f, F with finite Dirichlet integrals are associated with a positive $\lambda \in \Lambda(\Gamma)$, then

$$(3.2) \qquad \lambda \geqslant (K+1)/(K-1); \qquad \text{cf. [1].}$$

The functions $v = \operatorname{Im} f$, $V = \operatorname{Im} F$ are harmonic in complementary quasidisks D, D^* and have equal continuous boundary values on their common boundary Γ. Hence their Dirichlet integrals satisfy: $K^{-1}\mathcal{D}[v] \leqslant \mathcal{D}[V]$, cf. [1], or [3], p. 46. However, $\mathcal{D}[v] = \iint\limits_D |f'|^2$, $\mathcal{D}[V] = \iint\limits_{D^*} |F'|^2$ and this gives in view of (3.1): $K^{-1} \leqslant (\lambda-1)/(\lambda+1)$. This implies (3.2).

VII. If CHE f, F associated with a positive Fredholm eigenvalue $\lambda \in \Lambda(\Gamma)$ are locally univalent in D and D^*, resp., then Γ admits a K-qc reflection φ with $K = (\lambda+1)/(\lambda-1)$ which locally satisfies the relations

$$F \circ \varphi = l \circ f \quad \text{in } D, \qquad f \circ \varphi = L \circ F \quad \text{in } D^*. \tag{3.3}$$

For the proof and the uniqueness discussion see [6]. If CHE are univalent, then VII implies Theorem 7 in [9].

There are relatively few Jordan curves for which Fredholm eigenvalues and eigenfunctions are known. We give here some examples.

Example 1. Let E be the ellipse $\zeta = e^{i\theta} + ke^{-i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, $0 < k < 1$. Then $\Lambda(E) = \{\pm k^{-n}\colon n \in N\}$, cf. [14]. It is easily seen that $f(z) = (k-1)^{-1}z$, $F(Z) = (2k)^{-1}[Z-(Z^2-4k)^{1/2}]$ are CHE associated with $\lambda_1 = 1/k$. Let $P_n(z)$ be the sequence of polynomials defined as follows:

$$P_1(z) = z, \qquad P_2(z) = z^2 - 2k, \qquad P_{n+1}(z) = zP_n(z) - kP_{n-1}(z).$$

One can verify that

$$f_n(z) = (k^n - 1)^{-1} P_n(z), \qquad F_n(Z) = (F(Z))^n$$

are CHE associated with $\lambda_n = k^{-n}$.

Example 2. Let Γ be a circular wedge symmetric w.r.t. the real axis, with vertices -1, 1 and interior angles $\alpha\pi$, $0 < \alpha < 1$. The functions

$$f(z) = \frac{1}{\alpha}\log(1+z)/(1-z), \qquad F(Z) = \frac{1}{2-\alpha}\log(Z-1)/(Z+1)$$

are CHE associated with the eigenvalue $\lambda = (1-\alpha)^{-1}$. In this case CHE are unbounded and have infinite Dirichlet integrals although Γ is a quasicircle.

Example 3. Consider the analytic Jordan curve Γ:

$$\zeta = e^{i\theta}(1+ke^{-3i\theta})^{2/3}, \qquad 0 \leqslant \theta \leqslant 2\pi, \qquad 0 < k < 1.$$

As shown by Kühnau [8], $\lambda = k^{-2}$ is an eigenvalue of Γ. The functions

$$f(z) = (k^2-1)^{-1}(z^3-2k), \qquad F(Z) = (2k)^{-2}[Z^{3/2}-(Z^3-4k)^{1/2}]^2$$

associated with the positive eigenvalue k^{-2} are CHE which are not locally univalent which was the case in the preceding examples.

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