ANNALES
POLONICI MATHEMATICI
XLVI (1985)

Non-extendable holomorphic functions of bounded growth in Reinhardt domains

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Abstract. Let $D \subset C^n$ be a Reinhardt domain of holomorphy with $0 \in D$. We prove that, for every $N > 0$, there exists a function f holomorphic on D and non-extendable beyond D for which the function $\varrho_D^N \cdot f$ is bounded, where ϱ_D denotes the euclidean distance to ∂D (Theorem 1). We also present some characterizations of those domains D for which there exist bounded non-extendable holomorphic functions (Theorem 2) and for which there exist non-zero square-integrable holomorphic functions (Theorem 3).

1. Introduction. It is known that any domain of holomorphy $D \subset C^n$ carries, for any $\varepsilon > 0$, a holomorphic function f non-extendable across ∂D which satisfies the following growth condition

$$(*) \qquad \sup_{z \in D}\{|f(z)| \cdot [\min\{1, \operatorname{dist}(z, \partial D)\} \cdot (1+|z|^2)^{-1/2}]^{n+\varepsilon}\} < \infty.$$

This result, contained in [4], can be strengthened if the boundary of D is assumed to satisfy a general cone condition [5]. But, on the other hand, there is the exciting example due to N. Sibony [7] which shows that, in general, the bounded holomorphic functions do not characterize a domain of holomorphy. Note that the bounded functions can be regarded as those functions for which the above growth condition $(*)$ holds with the exponent zero instead of $(\eta+\varepsilon)$.

The results in [6], [8] imply that any bounded Reinhardt domain of holomorphy is an $H^\infty(D)$-domain of holomorphy (for this notion compare [7]). The present paper gives a characterization of those Reinhardt-domains which are $H^\infty(D)$-domains of holomorphy. And, additionally, we will show that, for any Reinhardt domain D of holomorphy and any $\varepsilon > 0$, there exists an holomorphic function f on D with the following two properties:

(α) $\sup_{z \in D} |f(z)| \cdot [\operatorname{dist}(z, \partial D)]^\varepsilon < \infty$,

(β) f cannot be holomorphically extended across ∂D.

2. Formulation of the main results. Let D be a domain in C^n; then, for $z \in D$, $\varrho_D(z)$ is defined as the euclidean boundary distance of the point $z \in D$. Then, for $N \geqslant 0$, $O^{(N)}(D, \varrho_D)$ denotes the space of holomorphic functions f on D with $\sup_{z \in D} |f(z)| \cdot \varrho_D^N(z) < \infty$. Functions of $O^{(N)}(D, \varrho_D)$ are called *ϱ_D-tempered functions* of order $\leqslant N$. For a complete discussion of the properties of the space $O^{(N)}(D, \varrho_D)$ compare the book of J.-P. Ferrier [1] or the work of M. Jarnicki [3].

Now we can formulate our first result.

THEOREM 1. *Let $D \subsetneq C^n$, $n \geqslant 2$, be a Reinhardt domain of holomorphy with $0 \in D$*(¹). *Then, for every $N > 0$, D is an $O^{(N)}(D, \varrho_D)$-domain of holomorphy.*

Let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy, and let $X \subset R^n$ denote its logarithmic image, i.e.

$$X := \{x = (x_1, \ldots, x_n) \in R^n\colon\ (e^{x_1}, \ldots, e^{x_n}) \in D\}.$$

Let E denote a vector subspace of R^n such that

(i) there exists $x^0 \in R^n$ with $x^0 + E \subset \bar{X}$,

(ii) if F is a vector subspace of R^n with $\tilde{x} + F \subset \bar{X}$ ($\tilde{x} \in R^n$), then $\dim_R F \leqslant \dim_R E$.

Because of the convexity of X it turns out that such a subspace E is uniquely defined by D; henceforth, it is denoted by E_D.

Now we use E_D to distinguish specific Reinhardt domain of holomorphy, namely:

DEFINITION 1. A Reinhardt domain of holomorphy $D \subsetneq C^n$ is called to be *of rational type* iff the space E_D admits a basis consisting of vectors in Q^n.

According to this notion we obtain the next theorem.

THEOREM 2. *Let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy. Then D is an $H^\infty(D)$-domain of holomorphy, if and only if, D is of rational type.*

Since there are Reinhardt domains of holomorphy, which are not of rational type, there are examples of Reinhardt domains D such that every bounded holomorphic function can be extended beyond D but, for any $N > 0$, there exists a non-extendable function $f \in O^{(N)}(D, \varrho_D)$.

