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An iterative method for solving operator equations

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1. Let us consider the equation

$$x = Tx, \tag{1.1}$$

where T is an operator $R \to R$ and R — a space with a metric in G (cf. [4]). Kurpiel in [2] gives some methods for solving such equations, e.g. he considers the equation

$$x_n = T_n(x_n, x_{n-1}), \quad n = 1, 2, \ldots, \tag{1.2}$$

adjoined to equation (1.1) (see [2], p. 36).

This paper deals with another iterative method which gives the solution of equation (1.1). For this scheme we shall state an analogous theorem to that given for iteration (1.2) in [2].

DEFINITION of the set G (cf. [4]). Let G be a set with the following properties:

1° G is a semi-ordered set and 0 is its minimal element, i.e. $0 \leqslant x$ for all $x \in G$,

2° For all $u, v \in G$ there is a uniquely defined sum $u+v \in G$ such that

(a) $u+v = v+u$, $u+0 = u$,

(b) if $u, v, w \in G$ and $u \leqslant v$, then $u+w \leqslant v+w$,

(c) from $u+v \leqslant w$ follows $u \leqslant w$,

3° Every non-increasing sequence $\{u_i\}$, where $u_i \in G$ and $u_{i+1} \leqslant u_i$, $i = 1, 2, \ldots$, has a limit $u \in G$ (we write $\lim\limits_{i\to\infty} u_i = u$ or $u_i \searrow u$) and the following conditions are satisfied:

(a) if $u_i = u$ $(i = 1, 2, \ldots)$, then $u_i \searrow u$,

(b) if $u_i \searrow u$ and $v_i \searrow v$, then $u_i + v_i \searrow u+v$,

(c) if $u_i \to u$, $v_i \to v$ and $u_i \leqslant v_i$, then $u \leqslant v$,

(d) the limit never changes when we drop off the initial terms.

The solution $\bar{x} \in [a, b]$ of the equation $x = \varphi(x)$ is called an *upper solution* on the interval $[a, b]$ if for all solutions $x \in [a, b]$ of this equation we have $x \leqslant \bar{x}$ (cf. [2], p. 20).

LEMMA. *Let $\varphi(u)$ be a function defined on the set $\Delta = [0, k] \in G$, non-decreasing and continuous $\varphi(u) \in G$ and let there exist $b \in \Delta$ such that*

$$\varphi(b) \leqslant b. \tag{1.3}$$

Then the equation

$$u = \varphi(u) \tag{1.4}$$

possesses on the interval $[0, b]$ an upper solution $m(\varphi, b)$. If, in addition, $0 \leqslant p \leqslant b$, and

$$p \leqslant \varphi(p), \tag{1.5}$$

then $p \leqslant m(\varphi, b)$. If ψ has the same properties as φ and $\varphi(u) \leqslant \psi(u)$ for all $u \in [0, b]$, $\varphi(b) \leqslant b$, $\psi(b) \leqslant b$, then $m(\varphi, b) \leqslant m(\psi, b)$ (cf. [2], p. 20).

Let R denote a space with the following properties:

1° R is an abstract space with a metric defined by elements of the space G;

2° For the sequence of elements $\{x_i\}$, $x_i \in R$ there is a uniquely defined limit $\lim_{n\to\infty} x_n = x$ $(x \in R)$ which does not depend on the initial terms, and if $x_i = s \in R$, $i = 1, 2, \ldots$, then $\lim_{i\to\infty} x_i = s$;

3° any sphere $S(\bar{x}, r)$, $\bar{x} \in R$, $r \in G$, is a closed set belonging to R;

4° The space R is complete (cf. [4]).

2. Suppose we are given a sequence $c_n \searrow 0$ and the equation

$$x_n = T_n(x_{n-1}, y_{n-1}), \quad n = 1, 2, \ldots, \tag{2.1}$$

where $T_n\colon R \times R \to R$.

We assume that the operators T_n fulfil the conditions

$$d\big(T_n(x, x), T(x)\big) \leqslant a_n \quad (a_n \searrow 0), \tag{2.2}$$

where x is a solution of equation (1.1), and let y_n be a sequence fulfilling the condition

$$d(x_n, y_n) \leqslant c_n, \quad n = 0, 1, 2, \ldots \tag{2.3}$$

THEOREM. *If, for all $x, y, z, t \in D$, we have the inequality*

$$d\big(T_n(x, y), T_n(z, t)\big) \leqslant \varphi_n\big(d(x, z), d(y, t)\big), \tag{2.4}$$

where $\varphi_n(c, d)\colon \Delta \times \Delta \to G$ $(\Delta = [0, k] \subset G)$, φ_n are non-decreasing, continuous functions (cf. [2], p. 36) *and conditions* (2.2), (2.3) *are satisfied, then for any $x_{n-1}, y_{n-1} \in D$ $(n = 1, 2, \ldots)$ equation* (2.1) *has solutions.*

