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Asymptotic properties of polynomials with auxiliary conditions of interpolation *

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Dedicated to the memory of Mieczysław Biernacki

It is especially appropriate that this note should be dedicated to the memory of M. Biernacki, because of his interest in, and important contributions to, the study of polynomials.

Polynomials $p_n(z) \equiv z^n + \ldots$ minimizing a norm on a closed bounded point set E in the z-plane are important in numerous investigations; of especial importance [e.g. 4, ch. 7] is the asymptotic relation (due to Fekete) which they satisfy:

$$\lim_{n\to\infty} \|p_n(z)\|^{1/n} = \tau(E) , \tag{1}$$

where $\tau(E)$ is the capacity (transfinite diameter) of E and the norm is taken in the sense of Tchebyscheff:

$$\|p_n(z)\| = [\max|p_n(z)| , \; z \text{ on } E] .$$

The question naturally arises as to whether (1) can be satisfied by a sequence of polynomials $p_n(z) \equiv z^n + \ldots$ which are required to satisfy suitable auxiliary conditions of interpolation in a finite number of points. If the assigned values of $p_n(z)$ are all zero [2], or if there is but a single condition of interpolation [1], the question has been answered in the affirmative; the object of the present note is to establish an affirmative answer in the general case:

Theorem 1. *Let E be a closed bounded set in the z-plane of positive capacity whose complement K is a region (necessarily containing $z = \infty$), let $g(z)$ be the generalized Green's function for K with pole at infinity, and let $h(z)$ be the harmonic function conjugate to $g(z)$ in K. Set $\varphi(z) \equiv e^{g(z)+ih(z)}$ in K. Let there be given a finite number of distinct points $z_1, z_2, z_3, \ldots, z_\nu$,*

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and assigned values $A_{n1}, A_{n2}, \ldots, A_{n\nu}$, $n = \nu, \nu+1, \nu+2, \ldots$ *A necessary and sufficient condition that there exist polynomials* $p_n(z) \equiv z^n + a_{n1}z^{n-1} + \ldots$, $n = \nu, \nu+1, \nu+2, \ldots$, *which satisfy the conditions* (1) *and*

$$(2) \qquad p_n(z_k) = A_{nk}, \quad k = 1, 2, \ldots, \nu,$$

is that we have

$$(3) \qquad \limsup_{n\to\infty} |A_{nk}|^{1/n} \leqslant \tau(E)\cdot|\varphi(z_k)|, \quad z_k \text{ in } K,$$

$$(4) \qquad \limsup_{n\to\infty} |A_{nk}|^{1/n} \leqslant \tau(E), \quad z_k \text{ in } E.$$

If all the points z_k of Theorem 1 lie in E, the hypothesis $\tau(E) > 0$ may be omitted, as the reader may verify.

THEOREM 2. *Theorem 1 remains valid if the assigned points z_k are not distinct, provided K is regular in the sense that it admits a classical Green's function $g(z)$ with pole at infinity, and provided conditions* (2) *are interpreted as prescribing in each multiple point z_k not merely the value of $p_n(z)$ but also the values of the first and perhaps other successive derivatives of $p_n(z)$; the total number of conditions involving z_k is the multiplicity of z_k in the enumeration. Conditions* (3) *and* (4) *remain unchanged in form when relating to those derivatives.*

A special case of Theorem 2 occurs if $z_k = 0$ for every k; conditions (2) prescribe merely the coefficients a_{nk}, $k = n-\nu+1, n-\nu+2, \ldots, n$ of $p_n(z) \equiv z^n + a_{n1}z^{n-1} + \ldots + a_{nn}$.

We proceed with the proof of Theorem 1; in [1] there is established the necessity of (3) and (4); in fact, the necessity of (4) is contained in (1), and the necessity of (3) follows from (1) by the generalized Bernstein lemma [4, § 4.6]. The sufficiency of (3) and (4) in the case $\nu = 1$ is also proved in [1]; to establish this sufficiency in the general case we assume Theorem 1 valid for prescription of assigned values in $\nu-1$ distinct points $z_1, z_2, \ldots, z_{\nu-1}$, and prove Theorem 1 as stated.

If $Q_{n-2}(z) \equiv z^{n-2} + \ldots$ denotes a polynomial of degree $n-2$ $(\geqslant \nu-1)$ still to be further restricted, we choose $z_\nu = 0$ which involves no loss of generality) and write

$$(5) \qquad p_n(z) \equiv zQ_{n-2}(z) + R_n(z),$$

where $R_n(z) \equiv z^n + \ldots + A_{n\nu}$ is a polynomial of degree n. The prescribed conditions (2) are

$$p_n(0) = A_{n\nu},$$
$$p_n(z_k) = A_{nk} = z_kQ_{n-2}(z_k) + R_n(z_k), \quad 1 \leqslant k \leqslant \nu-1,$$

so the interpolation conditions (necessary and sufficient for (2)) concerning $Q_{n-2}(z)$ are

$$Q_{n-2}(z_k) = \frac{A_{nk} - R_n(z_k)}{z_k}, \quad 1 \leqslant k \leqslant \nu - 1 . \tag{6}$$

Let $R_n(z)$ satisfy the analogue of (1), and also (as has been indicated) $R_n(0) = A_{n\nu}$; such a sequence $R_n(z)$ exists [1]. At an arbitrary point z in K we have by the generalized Bernstein lemma

$$\limsup_{n\to\infty} |R_n(z)|^{1/n} \leqslant \tau(E)\cdot|\varphi(z)| . \tag{7}$$

By (6), the analogues of (3) and (4) for $Q_{n-2}(z)$ are $(1 \leqslant k \leqslant \nu-1)$

$$\limsup_{n\to\infty} \left|\frac{A_{nk} - R_n(z_k)}{z_k}\right|^{1/n} \leqslant \tau(E)\cdot|\varphi(z_k)| , \quad z_k \text{ in } K ,$$

$$\limsup_{n\to\infty} \left|\frac{A_{nk} - R_n(z_k)}{z_k}\right|^{1/n} \leqslant \tau(E) , \quad z_k \text{ in } E ,$$

and these conditions are a consequence of (3), (4), (7), and the analogue of (1) for $R_n(z)$. It follows then from the induction hypothesis that $Q_{n-2}(z)$ exists satisfying (6), and satisfying also

$$\lim_{n\to\infty} \|Q_{n-2}(z)\|^{1/n} = \tau(E) , \quad z \text{ in } E ,$$

so $p_n(z)$ defined by (5) satisfies (1) and (2).

To prepare for the proof of Theorem 2 we establish

LEMMA 1. *Let E be a closed bounded point set whose complement K is connected. Suppose K is regular in the sense that the classical Green's function $g(z)$ for K with pole at infinity exists. Suppose $q_n(z) \equiv z^n + \dots$ is a sequence of polynomials of respective degrees n such that*

$$\lim_{n\to\infty} \|q_n(z)\|^{1/n} = \tau(E) , \quad z \textit{ in } E . \tag{8}$$

Then we have also

$$\lim_{n\to\infty} \|q_n'(z)\|^{1/n} = \tau(E) , \quad z \textit{ in } E , \tag{9}$$

$$\limsup_{n\to\infty} |q_n'(z)|^{1/n} \leqslant \tau(E)\cdot|\varphi(z)| , \quad z \textit{ in } K . \tag{10}$$

In the notation of Theorem 1, the critical points of $\varphi(z)$ have no limit point in K, so there exists a monotonic decreasing sequence $R_1, R_2, \dots \to 1$ such that no locus E_{R_k}: $|\varphi(z)| = R_k$ in K has a multiple point. If we set $\|q_n(z)\| = M_n$, it follows from the generalized Bernstein lemma [4, § 4.6] that we have

$$|q_n(z)| \leqslant M_n R_k^n , \quad z \text{ on } E_{R_k} . \tag{11}$$

Since E_{R_k} has no multiple points, there exists some r (>0) independent of z such that at each point z of E_{R_k} some circular disc with radius r is tangent to E_{R_k} there, and the closed disc lies in the closed interior of a Jordan curve belonging to E_{R_k}. It follows from (11), again by a lemma due to Bernstein, that we have in each closed disc and hence on E_{R_k} and on E

$$|q_n'(z)| \leqslant \frac{nM_nR_k^n}{r},$$

$$\limsup_{n\to\infty} \| q_n'(z) \|^{1/n} \leqslant \tau(E)\cdot R_k\,, \qquad \limsup_{n\to\infty} \| q_n'(z) \|^{1/n} \leqslant \tau(E)\,. \tag{12}$$

However, $\tau(E)$ may be defined by (1) where for each n the polynomial $p_n(z)$ is of smallest norm, hence for an arbitrary sequence of polynomials $p_n(z) \equiv z^n + \ldots$ we have

$$\liminf_{n\to\infty} \|p_n(z)\|^{1/n} \geqslant \tau(E);$$

the corresponding relation is valid for $q_n'(z) \equiv nz^{n-1} + \ldots$, so (9) follows from (12). Equation (9) implies (10) as in the proof of (11).

It is a consequence of (9) that the relation corresponding to (9) holds also for the sequence ($n = 1, 2, \ldots$) of j-th derivatives $q_n^{(j)}(z)$, $j > 1$, and for the sequence $q_n^{(j)}(z)/n(n-1)\ldots(n-j+1)$.

Lemma 1 possesses some independent interest; the study of such relations as (9) and (10) goes back to G. Faber (1920). Equation (9) is closely related also to the discussion given in [4, §§ 7.2-7.7] concerning interpolation in the zeros of the $q_n^{(j)}(z)$.

We proceed with the proof of Theorem 2. The necessity of conditions (3) and (4) follows from Theorem 1 so far as concerns the values of $p_n(z_k)$, and follows from (9) and (10) so far as concerns the prescribed values $p_n'(z_k), p_n''(z_k), \ldots, p_n^{(\mu_k)}(z_k)$ of the derivatives.

To prove the sufficiency of conditions (3) and (4) for values of $p_n(z)$ and its derivatives, we use induction. We suppose that prescribed interpolation conditions are given in distinct points $0, z_1, z_2, \ldots, z_\nu$:

$$\begin{aligned} p_n^{(j)}(0) &= A^{(j)}\,, \quad & j &= 0, 1, 2, \ldots, \mu\,; \\ p_n^{(j)}(z_k) &= A_k^{(j)}\,, \quad & j &= 0, 1, 2, \ldots, \mu_k\,, \end{aligned} \tag{13}$$

where the $A^{(j)}$ and $A_k^{(j)}$ depend on n and satisfy the analogues of (3) and (4). Thanks to Theorem 1, we may (and do) assume $\mu > 0$ without loss of generality. Our induction hypothesis is that polynomials of respective degrees n can be determined satisfying both the analogue of (1) and arbitrary interpolation conditions of form (13) with (3) and (4) valid, except with μ replaced by $\mu - 1$.

We define $p_n(z)$ in the form

$$p_n(z) \equiv zQ_{n-2}(z) + R_n(z)\,, \qquad n-3 \geqslant \mu + \sum \mu_k + \nu\,, \tag{14}$$

where $Q_{n-2}(z) \equiv z^{n-2} + \ldots$ and $R_n(z) \equiv z^n + \ldots$ are still to be defined. The prescribed conditions (13) are to be used in conjunction with (14) and equations derived from (14):

$$\begin{aligned} p_n'(z) &\equiv zQ_{n-2}'(z) + Q_{n-2}(z) + R_n'(z)\,, \\ p_n''(z) &\equiv zQ_{n-2}''(z) + 2Q_{n-2}'(z) + R_n''(z)\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ p_n^{(\mu)}(z) &\equiv zQ_{n-2}^{(\mu)}(z) + \mu Q_{n-2}^{(\mu-1)}(z) + R_n^{(\mu)}(z)\,. \end{aligned} \tag{15}$$

Let $R_n(z) \equiv z^n + \ldots$ be a sequence of polynomials (existent by the induction hypothesis) satisfying all the conditions (13) of interpolation except perhaps $R_n^{(\mu)}(0) = A^{(\mu)}$ and satisfying the analogue of (1). However, $R_n^{(\mu)}(0)$ satisfies the analogue of (3) or (4). Then equations (15) together with (13) and (14) define interpolation conditions for $Q_{n-2}(z)$ in the points $0, z_1, z_2, \ldots, z_\nu$, conditions which (since $\mu \neq 0$) determine successively $Q_{n-2}^{(j)}(0)$ for $j = \mu-1, \mu-2, \ldots, 1, 0$ and $Q_{n-2}^{(j)}(z_k) = 0$ for $j = 0, 1, 2, \ldots, \mu_k$, but not involving $Q_{n-2}^{(\mu)}(0)$. For instance $Q_{n-2}^{(\mu-1)}(0) = [A^{(\mu)} - R_n^{(\mu)}(0)]/\mu$. These conditions satisfy the analogue of (3) or (4) according as the point 0 or z_k concerned lies in K or E. Again by the induction hypothesis, the polynomials $Q_{n-2}(z)$ exist and satisfy the analogue of (1). Consequently the polynomials $p_n(z)$ defined by (14) exist satisfying both (1) and the prescribed conditions (13) of interpolation. Theorem 2 is established.

In the latter part of Theorems 1 and 2 we have shown the existence of polynomials $p_n(z)$ of respective degrees n which satisfy (1) and (2). It follows a fortiori that (1) is satisfied for the extremal polynomials $p_n(z) \equiv z^n + \ldots$ which satisfy (2) with the interpretation of Theorem 2, and minimize the norm $\|p_n(z)\|$.

A remark relative to the zeros of the $p_n(z)$ is of interest. If E satisfies the conditions of Theorem 1, if the $p_n(z) (\equiv z^n + \ldots)$ satisfy (1), and if a bounded subregion D of K contains no limit point of the zeros of the $p_n(z)$, the relation

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = \tau(E) \cdot |\varphi(z)|$$

holds uniformly [1, Theorem 3] in any closed subregion of D. Consequently if z_j lies in K and if we have

$$\limsup_{n_k\to\infty} |p_{n_k}(z_j)|^{1/n_k} < \tau(E) \cdot |\varphi(z_j)|\,,$$

the point z_j is a limit point of zeros of the $p_{n_k}(z)$.

It is appropriate to raise the question as to whether the number ν of points z_k in Theorems 1 and 2 may be allowed to become infinite with n. We indicate that *this is indeed possible provided the prescribed conditions* (2) *interpreted as in Theorem 2 are relaxed so as not to prescribe the precise number of interpolation conditions satisfied by each* $p_n(z)$. Let the conditions replacing (2) be given in the form

$$p_n(z_k) = A_{nk}\,, \quad k = 1, 2, \ldots, \tag{16}$$

only a finite number of which equations will be prescribed for a particular $p_n(z)$; of course we suppose (3) and (4) valid. It follows from Theorems 1 and 2 that there exist polynomials $p_{nk}(z) \equiv z^n + \ldots$ of respective degrees n $(\geqslant k)$ satisfying

$$p_{nk}(z_j) = A_{nj}\,, \quad j = 1, 2, \ldots, k\,,$$

$$\lim_{n\to\infty} \|p_{nk}(z)\|^{1/n} = \tau(E)\,.$$

In particular there exists N_1 such that we have for $n \geqslant N_1$

$$\|p_{n1}(z)\| \leqslant [\tau(E) + \tfrac{1}{2}]^n\,,$$

and more generally there exists N_k $(\geqslant k)$ such that we have for $n \geqslant N_k$

$$\|p_{nk}(z)\| \leqslant [\tau(E) + 2^{-k}]^n\,.$$

We suppose (as we may) $N_1 < N_2 < N_3 \ldots$ Let us now define $(k = 1, 2, \ldots)$

$$p_n(z) \equiv p_{n1}(z)\,, \quad n < N_2\,,$$

$$p_n(z) \equiv p_{nk}(z)\,, \quad N_k \leqslant n < N_{k+1}\,, \quad k > 1\,.$$

There follow the inequalities $(k > 1)$

$$\|p_n(z)\| \leqslant [\tau(E) + 2^{-k}]^n\,, \quad N_k \leqslant n < N_{k+1}\,,$$

and since k becomes infinite with n,

$$\limsup_{n\to\infty} \|p_n(z)\|^{1/n} \leqslant \tau(E)\,,$$

from which (1) follows. It will be noted that $p_n(z)$ satisfies the given conditions (16) of interpolation so far as conserns the k $(= k_n)$ points $z_1, z_2, \ldots, z_k$, where k (> 1) is determined by $N_k \leqslant n < N_{k+1}$, and k becomes infinite with n.

The remark just established is considered in [2] in some detail in the analogous case that E is a disc $|z| \leqslant R$ and the auxiliary conditions are not of form (16) but prescribe the first k coefficients of $p_n(z) \equiv z^n + a_{n1}z^{n-1} + a_{n2}z^{n-1} + \ldots$; some of the results (loc. cit.) concerning the latter problem are due to G. Szegö.

There is also treated in [2] the special problem obtained by replacing (16) with the auxiliary requirement

$$p_n(z_{nk}) = 0\,, \quad k = 1, 2, \ldots, k_n\,, \tag{17}$$

where the z_{nk} are bounded in their totality. Provided merely $k_n = o(n)$, it is proved that the $p_n(z)$ exist satisfying (17) and (1). We add here the additional remark that *if $p_n(z)$ is the polynomial $z^n + \ldots$ satisfying* (17) *and among all such polynomials minimizing $\|p_n(z)\|$, then all zeros of $p_n(z)$ other than the z_{nk} lie in the convex hull of E* (1). Indeed, suppose no z_{nk} belongs to E; we have

$$p_n(z) \equiv q_{n-k_n}(z) \cdot \prod_{k=1}^{k_n} (z - z_{nk}), \qquad |p_n(z)| \equiv \prod_{k=1}^{k_n} |z - z_{nk}| \cdot |q_{n-k_n}(z)|\,,$$

where the first factor in the second member does not vanish on E, and the conclusion follows by a well-known theorem due to Féjer; if some or all z_{nk} belong to E, a slight extension of this reasoning yields the conclusion. At every point of K exterior to the convex hull of E other than the limit points of the z_{nk} we have

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\varphi(z)|\,, \tag{18}$$

and throughout any closed bounded set of such points this relation is valid uniformly. The zeros of the $p_n(z)$ on any closed bounded subset K_1 of K may be factored out from $p_n(z)$ without altering the essence of (18) on K_1.

Throughout the present note we have used for the polynomials $p_n(z)$ the Tchebycheff norm. However, equation (1) for the Tchebycheff norm is equivalent [3] to that equation for any one of a wide category of norms; any of the latter may be used in Theorems 1 and 2. For instance, if E is a closed Jordan region, one may use the p-th root of

$$\iint_E |p_n(z)|^p dS \quad \text{or} \quad \int_C |p_n(z)|^p |dz|\,, \quad p > 0\,,$$

where C is the boundary (assumed rectifiable) of E. Likewise the remark concerning the position of the zeros of $p_n(z)$ satisfying (17) applies to these and other more general norms.

References

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[2] M. Fekete and J. L. Walsh, *Asymptotic behavior of restricted extremal polynomials and of their zeros*, Pacific Jour. of Math. 7 (1957), pp. 1037-1064.

(1) In fact this conclusion applies to an arbitrary infrapolynomial required to satisfy the auxiliary conditions.

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