Asking for the existence of holomorphic functions which are square-integrable we have the following characterization.

THEOREM 3. *Let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy. Then $E_D \neq \{0\}$, if and only if, $L^2(D) \cap O(D) = \{0\}$.*

Remark. (1) Theorem 3 shows that a Reinhardt domain of holomorphy $D \subsetneq C^n$ with $L^2(D) \cap O(D) \neq \{0\}$ is always an $H^\infty(D)$-domain of holomorphy.

(2) Theorem 1 leads to the question for which pseudoconvex domain in

(¹) In the sequel we shall always assume that $n \geqslant 2$ and $0 \in D$.

C^n such a result is also true. It would be interesting to know the answer, at least, for Sibony's example, because this is, in contrast to the present situation, a bounded domain of holomorphy which is not an $H^\infty(D)$-domain of holomorphy.

3. Proof of the theorems. We begin with a general remark.

LEMMA 1. *Let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy and let $F \subset O(D)$ be a Banach-space of holomorphic functions endowed with a topology stronger than the topology of uniform convergence on compact subsets of D. Assume that*

(*) *for any $f \in F$, for any $\Theta_1, \ldots, \Theta_n \in R$, the function g given by $g(z) := f(e^{i\Theta_1} z_1, \ldots, e^{i\Theta_n} z_n)$, $z = (z_1, \ldots, z_n) \in D$ belongs to F.*

Then the following equivalent conditions (i) *and* (ii):

(i) *D is an F-domain of holomorphy;*

(ii) *there exists a function $f_0 \in F$ such that f_0 cannot be extended beyond D;*

are the consequence of:

(iii) *for any $t = (t_1, \ldots, t_n) \in \partial D \cap (R_{>0})^n$, for which exists at least one k, $1 \leqslant k \leqslant n$, with $(t_1, \ldots, Dt_k, \ldots, t_n) \subset D$, D denotes the unit disc in C, there are a number j, $1 \leqslant j \leqslant n$, with $(t_1, \ldots, Dt_j, \ldots, t_n) \subset D$ and a function $f \in F$ such that the function $g(\lambda) := f(t_1, \ldots, \lambda \cdot t_j, \ldots, t_n)$ has a singularity at ∂D.*

Because the proof of this lemma is standard it is left for the reader.

Remark 1. Let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy. Then, for every $N \geqslant 0$, the space $O^{(N)}(D, \varrho_D)$ satisfies assumption (*) of Lemma 1; in particular, $H^\infty(D) = O^{(0)}(D, \varrho_D)$ does.

As a particular case of Theorem 2 we receive the following proposition.

PROPOSITION 1. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (R_+)^n_* := \{r = (r_1, \ldots, r_n) \in R^n - \{0\}: r_j \geqslant 0\}$, $c > 0$ *and let* $D := \{z = (z_1, \ldots, z_n) \in C^n: |z_1|^{\alpha_1} \ldots |z_n| \alpha_n < c\}$. *Then D is an $H^\infty(D)$-domain of holomorphy, if and only if,* $\alpha \in R(Q_+)^n$ (²).

Proof. By a suitable change of the coordinates we reduce the problem to the case when:

$$\alpha_1, \ldots, \alpha_k \text{ are positive and } \alpha_{k+1} = \ldots = \alpha_n = 0,\ c = 1\ (1 \leqslant k \leqslant n).$$

Then $D = D_0 \times C^{n-k}$, where

$$D_0 := \{(z_1, \ldots, z_k) \in C^k: |z_1|^{\alpha_1} \ldots |z_k|^{\alpha_k} < 1\}.$$

Note that D is an $H^\infty(D)$-domain of holomorphy iff D_0 is an $H^\infty(D_0)$-domain of holomorphy.

Consequently, without loss of generality, we may assume that $0 < \alpha_j \leqslant 1$ for $1 \leqslant j < n$ and $\alpha_n = 1$.

(²) Equivalently: there exist $v \in (Z_+)^n_*$ and $c' > 0$ such that $D = \{z \in C^n | |z^v| < c'\}$.