Furthermore, let $\varphi(c, d)$ be a non-decreasing continuous function $[0, b]^2 \to G$ such that

$$\varphi(b, b) + a_1 + c_1 \leqslant b, \tag{2.5}$$

where b is a diameter of the set D, and

$$\varphi_n(c, d) \leqslant \varphi(c, d) \quad \textit{for } c, d \in [0, b]^2. \tag{2.6}$$

Let us assume that the equation $x = \varphi(x, x)$ has a unique solution $x = 0$.

Then equation (1.1) has a unique solution on the set D equal to x which is a limit of the sequences of solutions $\{x_n\}$ $\{y_n\}$ of (2.1) for every $x_0, y_0 \in D$, and the following inequalities hold:

$$d(x_n, x) \leqslant B_n(b), \quad n = 1, 2, \ldots, \tag{2.7}$$

$$d(y_n, x) \leqslant C_n(b), \quad n = 1, 2, \ldots, \tag{2.8}$$

where $B_n(b)$ is an upper solution of

$$u = \varphi\big(u, C_{n-1}(b)\big) + a_n, \quad C_0(b) = b, \quad n = 1, 2, \ldots, \tag{2.9}$$

and $C_n(b)$ is an upper solution of the equation

$$u = \varphi\big(B_{n-1}(b), u\big) + a_n + c_n, \quad B_0(b) = b, \quad n = 1, 2, \ldots, \tag{2.10}$$

and $B_n(b) \searrow 0$, $C_n(b) \searrow 0$ for $n \to \infty$ on the interval $[0, b]$, and the inequalities

$$d(x_n, x) \leqslant m(\psi_n, b), \tag{2.11}$$

$$d(y_n, x) \leqslant m(\overline{\psi}_n, b), \tag{2.12}$$

where $m(\psi_n, b)$ is an upper solution of the equation

$$u = \psi_n(u) = \varphi_n\big(u, d(y_{n-1}, x)\big) + a_n \tag{2.13}$$

and $m(\overline{\psi}_n, b)$ — an upper solution on the interval $[0, b]$ of the equation

$$u = \overline{\psi}_n(u) = \varphi_n\big(d(x_{n-1}, x), u\big) + a_n + c_n. \tag{2.14}$$

Proof. Since $x \in D$ and $x_n, y_n \in D$ from the equality $x = Tx$, $x_n = T_n(x_{n-1}, y_{n-1})$ and from assumptions (2.2), (2.3), we have

$$d(x_n, x) = d\big(T_n(x_{n-1}, y_{n-1}), Tx\big) \leqslant d\big(T_n(x_{n-1}, y_{n-1}), T_n(x, x)\big) + \\ + d\big(T_n(x, x), Tx\big) \leqslant \varphi_n\big(d(x_{n-1}, x), d(y_{n-1}, x)\big) + a_n,$$

and since $d(x_n, x) \leqslant b$ and $d(y_n, x) \leqslant b$ for $n = 0, 1, \ldots$ from (2.2), (2.6), (2.5) and (2.3), we have

$$d(x_n, x) \leqslant \varphi\big(d(x_{n-1}, x), d(y_{n-1}, x)\big) + a_n \leqslant \varphi(b, b) + a_1 \leqslant b \tag{2.15}$$

and

$$d(y_n, x) \leqslant d(x_n, y_n) + d(x_n, x) \leqslant \varphi(b, b) + a_1 + c_1 \leqslant b. \tag{2.16}$$

For $n = 1$ we obtain

$$d(x_1, x) \leqslant \varphi\big(d(x_0, x), b\big) + a_1 \leqslant b, \quad d(y_1, x) \leqslant \varphi\big(b, d(y_0, x)\big) + a_1 + c_1 \leqslant b,$$

and from the lemma we have

$$d(x_1, x) \leqslant B_1(b) \leqslant b = B_0(b), \qquad d(y_1, x) \leqslant C_1(b) \leqslant b = C_0(b).$$

