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Smoothness of solutions for a system of first order non-linear partial differential equations

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1. In this paper we shall deal with the system of partial differential equations of the type

$$u_x^i = f^i(x, y, u^1, \ldots, u^n, u_y^i), \quad i = 1, 2, \ldots, n. \tag{1}$$

We show (Theorem 1) that first order derivatives of solutions for (1) are Lipschitz-continuous provided f^i satisfy suitable inequalities. We give (Theorem 2) a simple estimation for the Lipschitz constant.

This result is non-trivial even for solutions of class C^∞. Analogous problems for a single equation were treated by T. Ważewski in [4], A. Pliś in an unpublished paper and S. Bal in [1].

2. We now give the idea for the proof of Theorem 2. We have to prove that a function g is Lipschitz-continuous. Consider the function

$$m(l) = \max[|g(x, z) - g(x, y)|/(z - y + \delta)]$$

for $(x, y, z) \in P_l$, where P_l is a suitable subset of the semi-space $z \geqslant y$, P_l depends on the non-negative variable l and $g(x, y)$ is a continuous function. We use a sufficiently small positive number δ to avoid the singularity of the quotient under consideration for $z = y$, $\delta = 0$. Set P_l is defined in such a manner that the function $m(l)$ satisfies a differential inequality of the type $y' \geqslant ky^2$ $(k > 0)$.

From the asymptotical properties of solutions for the equation $y' = ky^2$ and from a theorem on differential inequalities it follows that the value $m(0)$ must be bounded. For $\delta \to 0$ this implies that g satisfies the Lipschitz condition.

3. Now we shall prove the following lemma:

If a function $m(l)$, continuous and non-decreasing on $[0, a]$, satisfies on $(0, a)$ the differential inequality

$$D_+ y(l) \geqslant B_0 y^2 - B_1 y - B_2 \quad \textit{for } y > K, \tag{2}$$

where B_0, B_1, B_2, a *are positive constants, then* $m(0) \leqslant K$, *where* $K = \max[2/B_0 a, (B_1 + \sqrt{B_1^2 + 2B_0B_2})/B_0]$, *and*

$$D_+ y(l) = \limsup_{\Delta l \to +0} \frac{y(l+\Delta l) - y(l)}{\Delta l}.$$

Proof. Suppose the contrary: $m(0) > K$. This implies that $m(l) > K$ on the whole interval $[0, a]$. Inequality (2) can be written in the form $D_+ y \geqslant B_0 y^2/2 + (B_0 y^2/2 - B_1 y - B_2)$, where $D_+ y$ denotes a right-hand derivative of the function y.

For $m(l) > K$ the expression in parenthesis is positive; therefore we have on $[0, a)$ the inequality

$$D_+ m(l) > \tfrac{1}{2} B_0 m^2(l) \quad \text{for } 0 \leqslant l < a.$$

Consider the equation $p'(l) = \frac{1}{2} B_0 p^2(l)$.

The solution of this equation with the initial condition $p(0) = m(0)$ is

$$p = 2p(0)/\big(2 - B_0 l p(0)\big). \tag{3}$$

The denominator vanishes for $l_0 = 2/B_0 p(0)$.

In virtue of $p(0) = m(0)$, $K \geqslant 2/B_0 a$, we have $0 < l_0 = 2/B_0 p(0) < a$. On $[0, l_0]$, by the theorem on ordinary differential inequalities [3], we obtain the inequality $p(l) \leqslant m(l)$ for $0 \leqslant l < l_0$; therefore $p(l)$ is bounded on $0 \leqslant l < l_0$. We have got a contradiction of (3).

4. Now we shall formulate and prove two theorems. The first one deals with the smoothness of derivatives for a solution. The second one gives an estimation for the Lipschitz constant.

THEOREM 1. *If functions* $f^i(x, y, u^1, \ldots, u^n, q)$ *are of class* C^2 *on an open subset* S *of* R^{n+3} *and* $f^i_{qq} \neq 0$, *then the derivatives* $u^1_y(x, y), \ldots, u^n_y(x, y)$ *of an arbitrary vector function* $u(x, y) = \big(u^1(x, y), \ldots, u^n(x, y)\big)$ *which are of class* C^1 *on a certain open set* ω, $\omega \subset R^2$ *and satisfy system* (1) *are Lipschitz-continuous on* ω.

THEOREM 2. *Suppose that the functions* $f^i(x, y, s, q^i)$ *are of class* C^2 *on the set*

$$Q = \{(x, y, s, q): (x, y) \in P, |s^i - s^i_0| \leqslant h, |q^i - q^i_0| \leqslant h, i = 1, 2, \ldots, n\},$$

where $s = (s^1, \ldots, s^n)$, $q = (q^1, \ldots, q^n)$,

$$P = \{(x, y): |x - x_0| \leqslant a, |y - y_0| \leqslant b + Ca\},$$

a, b, h, C *are positive constants and satisfy the inequalities* $|f^i_q|$, $|f^i + q^i f^i_q|$, $|f^i_y + q^k f^i_k| \leqslant C$ *on* Q, *where* $f^i_k = f^i_{s^k}$, $f^i_q = f^i_{q^i}$, $i, k = 1, 2, \ldots, n$, *and* $f^i_{q^i q^i} \neq 0$, $i = 1, 2, \ldots, n$, *on* Q.

The derivatives $u^1_y(x, y), \ldots, u^n_y(x, y)$ *of an arbitrary solution of* (1), *of class* C^1 *on* P *and satisfying the inequalities*

$$|u^1(x,y)-s_0^1| \leqslant h, \quad \ldots, \quad |u^n(x,y)-s_0^n| \leqslant h,$$

$$|u_y^1(x,y)-q_0^1| \leqslant h, \quad \ldots, \quad |u_y^n(x,y)-q_0^n| \leqslant h$$

on the rectangle P, satisfy the inequalities

$$|u_y^i(x_0,z)-u_y^i(x_0,y)| \leqslant K|z-y|, \qquad i=1,2,\ldots,n,$$

$$K = \max[2/C_0 a, (C_1+\sqrt{C_1^2+2C_0C_2})/C_0],$$

where

$$C_0 = \min[|f_{qq}^1(x,y,s,q^1)|, \ldots, |f_{qq}^n(x,y,s,q^n)|] \quad \textit{on } Q,$$

$$C_1 = \max|f_{yq}^i| + \sum_{j=1}^n \max|q^j|\,|f_{jq}^i| + \max|f_{qy}^i| + \sum_{k=1}^n \max|f_{qk}^i|\,|q^k| + \sum_{j=1}^n \max|f_j^i| \quad \textit{on } Q,\ i=1,2,\ldots,n,$$

$$C_2 = \max|f_{yy}^i| + \sum_{k=1}^n \max|f_{yk}^i|\,|q^k| + \sum_{j=1}^n \max|q^j|\,|f_{jy}^i| + \sum_{j=1}^n \max|q^j| \sum_{k=1}^n |f_{jk}^i|\,|q^k| \quad \textit{on } Q,\ i=1,2,\ldots,n,$$

where $f_{yk}^i = f_{ys^k}^i$, $f_{kj}^i = f_{s^k s^j}^i$, $f_{qq}^i = f_{q^i q^i}^i$, $f_{yq}^i = f_{yq^i}^i$; $f_{kq}^i = f_{s^k q^i}^i$.

5. Proof. Let us define a function of three variables

$$W(x,y,z) = \frac{\max[|u_y^1(x,z)-u_y^1(x,y)|, \ldots, |u_y^n(x,z)-u_y^n(x,y)|]}{z-y+\delta}$$

on $R = \{(x,y,z)\colon (x,y)\in P, (x,z)\in P, y \leqslant z\}$, where $u = \big(u^1(x,y), \ldots \ldots, u^n(x,y)\big)$ is a given solution of (1) and δ a positive constant. The function $W(x,y,z)$ is continuous on R with respect to all three variables.

Now consider the following function of a real variable l: $y(l) = \max\limits_{(x,y,z)\in P_l} W(x,y,z)$, where

$$P_l = \{(x,y,z)\colon |x-x_0| \leqslant l, |y-y_0|, |z-y_0| \leqslant b+Cl, y \leqslant z, 0 \leqslant l \leqslant a\}.$$

$y(l)$ is a function continuous on $[0,a]$ because it is a maximum of a continuous function. Hence follows the boundedness of the function $y(l)$.

The definition implies that $y(l)$ is non-decreasing. We shall show that if $y(l) > K$ for $l\in[0,a)$, then the function $y(l)$ satisfies the differential inequality

$$D_+ y(l) \geqslant C_0 y^2 - C_1 y - C_2. \tag{4}$$

From the definition of $y(l)$ it follows that $y(l) = W(\bar{x},\bar{y},\bar{z})$, where $(\bar{x},\bar{y},\bar{z})\in P_l$.

In virtue of the inequality $K > 0$ and the equality $W(x,y,z) = 0$ for $z = y$, we have $\bar{y} < \bar{z}$.

There exists a subscript i such that

$$W(\bar{x}, \bar{y}, \bar{z}) = \frac{|u_y^i(\bar{x}, \bar{z}) - u_y^i(\bar{x}, \bar{y})|}{\bar{z} - \bar{y} + \delta}.$$

Consider the equation

$$y' = -f_q^i\big(x, y, u^1(x, y), \ldots, u^n(x, y), u_y^i(x, y)\big). \tag{5}$$

From the theorem on characteristics [2] it follows that there exist two systems of functions, $y(x), s^i(x), q^i(x)$ and $z(x), w^i(x), p^i(x)$, satisfying the following system of ordinary differential equations:

$$\begin{aligned} y' &= -f_q^i\big(x, y, u^1(x, y), \ldots, u^n(x, y), q^i\big), \\ q^{i\prime} &= f_y^i\big(x, y, u^1(x, y), \ldots, u^n(x, y), q^i\big) + \\ &\quad + \sum_{k=1}^{n} u_y^k(x, y) f_k^i\big(x, y, u^1(x, y), \ldots, u^n(x, y), q^i\big), \end{aligned}$$

where $y = y(x), z = z(x)$ are two solutions of (5) satisfying the initial conditions $\bar{y} = y(\bar{x}), \bar{z} = z(\bar{x})$, and defined on a neighbourhood of $\bar{x}$. These functions satisfy the inequalities on a neighbourhood of $\bar{x}$.

The functions $s^i(x), q^i(x), w^i(x), p^i(x)$ satisfy the initial conditions

$$\begin{aligned} s^i(\bar{x}) &= u^i\big(\bar{x}, y(\bar{x})\big) = u^i(\bar{x}, \bar{y}), \quad & q^i(\bar{x}) &= u_y^i\big(\bar{x}, y(\bar{x})\big) = u_y^i(\bar{x}, \bar{y}), \\ w^i(\bar{x}) &= u^i\big(\bar{x}, z(\bar{x})\big) = u^i(\bar{x}, \bar{z}), \quad & p^i(\bar{x}) &= u_y^i\big(\bar{x}, z(\bar{x})\big) = u_y^i(\bar{x}, \bar{z}) \end{aligned}$$

and the identities

$$\begin{aligned} s^i(x) &= u^i\big(x, y(x)\big), \quad & w^i(x) &= u^i\big(x, z(x)\big), \\ q^i(x) &= u_y^i\big(x, y(x)\big), \quad & p^i(x) &= u_y^i\big(x, z(x)\big) \end{aligned}$$

on a neighbourhood of $\bar{x}$.

These functions satisfy the inequalities

$$\begin{aligned} |s^i(x) - s_0^i| &\leqslant h, \quad & |q^i(x) - q_0^i| &\leqslant h, \\ |w^i(x) - s_0^i| &\leqslant h, \quad & |p^i(x) - q_0^i| &\leqslant h \end{aligned}$$

on a neighbourhood of $\bar{x}$.

Under our assumptions the inequality $y(x) \leqslant z(x)$ is satisfied on a certain neighbourhood of $\bar{x}$.

We shall use the following notation:

$$\begin{aligned} s^k(x) &= u^k\big(x, y(x)\big), \quad & w^k(x) &= u^k\big(x, z(x)\big), \\ q^k(x) &= u_y^k\big(x, y(x)\big), \quad & p^k(x) &= u_y^k\big(x, z(x)\big), \end{aligned}$$

$k = 1, 2, \ldots, n$, on a neighbourhood of $\bar{x}$.

The functions

$$v^k(x) = \frac{p^k(x)-q^k(x)}{z(x)-y(x)+\delta}, \qquad k=1,2,\ldots,n,$$

are of class C^1 on a neighbourhood of $\bar{x}$.

We shall denote $v^i(x)$ by $v(x)$, omitting the superscript i.

We shall show that the function $v(x)$ satisfies the inequality $v'(\bar{x}) \geqslant C_0\big(v(\bar{x})\big)^2 - C_1|v(\bar{x})| - C_2$.

We have

$$\begin{aligned} v'(x) &= \left(\frac{p^i(x)-q^i(x)}{z(x)-y(x)+\delta}\right)' \\ &= \frac{(p^i-q^i)'(z-y+\delta)-(p^i-q^i)(z-y)'}{(z-y+\delta)^2} \\ &= \frac{1}{z-y+\delta}\Big[f_y^i(M)-f_y^i(S)+\sum_{k=1}^{n}\big(p^k f_k^i(M)-q^k f_k^i(S)\big)\Big]+ \\ &\quad + \frac{v(x)}{z-y+\delta}[f_q^i(M)-f_q^i(S)], \end{aligned}$$

where

$$S = \big(x, y(x), s^1(x), \ldots, s^n(x), q^i(x)\big), \quad M = \big(x, z(x), w^1(x), \ldots, w^n(x), p^i(x)\big).$$

We apply the mean value theorem:

$$\begin{aligned} v'(x) &= \frac{1}{z-y+\delta}\Big[f_{yy}^i(\tilde{S})(z-y)+\sum_{k=1}^{n} f_{yk}^i(\tilde{S})(w^k-s^k)+f_{yq}^i(\tilde{S})(p^i-q^i)\Big]+ \\ &\quad + \frac{1}{z-y+\delta}\sum_{j=1}^{n}\big[p^j\big(f_j^i(M)-f_j^i(S)\big)+(p^j-q^j)f_j^i(S)\big]+ \\ &\quad + \frac{v(x)}{z-y+\delta}\Big[f_{qy}^i(\tilde{M})(z-y)+\sum_{k=1}^{n} f_{qk}^i(\tilde{M})(w^k-s^k)+f_{qq}^i(\tilde{M})(p^i-q^i)\Big], \end{aligned}$$

where $\tilde{S} = (x, \tilde{y}, \tilde{s}^1, \ldots, \tilde{s}^n, \tilde{q}^i)$, $\tilde{M} = (x, \tilde{z}, \tilde{w}^1, \ldots, \tilde{w}^n, \tilde{p}^i)$ are on the segment between S and M,

$$\begin{aligned} v'(x) &= \frac{1}{z-y+\delta}\Big[f_{yy}^i(\tilde{S})(z-y)+\sum_{k=1}^{n} f_{yk}^i(\tilde{S})(w^k-s^k)+f_{yq}^i(\tilde{S})(p^i-q^i)\Big]+ \\ &\quad + \frac{1}{z-y+\delta}\sum_{j=1}^{n}\Big[p^j f_{jy}^i(N_j)(z-y)+p^j\sum_{k=1}^{n} f_{jk}^i(N_j)(w^k-s^k)+ \\ &\qquad\qquad + p^j f_{jq}^i(N_j)(p^i-q^i)+(p^j-q^j)f_j^i(S)\Big]+ \\ &\quad + \frac{v(x)}{z-y+\delta}\Big[f_{qy}^i(\tilde{M})(z-y)+\sum_{k=1}^{n} f_{qk}^i(\tilde{M})(w^k-s^k)+f_{qq}^i(\tilde{M})(p^i-q^i)\Big], \end{aligned}$$

where $N_j = (x, y_j, s_j^1, \ldots, s_j^n, q_j^i), j = 1, 2, \ldots, n,$ are on the segment between the points S and M.

After rearranging the terms, we get

$$v'(x) = f^i_{qq}(\tilde{M})\big(v(x)\big)^2 +$$

$$+v(x)\left[f^i_{yq}(\tilde{S}) + \sum_{j=1}^{n} p^j f^i_{jq}(N_j) + f^i_{qy}(\tilde{M})\frac{z-y}{z-y+\delta} + \sum_{k=1}^{n} f^i_{qk}(\tilde{M}) q_1^k \frac{z-y}{z-y+\delta}\right] +$$

$$+\sum_{j=1}^{n} v^j f^i_j(S) + f^i_{yy}(\tilde{S})\frac{z-y}{z-y+\delta} + \sum_{k=1}^{n} f^i_{yk}(\tilde{S}) q^k \frac{z-y}{z-y+\delta} +$$

$$+\sum_{j=1}^{n} p^j f^i_{jy}(N_j)\frac{z-y}{z-y+\delta} + \sum_{j=1}^{n} p^j \sum_{k=1}^{n} f^i_{jk}(N_j) q_2^k \frac{z-y}{z-y+\delta},$$

where $q_1^k = u_y^k(x, y_1)$, $q_2^k = u_y^k(x, y_2)$, y_1, y_2 are points on the suitable segments.

Consider the case where $f^i_{qq} > 0$ on Q.

We obtain on a neighbourhood of $\bar{x}$ the inequality

$$v'(x) \geqslant f^i_{qq} v^2 - |v(x)|\Big[|f^i_{yq}| + \sum_{j=1}^{n} |p^j|\,|f^i_{jq}| + |f^i_{qy}| + \sum_{k=1}^{n} |f^i_{qk}|\,|q_1^k|\Big] -$$

$$-\sum_{j=1}^{n} |v^j|\,|f^i_j| - |f^i_{yy}| - \sum_{k=1}^{n} |f^i_{yk}|\,|q^k| - \sum_{j=1}^{n} |p^j|\,|f^i_{jy}| - \sum_{j=1}^{n} |p^j| \sum_{k=1}^{n} |f^i_{jk}|\,|q_2^k|.$$

Therefore $v'(x) \geqslant C_0 v^2 - C_1|v| - C_2$ on a neighbourhood of $\bar{x}$, where

$$C_0 = \min[|f^1_{qq}|, \ldots, |f^n_{qq}|] \quad \text{on } Q,$$

$$C_1 = \max|f^i_{yq}| + \sum_{j=1}^{n} \max|q^j|\,|f^i_{jq}| + \max|f^i_{qy}| + \sum_{k=1}^{n} \max|f^i_{qk}|\,|q^k| + \sum_{j=1}^{n} \max|f^i_j| \quad \text{on } Q,\ i = 1, 2, \ldots, n,$$

$$C_2 = \max|f^i_{yy}| + \sum_{k=1}^{n} \max|f^i_{yk}|\,|q^k| + \sum_{j=1}^{n} \max|q^j|\,|f^i_{jy}| + \sum_{j=1}^{n} \max|q^j| \sum_{k=1}^{n} |f^i_{jk}|\,|q^k| \quad \text{on } Q,\ i = 1, 2, \ldots, n.$$

Obviously such constants C_0, C_1, C_2 exist because the functions f^i are of class C^2 on Q.

Now we shall show the inequality $D_+ y(l) \geqslant v'(\bar{x})$.

The definitions of y, v imply that $y(l) = |v(\bar{x})| > K$.

Consider the difference quotient

$$\frac{y(l+\Delta l) - y(l)}{\Delta l}, \quad \text{where } \Delta l > 0 \text{ and } l + \Delta l < a.$$

From the definition of P_l and the inequality $|y'| = |f_q^i| \leqslant C$, $|z'| = |f_q^i| \leqslant C$ it follows that $(\bar{x}+\Delta x, y(\bar{x}+\Delta x), z(\bar{x}+\Delta x)) \in P_{l+\Delta l}$, where $|\Delta x| = \Delta l$. If $y(l) = v(\bar{x}) > K$ and $\Delta x = \Delta l$, then we have the inequality

$$\frac{y(l+\Delta l)-y(l)}{\Delta l} \geqslant \frac{v(\bar{x}+\Delta x)-v(\bar{x})}{\Delta x}.$$

If $y(l) = -v(\bar{x})$, then for $\Delta l > 0$ and $\Delta l = -\Delta x$ we have

$$\frac{y(l+\Delta l)-y(l)}{\Delta l} \geqslant \frac{-v(\bar{x}+\Delta x)+v(\bar{x})}{\Delta x} = \frac{v(\bar{x}+\Delta x)-v(\bar{x})}{\Delta x}.$$

Therefore in both cases we get the inequality $D_+y(l) \geqslant v'(\bar{x})$. Thus, we have

$$D_+y(l) \geqslant v'(\bar{x}) \geqslant C_0(v(\bar{x}))^2 - C_1|v(\bar{x})| - C_2 = C_0(y(l))^2 - C_1 y(l) - C_2 > 0.$$

In the case where $f_{qq}^i < 0$ on Q we can get the inequality

$$v'(\bar{x}) \leqslant -C_0(v(\bar{x}))^2 + C_1|v(\bar{x})| + C_2, \qquad -v'(\bar{x}) \geqslant C_0(v(\bar{x}))^2 - C_1|v(\bar{x})| - C_2$$

and

$$D_+y(l) \geqslant -v'(\bar{x}) \geqslant C_0(v(\bar{x}))^2 - C_1|v(\bar{x})| - C_2 = C_0(y(l))^2 - C_1 y(l) - C_2 > 0.$$

In all cases we have

$$D_+y(l) \geqslant C_0(y(l))^2 - C_1 y(l) - C_2 > 0. \tag{6}$$

Since l is arbitrary $0 \leqslant l < a$, inequality (6) holds on the whole interval $[0, a)$, and in virtue of the Lemma we get $y(0) \leqslant K$.

Therefore

$$|u_y^k(x_0, z) - u_y^k(x_0, y)| \leqslant K(z-y+\delta), \qquad z > y,\ k = 1, 2, \ldots, n.$$

Passing to the limit we get

$$|u_y^k(x_0, z) - u_y^k(x_0, y)| \leqslant K|z-y|.$$

The proof of Theorem 2 is thus complete.

Theorem 1 is a corollary to Theorem 2.

We consider a neighbourhood with its closure contained in Ω. In virtue of the regularity of the respective functions there exist constants C_0, C_1, C_2, C_3. Using these constants we can define a rectangle P and a set Q contained in the neighbourhood. The assumptions of Theorem 2 being satisfied, we obtain Theorem 1.

References

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