Suppose now that $\alpha \notin \boldsymbol{R}\cdot(\boldsymbol{Q}_+^n)$, so at least one of the numbers $\alpha_1, \ldots, \alpha_{n-1}$ is irrational. We can assume that $\alpha_1, \ldots, \alpha_k \in \boldsymbol{Q}$ and $\alpha_{k+1}, \ldots, \alpha_{k-1} \notin \boldsymbol{Q}$ $(0 \leqslant k \leqslant n-2)$.

Let $f \in H^\infty(D)$ with $\|f\|_\infty = 1$ and let $f(z) = \sum_{\nu \in (\boldsymbol{Z}_+)^n} a_\nu z^\nu$ the power series expansion of f on D. Then, for every $t = (t_1, \ldots, t_n) \in (\partial D) \cap (\boldsymbol{R}_{>0})^n$, according to Cauchy's inequalities, we get:

$$|a_\nu| \leqslant t^{-\nu} \quad \text{for } \nu \in (\boldsymbol{Z}_+)^n; \quad \text{in particular:} \quad |a_\nu| \leqslant 1.$$

Note that $t_n = t_1^{-\alpha_1} \ldots t_{n-1}^{-\alpha_{n-1}}$, thus $|a_\nu| \leqslant t^{\alpha_1 \nu_n - \nu_1} \ldots t_{n-1}^{\alpha_{n-1}\nu_n - \nu_{n-1}}$. Since $\alpha_j \notin \boldsymbol{Q}$ for $k+1 \leqslant j \leqslant n-1$ we obtain $\alpha_j \nu_n - \nu_j \neq 0$ and therefore, letting $t_j \to 0$ (resp. $t_j \to +\infty$) we obtain $a_\nu = 0$ for any ν with $\nu_j > 0$, $k+1 \leqslant j \leqslant n-1$.

Hence we have received:

$$f(z) = \sum_{\alpha \in (\boldsymbol{Z}_+)^n} a_\nu z^\nu = \sum_{\mu \in (\boldsymbol{Z}_+)^k} a_{(\mu, 0, \ldots, 0)} z_1^{\mu_1} \ldots z_k^{\mu_k}.$$

Since $|a_\nu| \leqslant 1$ this series converges at least in $\boldsymbol{D}^k \times \boldsymbol{C}^{n-k}$, where $\boldsymbol{D}$ denotes the unit disc in $\boldsymbol{C}$ (³). Observing, that $D \cap (\boldsymbol{D}^k \times \boldsymbol{C}^{n-k})$ is connected, we have constructed an holomorphic extension of f from D to $D \cup (\boldsymbol{D}^k \times \boldsymbol{C}^{n-k})$. Hence D is not an $H^\infty(D)$-domain of holomorphy.

In order to prove the converse we assume $\alpha_j = p_j/q$ with p_j, $q \in \boldsymbol{N}$ $(1 \leqslant j \leqslant n-1)$. It suffices to prove (iii) of Lemma 1. We define

$$f_0(z) := \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} (z_1^{p_1} \ldots z_{n-1}^{p_{n-1}} z_n^q)^\mu \quad \text{for } z = (z_1, \ldots, z_n) \in D.$$

It is obvious that f_0 is well defined and bounded on D.

Fix $t = (t_1, \ldots, t_n) \in (\partial D) \cap (\boldsymbol{R}_{>0})^n$, then $t_n = t_1^{-p_1/q_1} \ldots t_{n-1}^{-p_{n-1}/q_{n-1}}$ and so it is clear that the function

$$g(\lambda) = f_0(t_1, \ldots, t_{n-1}, \lambda \cdot t_n) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \lambda^\mu$$

has a singularity on $\partial \boldsymbol{D}$. In view of Lemma 1 it is proved that D is an $H^\infty(D)$-domain of holomorphy. Thus the proof of Proposition 1 is finished.

Before proceeding we remind the following simple geometrical fact for logarithmically convex Reinhardt domains.

LEMMA 2. *Let $D \subsetneq \boldsymbol{C}^n$ be a Reinhardt domain of holomorphy. Then, for every $a \in (\partial D) \cap (\boldsymbol{C}_*)^n$ there exist $\alpha \in (\boldsymbol{R}_+)^n$ and $c > 0$ such that*

$$D \subset \{z \in \boldsymbol{C}^n\colon |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} < c\} =: G \quad \textit{and} \quad a \in \partial G.$$

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Fix $a \in (\partial D) \cap (\boldsymbol{C}_*)^n$ and let G as in Lemma 2.

(³) In the case $k = 0$, this means that $H^\infty(D) \approx \boldsymbol{C}$.

Note that $\varrho_D(z) \leqslant \varrho_G(z)$ for $z \in D$ which implies: $O^{(N)}(G, \varrho_G)|_D \subset O^{(N)}(D, \varrho_D)$.

Hence it is sufficient to construct a function $f \in O^{(N)}(G, \varrho_G)$ which cannot be holomorphically extended through a.

Thus, without loss of generality, we may assume that

$$D = \{z \in \boldsymbol{C}^n\colon |z_1|^{\alpha_1} \ldots |z_1|^{\alpha_n} < c\} \quad \text{with } \alpha \in (\boldsymbol{R}_+)^n_*, \ c > 0.$$

Since the space $O^{(N)}(D, \varrho_D)$ is invariant with respect to affine isomorphisms and since a domain $D_0 \times \boldsymbol{C}^k$ is an $O^{(N)}(D_0 \times \boldsymbol{C}^k, \varrho_{D_0 \times \boldsymbol{C}^k})$-domain of holomorphy iff D_0 is an $O^{(N)}(D_0, \varrho_{D_0})$-domain of holomorphy we can reduce the problem to the case $0 < \alpha_j \leqslant 1$ for $1 \leqslant j \leqslant n-1$, $\alpha_n = 1$ and $c = 1$.

Since for $\alpha \in (\boldsymbol{Q}_+)^n$ the result is a direct consequence of Proposition 1 and the fact that $H(D)^\infty \subset O^{(N)}(D, \varrho_D)$ we can assume that $\alpha_1, \ldots, \alpha_{\tilde{m}} \in \boldsymbol{Q}_{>0}$ and $\alpha_{\tilde{m}+1}, \ldots, \alpha_{n-1} \notin \boldsymbol{Q}$ $(0 \leqslant \tilde{m} < n-1)$.

Since $\alpha_j > 0$ then according to the Dirichlet pigeonhole principle [2] there are sequences of natural numbers $(P_{j\mu})_{\mu=1}^\infty$ $(1 \leqslant j \leqslant n-1)$ and $(q_\mu)_{\mu=1}^\infty$ and a integer m with $\tilde{m} \leqslant m \leqslant n-1$ such that:

(1) $$0 \leqslant \frac{p_{j,\mu}}{q_\mu} - \alpha_j \leqslant \frac{\alpha_j N}{(n-1) K q_\mu} \quad \text{for any } \mu \in \boldsymbol{N},\ 1 \leqslant j \leqslant m;$$

(2) $$0 < \alpha_j - \frac{p_{j,\mu}}{q_\mu} \leqslant \frac{\alpha_j \cdot N}{(n-1) K q_\mu} \quad \text{for any } \mu \in \boldsymbol{N},\ m+1 \leqslant j \leqslant n-1;$$

(3) $$q_\mu \to \infty \quad \text{as } \mu \to \infty.$$

The denominator K is an integer chosen so that $K \geqslant 2$ and

$$K \cdot \min_{\tilde{m} < \mu \leqslant n-1} (1 - \alpha_\mu) > \frac{n-1}{n}.$$

Fix $p \in \boldsymbol{N}$ such that

(4) $$p \geqslant N(1/\alpha_1 + \ldots + 1/\alpha_{n-1})$$

and let, for $|\Theta| < 1$,

(5) $$\frac{\Theta^p}{(1-\Theta)^N} = \sum_{\mu=1}^{\infty} c_\mu \Theta^\mu.$$

Note that $c_\mu > 0$ (for any $\mu \in \boldsymbol{N}$) and $\sqrt[\mu]{c_\mu} \to 1$.

We define, for $z = (z_1, \ldots, z_n) \in D$,

$$f(z) = \sum_{\mu=1}^{\infty} c_{q_\mu} z_1^{p_{1,\mu}} \ldots z_{n-1}^{p_{n-1,\mu}} z_n^{q_\mu}$$

and we observe, that for every $t = (t_1, \ldots, t_n) \in (\partial D) \cap (\boldsymbol{R}_{>0})^n$ and for every

$\lambda \in \boldsymbol{D}$ the series taken at the point $(t_1, \ldots, t_{n-1}, \lambda t_n)$ is equal $\sum_{\mu=1}^{\infty} b_\mu(t_1, \ldots, t_{n-1}) \lambda^{q_\mu}$ with

$$b_\mu(t_1, \ldots, t_{n-1}) := c_{q_\mu} t_1^{p_{1,\mu} - \alpha_1 q_\mu} \ldots t_{n-1}^{p_{n-1,\mu} - \alpha_{n-1} q_\mu}.$$

In view of (1), (2) and (5) one obtains $\sqrt[q_\mu]{b_\mu(t_1, \ldots, t_{n-1})} \to 1$; hence f is well-defined and satiesfies (iii) of Lemma 1.

It remains to show that there exists $c > 0$ such that, for every $(t_1, \ldots, t_n) \in (\partial D) \cap (\boldsymbol{R}_{>0})^n$ and every $0 \leqslant \Theta < 1$, the following inequality is true:

$$\varrho_D^N(t_1, \ldots, t_{n-1}, \Theta t_n) \cdot f(t_1, \ldots, t_{n-1}, \Theta t_n) \leqslant c.$$

Without loss of generality we may assume that there are integers k $(0 \leqslant k \leqslant m)$ and s $(0 \leqslant s \leqslant n-1-m)$ such that:

$$t_1, \ldots, t_k, t_{m+1}, \ldots, t_{m+s} \leqslant 1 < t_{k+1}, \ldots, t_m, t_{m+s+1}, \ldots, t_{n-1}.$$

Then we estimate the function f as

$$\begin{aligned} &f(t_1, \ldots, t_{n-1}, \Theta t_n) \\ &= \sum_{\mu=1}^{\infty} c_{q_\mu} t_1^{p_{1,\mu} - \alpha_1 q_\mu} \ldots t_m^{p_{m,\mu} - \alpha_m q_\mu} (1/t_{m+1})^{\alpha_{m+1} q_\mu - p_{m+1,\mu}} \ldots \\ &\qquad \ldots (1/t_{n-1})^{\alpha_{n-1} q_\mu - p_{n-1,\mu}} \Theta^{q_\mu} \\ &\leqslant t_{k+1}^{\alpha_{k+1} N/(n-1)K} \ldots t_m^{\alpha_m N/(n-1)K} \cdot (1/t_{m+1})^{\alpha_{m+1} N/(n-1)K} \ldots \\ &\qquad \ldots (1/t_{m+s})^{\alpha_{m+s} N/(n-1)K} \frac{\Theta^p}{(1-\Theta)^N} \\ &\leqslant [t_{k+1}^{\alpha_{k+1}} \ldots t_m^{\alpha_m} (1/t_{m+1})^{\alpha_{m+1}} \ldots (1/t_{m+s})^{\alpha_{m+s}}]^{N/(n-1)K} \cdot \frac{\Theta^p}{(1-\Theta)^N}. \end{aligned}$$

Observe that the points $(t_1, \ldots, t_n)$ and $(t_1, \ldots, t_{j-1}, t_j \Theta^{-1/\alpha_j}, t_{j+1}, \ldots, \Theta t_n)$ are boundary points of D. Thus we get

$$\begin{aligned} &\varrho_D^{k+s+1}(t_1, \ldots, t_{n-1}, \Theta t_n) \\ &\leqslant t_1(\Theta^{-1/\alpha_1} - 1) \ldots t_k(\Theta^{-1/\alpha_k} - 1) t_1^{-\alpha_1} \ldots \\ &\qquad \ldots t_{n-1}^{-\alpha_{n-1}} (1-\Theta) \cdot t_{m+1}(\Theta^{-1/\alpha_{m+1}} - 1) \ldots t_{m+s}(\Theta^{-1/\alpha_{m+s}} - 1) \\ &\leqslant \Big[\prod_{\nu=1}^{k} t_\nu(\Theta^{-1/\alpha_\nu} - 1)\Big] \cdot \Big[\prod_{\mu=1}^{n-1} t_\mu^{-\alpha_\mu}\Big] \cdot (1-\Theta) \cdot \Big[\prod_{\lambda=1}^{s} t_{m+\lambda}(\Theta^{-1/\alpha_{m+\lambda}} - 1)\Big] \\ &= (1-\Theta) \cdot \Big(\prod_{\nu=k+1}^{m} t^{-\alpha_\mu}\Big) \cdot \Big(\prod_{\mu=1}^{s} t_{m+\mu}^{1-\alpha_{m+\mu}}\Big) \cdot \prod_{\substack{\nu=1,\ldots,k \\ \nu=m+1,\ldots,m+s}} (\Theta^{-1/\alpha_\nu} - 1). \end{aligned}$$

Therefore we find:

$$f(t_1, \ldots, t_{n-1}, \Theta t_n)\varrho_D^N(t_1, \ldots, t_{n-1}, \Theta t_n)$$

$$\leqslant \frac{\Theta^p}{(1-\Theta)^N} \cdot [(1-\Theta)(\Theta^{-1/\alpha_1}-1)\ldots(\Theta^{-1/\alpha_K}-1)(\Theta^{-1/\alpha_{m+1}}-1)\ldots$$

$$\ldots(\Theta^{-1/\alpha_{m+s}}-1)]^{N/(k+s+1)} \times$$

$$\times \Big[\prod_{v=k+1}^{m} t_v^{\alpha_v(\frac{1}{(n-1)K}-\frac{1}{k+s+1})N}\Big] \cdot \Big[\prod_{\mu=1}^{s} t_{m+\mu}^{N(\frac{1}{k+s+1}(1-\alpha_{m+\mu})-\frac{1}{(n-1)K}\cdot\alpha_{m+\mu})}\Big].$$

Fix v between $k+1$ and m we observe, since $K \geqslant 2$, that the exponent of t_v is non-positive. Using the assumption on K we also find the exponents of $t_{m+\mu}$ $(1 \leqslant \mu \leqslant s)$ to be positive.

It remains to estimate

$$\frac{\Theta^p}{(1-\Theta)^N}[(1-\Theta)(\Theta^{-1/\alpha_1}-1)\ldots(\Theta^{-1/\alpha_k}-1)(\Theta^{-1/\alpha_{m+1}}-1)\ldots$$

$$\ldots(\Theta^{-1/\alpha_{m+s}}-1)]^{n/(k+s+1)}$$

$$\leqslant \left[\frac{(1-\Theta^{1/\alpha_1})}{1-\Theta}\cdots\frac{(1-\Theta^{1/\alpha_k})}{1-\Theta}\,\frac{(1-\Theta^{1/\alpha_{m+1}})}{1-\Theta}\cdots\frac{(1-\Theta^{1/\alpha_{m+s}})}{1-\Theta}\right]^{N/(k+s+1)} \quad (^4)$$

which is bounded for $0 \leqslant \Theta < 1$.

Hence we have shown that the function f belongs to the space $O^{(N)}(D, \varrho_D)$ which completes the proof of Theorem 1.

Now we turn to the proof of Theorem 2. Preparatory to this we state without proof the following simple geometrical fact.

Lemma 3. *Let E be a vector subspace of $\mathbf{R}^n$; let $E^\perp$ be its orthogonal complement. Then, for every open convex subset $X \subset \mathbf{R}^n$, the following conditions are equivalent*:

(i) *there is a point $x^0 \in \mathbf{R}^n$ with $x^0 + E \subset \bar{X}$;*

(ii) *there exists a convex set $Y \subset E^\perp$ open in the topology of $E^\perp$ such that $X = E + Y$;*

(iii) *for every $x \in X$ it holds that $x + E \subset X$;*

(iv) *for any $x \in \partial X$: $x + E \subset \partial X$.*

The claim of Lemma 3 then easily leads to the following conclusion.

Corollary 1. *Let $D \subsetneqq \mathbf{C}^n$ be a Reinhardt domain of holomorphy and let X be its logarithmic image. Then*:

(i) *if F is as in condition* (ii) *in 2, then $F \subset E_D$; in particular, E_D is uniquely defined;*

(^{4}) In the case $k = s = 0$ we get $\dfrac{\Theta^p}{(1-\Theta)^N}\cdot(1-\Theta)^N \leqslant 1$.

(ii) *for any* $x \in X$ *one has*: $x + E_D \subset X$;

(iii) *for any* $x \in \partial X$: $x + E_D \subset \partial D$.

EXAMPLE 1. Let $D = D_1 \cap \ldots \cap D_k$, where $D_j = \{z \in \mathbf{C}^n: |z_1|^{\alpha_1^j} \ldots |z_n|^{\alpha_n^j} < c_j\}$ with $\alpha^j \in (\mathbf{R}_+)^n_*$, $c_j > 0$, $1 \leqslant j \leqslant k$. We also assume that the intersection is minimal, i.e. $D \subsetneqq D_1 \cap \ldots \cap D_{j-1} \cap D_{j+1} \cap \ldots \cap D_k$ for any j, $1 \leqslant j \leqslant k$. Then it is easy to see that

$$E_D = \{x \in \mathbf{R}^n: \langle x, \alpha^j \rangle = 0 \text{ for } 1 \leqslant j \leqslant k\}.$$

In particular, $\dim E_D = n\text{-rank}\begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{bmatrix}$.

Modifying Definition 1 we put:

DEFINITION 2. We shall say that a vector subspace $E \subset \mathbf{R}^n$ is of *rational type* if E admits a basis consisting of vectors from $\mathbf{Q}^n$.

Remark 2. If D is a Reinhardt domain of holomorphy, we have: D is of rational type iff E_D is of rational type.

Then, from linear algebra, using Cramer's rule we can adopt the following result.

LEMMA 4. *A vector subspace $E \subset \mathbf{R}^n$ is of rational type iff its orthogonal complement is of rational type.*

Therefore, as a direct consequence, we obtain the following conclusion.

COROLLARY 2. *If $E = \{x \in \mathbf{R}^n: \langle x, \alpha \rangle = 0\}$, $\alpha \in (\mathbf{R}^n)_*$, then E is of rational type if and only if $\alpha \in \mathbf{R} \cdot (\mathbf{Q}^n)_*$.*

Remark 3. Let $D \subsetneqq \mathbf{C}^n$ be a Reinhardt domain of holomorphy. Then $\dim E_D = n-1$ iff $D = \{z \in \mathbf{C}^n: |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} < c\}$ with $\alpha \in (\mathbf{R}_+)^n_*$, $c > 0$ (cf. Lemma 3 and Example 1).

EXAMPLE 2. Let $D = D_1 \cap \ldots \cap D_k$ be as in Example 1. Then, we claim, that the condition "D is of rational type" may be numerically characterized.

Proof. We already know that

$$E_D = \{x \in \mathbf{R}^n: \langle x, \alpha^j \rangle = 0 \text{ for } 1 \leqslant j \leqslant k\}.$$

Let $k = \text{rank}\begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{bmatrix}$ and assume that

$$d := \det \begin{bmatrix} \alpha_1^1 & \ldots & \alpha_k^1 \\ \vdots & & \vdots \\ \alpha_1^k & \ldots & \alpha_k^k \end{bmatrix} \neq 0.$$

Then the vectors

$$v^p := \left(\frac{d_{1,p}}{d}, \ldots, \frac{d_{k,p}}{d}, 0, \ldots, 0, -1, 0, \ldots, 0\right) - k+1 \leqslant p \leqslant n,$$

span $E^{\perp}$, where

$$d_{j,p} := \det \begin{bmatrix} \alpha_1^1 & \dots & \alpha_{j-1}^1 & \alpha_p^1 & \alpha_{j+1}^1 & \dots & \alpha_k^1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^k & \dots & \alpha_{j-1}^k & \alpha_p^k & \alpha_{j+1}^k & \dots & \alpha_k^k \end{bmatrix}.$$

In view of Lemma 4, E is of rational type iff there exists a non-singular $(n-k)\times(n-k)$-dimensional matrix A such that all entries of the matrix

$$A\begin{bmatrix} v^{k+1} \\ \vdots \\ v^n \end{bmatrix} = \left[A \cdot \begin{bmatrix} v_1^{k+1} & \dots & v_k^{k+1} \\ \vdots & & \vdots \\ v_1^n & \dots & v_k^n \end{bmatrix} \; \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \; -A \right]$$

are rational numbers. In particular, A has to be rational therefore the matrix

$$\begin{bmatrix} v_1^{k+1} & \dots & v_k^{k+1} \\ \vdots & & \vdots \\ v_1^n & \dots & v_k^n \end{bmatrix} \quad \text{has to be rational.}$$

Finally, we get the following condition: D is of rational type iff $\frac{d_{j,p}}{d} \in \mathbf{Q}$ for any $j = 1, \dots, k$; $p = k+1, \dots, n$.

To show that any Reinhardt domain of holomorphy, which is an $H^{\infty}(D)$-domain of holomorphy is of rational type the following lemma is essential.

Lemma 5. *Let E be a k-dimensional subspace of $\mathbf{R}^n$ $(1 \leqslant k \leqslant n-1)$, which is not of rational type. Then the subspace $F := (\mathbf{Q}^n \cap E^{\perp})^{\perp}$ has a dimension which is larger than k.*

Proof. It suffices to show that $\dim H \leqslant n-k-1$, where $H := \mathbf{Q}^n \cap E^{\perp}$. But this is clear, since $E^{\perp}$ is not of rational type.

Now we are able to present the first part of the proof of Theorem 2:

Let $D \subsetneq \mathbf{C}^n$ be a Reinhardt domain of holomorphy which is an $H^{\infty}(D)$-domain of holomorphy and assume that D is not rational type. Let $X = \log D$ and $k := \dim E_D$. Because the case $k = n-1$ was considered in Remark 3 we may assume $0 \leqslant k \leqslant n-2$.

Let $u^1, \dots, u^k$ be a basis of E_D and fix $f \in H^{\infty}(D)$, $\|f\|_{\infty} = 1$ such that f cannot holomorphically continued beyond D. Let $f(z) = \sum_{\nu \in (\mathbf{Z}_+)^n} a_\nu z^\nu$ its power series expansion in D. Fixing $x^0 \in X$ the Cauchy inequalities deliver for $\nu \in (\mathbf{Z}_+)^n$, $\tau_j \in \mathbf{R}$, since $x^0 + E_D \subset X$,

$$|a_\nu| \leqslant \exp[-\langle \nu, x^0 \rangle - \tau_1 \langle \nu, u^1 \rangle - \dots - \tau_k \langle \nu, u^k \rangle].$$

Define $M := \{\nu \in (\mathbf{Z}_+)^n : \langle \nu, u^j \rangle = 0 \text{ for } 1 \leqslant j \leqslant k\}$ we get

$$f(z) = \sum_{\nu \in M} a_\nu z^\nu.$$

Since f cannot be continued, so the domain of convergence of the last series is equal D.

On the other hand, let F be as in Lemma 5. Then it is obvious that, for $v \in F$, the series $\sum_{v \in M} a_v (e^{x^0+v})^v = \sum_{v \in M} a_v e^{\langle x^0, v\rangle}$ is convergent, so $x^0 + F \subset X$ and therefore $\dim E_D \geqslant k+1$. This is a contradiction. Hence it is shown, that any Reinhardt domain of holomorphy which is also an $H^\infty(D)$-domain of holomorphy is of rational type.

In order to prove the inverse direction of the claim of Theorem 2 we need the following geometrical fact.

Lemma 6. *Let C be an open convex cone in $\boldsymbol{R}^n$ with vertex at x^0. Assume that C does not contain any affine line. Then there exists a non-empty open set $U \subset \boldsymbol{R}^n$ such that, for any $\beta \in U$, C is contained in $\{x \in \boldsymbol{R}^n\colon \langle x - x_0, \beta\rangle \neq 0\}$.*

For a proof compare the discussion of the dual cone in [9].

Now we are going to apply Lemma 6 to complete the proof of Theorem 2. Again, let $D \subsetneq C^n$ be a Reinhardt domain of holomorphy which is of rational type, and we can assume that $\dim E_D = k$ with $0 \leqslant k \leqslant n-2$. In view of Proposition 1 it suffices to prove: for any $x^0 \in \partial X$ and any $\varepsilon > 0$ there exists $x^1 \in \boldsymbol{R}^n$ and an $(n-1)$-dimensional subspace $P \subset \boldsymbol{R}^n$ of rational type such that:

(i) $X \cap (x^1 + P) = \emptyset$;

(ii) $\operatorname{dist}(x^0, x^1 + P) < \varepsilon$.

Because of Lemma 3, $X = E + Y$, where Y is an open convex subset in $E^\perp$ and Y does not contain any affine line. Note that in the case $k = 0$ one has $E^\perp = \boldsymbol{R}^n$, $Y = X$. Fix $x^0 \in \partial X$ and let $x^1 \in E^\perp \setminus \bar{Y}$ be such that $\|x^1 - \mathrm{pr}_{E^\perp}(x^0)\| < \varepsilon$. In the space $E^\perp$ we denote by C the open cone generated by $Y \cup \{x^1\}$. Then, by Lemma 6, there exists an open subset U in $E^\perp$, $U \neq \emptyset$, such that for any $\beta \in U$:

$$Y \subset \{x \in E^\perp\colon \langle x - x^1, \beta\rangle \neq 0\}.$$

This means, since $E^\perp$ is rational, there is an $(n-k-1)$-dimensional subspace $V \subset E^\perp$ of rational type such that $(x^1 + V) \cap Y = \emptyset$. