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The translation equation on a direct product of groups

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Abstract. Let X be an arbitrary set and G an arbitrary group. We consider the functional equation

$$(1)\qquad F(F(x, a), \beta) = F(x, \beta a) \quad \text{for } x \in X,\ a, \beta \in G,$$

where $F\colon X \times G \to X$ is the function which we look for.

Theorem 1. *Let $G = G_1 \times G_2$ be the direct product of the groups G_1, G_2. Then the general solution $F\colon X \times G \to X$ of equation* (1) *can be written in the form $F(x, \langle a, \beta\rangle) = F_2(F_1(x, a), \beta)$ for $x \in X$, $a \in G_1$, $\beta \in G_2$, where $F_1\colon X \times G_1 \to X$ and $F_2\colon X \times G_2 \to X$ satisfy equation* (1) *and the following condition*:

$$(2)\qquad F_2(F_1(x, a), \beta) = F_1(F_2(x, \beta), a) \quad \text{for } x \in X,\ a, \beta \in G.$$

Theorem 2. *Let $G = G_1 \times G_2$ be the direct product of groups G_1, G_2 and let $F\colon X \times G \to X$ be a solution of equation* (1) *such that the function F_2 occurring in* (2) *is transitive (i.e., satisfies the condition $\bigwedge_{x,y\in X} \bigvee_{a\in G} F(x, a) = y$) and commutative (i.e., satisfies the condition $F(F(x, a), \beta) = F(F(x, \beta), a)$ for $x \in X$, $a, \beta \in G$). Then there exists a normal subgroup G_2^* of G_2 and a homomorphism $\varphi\colon G_1 \to G_2/G_2^*$ such that $F(x, \langle a, \beta\rangle) = F_2(x, \beta\varphi(a))$ for $x \in X$, $a \in G_1$, $\beta \in G_2$.*

We also give a sufficient condition for the solution of equation (1) to be continuous.

1. In the present paper X denotes a non-empty set and G denotes a group. The functional equation

$$(1)\qquad F(F(x, a), \beta) = F(x, \beta a),$$

where $F\colon X \times G \to X$ is the function which we look for, is called the *translation equation.*

Let $G = G_1 \times G_2$ be the direct product of groups G_1, G_2. We shall investigate in this paper the relations between the solutions of the translation equation on the groups G_1, G_2 and on the group G. In particular, we shall express the general solution of the translation equation on the group G by the solutions of this equation on the groups G_1, G_2. A sufficient condition for the solution of equation (1) on G to be continuous will also be given.

The solution $F\colon X\times G\to X$ of equation (1) is called an *algebraic object* (cf. [8], p. 69) whenever $F(x,1)=x$ for $x\in X$. An algebraic object $F\colon X\times G\to X$ is called *transitive* if for every $x,y\in X$ there exists an $a\in G$ such that $F(x,a)=y$. An algebraic object $F\colon X\times G\to X$ is called *commutative* (cf. [1], p. 19) whenever

$$F\big(F(x,a),\beta\big)=F\big(F(x,\beta),a\big)\quad \text{for } x\in X,\ a,\beta\in G.$$

Let $F\colon X\times G\to X$ be an algebraic object. Let us write

$$N=\big\{a\in G\colon \underset{x\in X}{\forall}\ \big(F(x,a)=x\big)\big\}. \tag{2}$$

M. Kania and Z. Moszner have proved (cf. [2]) that the object F is commutative iff the commutant $K(G)$ of G is included in N.

Moszner has shown (cf. [5], p. 51) that a function $F\colon X\times G\to X$ is a transitive algebraic object iff it is of the form

$$F(x,a)=g^{-1}\big(ag(x)\big)\quad \text{for } x\in X,\ a\in G, \tag{3}$$

where g is a bijection of the set X onto the left cosets of the group G with respect to some subgroup G^* (not necessarily normal).

Let G^* be a normal subgroup of G. Then the function F of form (3) is constant on every coset $A\in G/G^*$. This proves that for every fixed $x\in X$ the mapping

$$G/G^*\ni A\to F(x,a),\qquad a\in A,$$

is well defined. Furthermore, since g is a bijection, it is a bijection of G/G^* onto X. In consequence, we may define a new object $\tilde{F}\colon X\times G/G^*\to X$ in the following way (cf. [5], p. 57):

$$F(x,A)=F(x,a)\quad \text{for } a\in A\in G/G^*.$$

It can be verified that if F is transitive (commutative), then so is $\tilde{F}$. We shall use the same letter F for the objects F and $\tilde{F}$.

For the next section we need the following simple

Lemma 1. *Let $F\colon X\times G\to X$ be a transitive and commutative algebraic object and let F be written in form* (3). *Then the subgroup G^* is normal.*

Proof. We have $N=\bigcap_{a\in G}a^{-1}G^*a$ (cf. [6]), which proves that $N\subset G^*$. By the theorem of Kania and Moszner (cf. [2]) $K(G)\subset N$ and hence $K(G)\subset G^*$. This implies directly that G^* is normal.

2. We shall prove the following

Theorem 1[1]. *Let $G=G_1\times G_2$ be the direct product of groups G_1,G_2.*

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[1] This theorem for commutative algebraic objects has been proved by Kania and Moszner (cf. [2]).

A function $F\colon X\times G\to X$ *satisfies the translation equation iff* F *can be written in the form*

$$(4)\qquad F(x,\langle\alpha,\beta\rangle)=F_2\big(F_1(x,\alpha),\beta\big)\quad \text{for } x\in X,\ \alpha\in G_1,\ \beta\in G_2,$$

where $F_1\colon X\times G_1\to X$ *and* $F_2\colon X\times G_2\to X$ *satisfy the translation equation and the following condition*

$$(5)\qquad F_2\big(F_1(x,\alpha),\beta\big)=F_1\big(F_2(x,\beta),\alpha\big)\quad \text{for } x\in X,\ \alpha\in G_1,\ \beta\in G_2.$$

Proof. Suppose that a function $F\colon X\times G\to X$ satisfies the translation equation. We put

$$(6)\qquad F_1(x,\alpha)=F(x,\langle\alpha,1\rangle)\quad \text{for } x\in X,\ \alpha\in G_1,$$

$$(7)\qquad F_2(x,\beta)=F(x,\langle 1,\beta\rangle)\quad \text{for } x\in X,\ \beta\in G_2.$$

It is immediately seen that F_1 and F_2 satisfy the translation equation on the sets $X\times G_1$, $X\times G_2$, respectively. Furthermore, we have for $x\in X$, $\alpha\in G_1$, $\beta\in G_2$:

$$\begin{aligned} F_2\big(F_1(x,\alpha),\beta\big) &= F\big(F(x,\langle\alpha,1\rangle),\langle 1,\beta\rangle\big)=F(x,\langle 1,\beta\rangle\cdot\langle\alpha,1\rangle)\\ &= F(x,\langle\alpha,\beta\rangle)=F(x,\langle\alpha,1\rangle\cdot\langle 1,\beta\rangle)\\ &= F\big(F(x,\langle 1,\beta\rangle),\langle\alpha,1\rangle\big)=F_1\big(F_2(x,\beta),\alpha\big).\end{aligned}$$

Hence, condition (5) holds.

Conversely, let us suppose that the functions $F_1\colon X\times G_1\to X$ and $F_2\colon X\times G_2\to X$ satisfy the translation equation and condition (5). We shall show that a function F of form (4) satisfies the translation equation. Applying (4) and (5) we obtain for $x\in X$, $\langle\alpha_1,\beta_1\rangle,\langle\alpha_2,\beta_2\rangle\in G_1\times G_2$

$$\begin{aligned} F\big(F(x,\langle\alpha_1,\beta_1\rangle),\langle\alpha_2,\beta_2\rangle\big) &= F_2\Big(F_1\Big(F_2\big(F_1(x,\alpha_1),\beta_1\big),\alpha_2\Big),\beta_2\Big)\\ &= F_2\Big(F_2\Big(F_1\big(F_1(x,\alpha_1),\alpha_2\big),\beta_1\Big),\beta_2\Big)=F_2\big(F_1(x,\alpha_2\alpha_1),\beta_2\beta_1\big)\\ &= F(x,\langle\alpha_2\alpha_1,\beta_2\beta_1\rangle)=F(x,\langle\alpha_2,\beta_2\rangle\cdot\langle\alpha_1,\beta_1\rangle).\end{aligned}$$

This completes the proof.

We have proved in Theorem 1 that every solution F of equation (1) on the direct group $G=G_1\times G_2$ can be written in form (4). We shall prove in the following theorem that under some assumptions on F formula (4) can be simplified.

THEOREM 2. *Let* $G=G_1\times G_2$ *be the direct product of groups* G_1, G_2. *Let* $F\colon X\times G\to X$ *be a solution of the translation equation such that the object* F_2 *defined by* (7) *is transitive and commutative. Then there exists a normal subgroup* G_2^* *of the group* G_2 *and a homomorphism* $\varphi\colon G_1\to G_2/G_2^*$ *such that*

$$(8)\qquad F(x,\langle\alpha,\beta\rangle)=F_2\big(x,\beta\varphi(\alpha)\big)\quad \text{for } x\in X,\ \alpha\in G_1,\ \beta\in G_2.$$

Proof. The object F_2 can be represented in form (3), i.e., in the form

$$F_2(x, \beta) = g_2^{-1}\big(\beta g_2(x)\big) \quad \text{for } x \in X, \ \beta \in G_2, \tag{9}$$

where g_2 is a bijection of the set X onto the set G_2/G_2^* of the left cosets in G_2 with respect to some subgroup $G_2^* \subset G_2$. In virtue of Lemma 1, G_2^* is a normal subgroup of G_2. Let $x_0 \in X$ be arbitrarily fixed. As we have observed in 2, the mapping $G_2/G_2^* \ni B \mapsto F_2(x_0, \beta)$, $\beta \in G_2$ is a bijection of G_2/G_2^* onto X. Hence, for every $\alpha \in G_1$ there exists exactly one element $\varphi(\alpha) \in G_2/G_2^*$ such that

$$F_2\big(x_0, \varphi(\alpha)\big) = F_1(x_0, \alpha), \tag{10}$$

where F_1 is defined by formula (6). By Theorem 1, condition (5) holds. From this condition, from (8) and from the commutativity of F_2 we obtain for $\alpha_1, \alpha_2 \in G_1$:

$$\begin{aligned} F_1(x_0, \alpha_1\alpha_2) &= F_1\big(F_1(x_0, \alpha_2), \alpha_1\big) = F_1\Big(F_2\big(x_0, \varphi(\alpha_2)\big), \alpha_1\Big) \\ &= F_2\big(F_1(x_0, \alpha_1), \varphi(\alpha_2)\big) = F_2\Big(F_2\big(x_0, \varphi(\alpha_1)\big), \varphi(\alpha_2)\Big) \\ &= F_2\Big(F_2\big(x_0, \varphi(\alpha_2)\big), \varphi(\alpha_1)\Big) = F_2\big(x_0, \varphi(\alpha_1)\cdot\varphi(\alpha_2)\big). \end{aligned}$$

Hence, it follows from the above equality and from the definition of φ that $\varphi(\alpha_1\alpha_2) = \varphi(\alpha_1)\varphi(\alpha_2)$, i.e., that φ is a homomorphism.

We shall show that equality (8) holds. Let us consider the arbitrary elements $x \in X$, $\alpha \in G_1$, $\beta \in G_2$. Let $\gamma \in G_2$ be such an element that

$$x = F_2(x_0, \gamma). \tag{10}$$

Since object F_2 is commutative, we obtain from (5) and (10),

$$\begin{aligned} F_1(x, \alpha) &= F_1\big(F_2(x_0, \gamma), \alpha\big) = F_2\big(F_1(x_0, \alpha), \gamma\big) \\ &= F_2\Big(F_2\big(x_0, \varphi(\alpha)\big), \gamma\Big) = F_2\big(F_2(x_0, \gamma), \varphi(\alpha)\big) = F_2\big(x, \varphi(\alpha)\big). \end{aligned} \tag{11}$$

This equality, (4) and (11) give

$$\begin{aligned} F(x, \langle\alpha, \beta\rangle) &= F_2\big(F_1(x, \alpha), \beta\big) = F_2\Big(F_2\big(x, \varphi(\alpha)\big), \beta\Big) \\ &= F_2\big(x, \beta\varphi(\alpha)\big), \end{aligned}$$

which completes the proof.

As an immediate consequence of Theorem 2, we find, that under the assumption of this theorem function F can be written in the form

$$F(x, \langle\alpha, \beta\rangle) = g_2^{-1}\big(\beta\varphi(\alpha)g_2(x)\big) \quad \text{for } x \in X, \ \alpha \in G_1, \ \beta \in G_2, \tag{12}$$

where g_2 is the same function as in formula (9) and $\varphi\colon G_1 \to G_2/G_2^*$ is a homomorphism. It can also be seen that every function F defined by (12) satisfies the translation equation.

Let the assumptions of Theorem 2 be fulfilled and additionaly let the object F_2 be effective, i.e., let

$$\underset{\beta_1,\beta_2\in G_2}{\forall}\left[\underset{x\in X}{\forall}\big(F_2(x,\beta_1) = F_2(x,\beta_2)\big) \Rightarrow (\beta_1 = \beta_2)\right].$$

Then we have (cf. [5], p. 59) the relation

$$\bigcap_{\beta\in G_2} \beta^{-1}G_2^*\beta = \{1\}$$

and consequently, since G_2^* is normal, $G_2^* = \{1\}$. Hence, we may look at function φ occurring in (10) as a homomorphism of G_1 into G_2.

3. Let X be a topological space and let G_1 and G_2 be topological groups. Then the direct product $G = G_1 \times G_2$ is also a topological group. Under these assumptions one may look for a continuous solution of equation (1). We have the following

THEOREM 3. *Let X be a Hausdorff topological space locally compact at some point $x \in X$. Let G_1 and G_2 be topological groups and let G_2 be a countable union of compact sets. Let F: $X \times G \to X$ satisfy* (1) *and let F_2 defined by* (7) *be a transitive and commutative algebraic object. Let F_2 satisfy additionally the following conditions*:

(a) *there exists an $x_0 \in X$ such that the function $G_2 \ni \beta \mapsto F_2(x_0, \beta)$ is continuous,*

(b) *for every fixed β the function $X \ni x \mapsto F_2(x, \beta)$ is continuous.*

Let there exist $x_1 \in X$, $\beta_1 \in G_2$ such that the function $G_1 \ni \alpha \mapsto F(x_1, \langle\alpha, \beta_1\rangle)$ is continuous at some point $\alpha_1 \in G_1$.

Then the function φ occurring in formula (8) *is continuous and F is continuous.*

Proof. By a theorem of Moszner (cf. [4], p. 90) the function F_2 can be written in form (9) in such a manner that g_2 is a homeomorphism. We obtain from formula (12)

$$\varphi(\alpha) = \beta_1 g_2\big(F(x_1, \langle\alpha, \beta_1\rangle)\big)\cdot\big(g_2(x)\big)^{-1}. \tag{13}$$

We have for some point $x \in X$

$$G_2^* = g_2^{-1}(\{x\}).$$

Since X, as a Hausdorff space, is a T_1-space, the set $\{x\}$ is closed. In consequence, G_2^* is a topological subgroup of G_2, which implies that the multiplications $\beta\cdot A$ and $A\cdot B$, where $\beta \in G_2$, A, $B \in G_2/G_2^*$, are continuous (cf. [7], p. 101). Thus it follows from (13) and from our last assumption that φ is continuous at some point $\alpha_1 \in G_1$. Furthermore φ is a homomorphism. As is well known, a homomorphism continuous at one point is continuous everywhere([2]). Thus, φ is continuous. Now, it follows from formula (12) that F is a composition of continuous functions. This shows that F is continuous and this completes the proof.

We shall illustrate our considerations by the following simple

EXAMPLE. Let us put

$$F(x, a+\beta i) = x+c_1 a+c_2\beta \quad \text{for } x \in R,\ a+\beta i \in C,$$

where C denotes the additive group of complex numbers and c_1, c_2 are real constants different from zero. We treat R and C as topological spaces with the usual topologies. Obviously, C is the direct product of the topological group R by itself. It can be verified that F satisfies (1). We have

$$F_1(x, a) = x+c_1 a \quad \text{for } x \in R,\ a \in R,$$
$$F_2(x, \beta i) = x+c_2\beta \quad \text{for } x \in R,\ \beta \in R.$$

The object F_2 is commutative and effective. Hence, $G_2^* = \{0\}$. It is obvious that the object F_2 can be written in form (11), where $\varphi(a) = \dfrac{c_1}{c_2} a$ for $a \in R$.

([2]) This theorem has been proved in [7], p. 109, for the unity of the group under consideration. A proof for an arbitrary fixed element can be realized in a similar way.

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