Let us assume that

$$d(x_n, x) \leqslant B_n(b) \leqslant b, \qquad d(y_n, x) \leqslant C_n(b) \leqslant b$$

for a certain index n. From conditions (2.15) (2.16) we get

$$d(x_{n+1}, x) \leqslant \varphi\big(d(x_n, x), d(y_n, x)\big) + a_n \leqslant \varphi\big(d(x_n, x), C_n(b)\big) + a_n \leqslant b,$$

$$d(y_{n+1}, x) \leqslant \varphi\big(d(x_n, x), d(y_n, x)\big) + a_n + c_n \leqslant \varphi\big(B_n(b), d(y_n, x)\big) + a_n + c_n \leqslant b,$$

i.e.

$$d(x_{n+1}, x) \leqslant B_{n+1}(b) \leqslant b, \qquad d(y_{n+1}, x) \leqslant C_{n+1}(b) \leqslant b,$$

and by induction — for all $n = 0, 1, 2, \ldots$ we have

$$d(x_n, x) \leqslant B_n(b) \leqslant b, \tag{2.17}$$

$$d(y_n, x) \leqslant C_n(b) \leqslant b. \tag{2.18}$$

Let us see that $B_n(b) \searrow 0$ and $C_n(b) \searrow 0$. We have proved that

$$B_1(b) \leqslant B_0(b), \qquad C_1(b) \leqslant C_0(b).$$

Let us assume that

$$B_n(b) \leqslant B_{n-1}(b), \qquad C_n(b) \leqslant C_{n-1}(b).$$

Since $\varphi(c, d)$ is non-decreasing and $a_n \leqslant a_{n-1}$, $c_n \leqslant c_{n-1}$, we have

$$\psi_n(u) = \varphi\big(u, C_n(b)\big) + a_n \leqslant \varphi\big(u, C_{n-1}(b)\big) + a_{n-1} = \psi_{n-1}(u)$$

for all $u \in [0, b]$. From the lemma we have

$$m\big(\psi_n(u), b\big) \leqslant m(\psi_{n-1}, b),$$

i.e. $B_{n+1}(b) \leqslant B_n(b)$ and similarly $C_{n+1}(b) \leqslant C_n(b)$. The elements of the sequences B_n and C_n belong to G, and thus there exist limits c, d

$$B_n(b) = \varphi\big(B_n(b), C_{n-1}(b)\big) + a_n, \qquad C_n(b) = \varphi\big(B_{n-1}(b), C_n(b)\big) + a_n + c_n,$$

$$c = \varphi(c, d), \qquad d = \varphi(c, d), \qquad \text{i.e. } c = d = 0.$$

Inequalities (2.11) and (2.12) follow from the conditions

$$d(x_n, x) \leqslant \varphi_n\big(d(x_{n-1}, x), d(y_{n-1}, x)\big) + a_n = \psi_n\big(d(x_{n-1}, x)\big) \leqslant m(\psi_n, b)$$

and

$$d(y_n, x) \leqslant \varphi_n\big(d(x_{n-1}, x), d(y_{n-1}, x)\big) + a_n + c_n = \overline{\psi}_n\big(d(y_{n-1}, x)\big) \leqslant m(\overline{\psi}_n, b).$$

This ends the proof.

3. Here we give two examples to which iteration (2.1) can be applied.

Example 1 (cf. [3]). Let us consider the equation

$$Fx = 0, \tag{3.1}$$

and suppose we are given the iteration

$$x_n = x_{n-1} - \Gamma_n(y_{n-1})Fx_{n-1} \quad (n = 1, 2, \ldots), \tag{3.2}$$

where Γ_n differs little from $[F']^{-1}$. Then

$$T_n(x_{n-1}, y_{n-1}) = x_{n-1} - \Gamma_n(y_{n-1})Fx_{n-1}.$$

Example 2 (cf. [1]). For the equation $f(X) = 0$, where $X = (x^1, x^2, \ldots, x^p)$, $Y = (y^1, \ldots, y^p)$, $f = (f_1, \ldots, f_p)$ and the iteration

$$x_n^i = \frac{x_{n-1}^i f(Y_{n-1}) - y_{n-1}^i f(X_{n-1})}{f(Y_{n-1}) - f(X_{n-1})} \tag{3.3}$$

for $f(Y_{n-1}) \neq f(X_{n-1})$ $(i = 1, \ldots, p)$, we have

$$T_n(X_{n-1}, Y_{n-1}) = \begin{bmatrix} \dfrac{x_{n-1}^1 f(Y_{n-1}) - y_{n-1}^1 f(X_{n-1})}{f(Y_{n-1}) - f(X_{n-1})} \\ \vdots \\ \dfrac{x_{n-1}^p f(Y_{n-1}) - y_{n-1}^p f(X_{n-1})}{f(Y_{n-1}) - f(X_{n-1})} \end{bmatrix}.$$

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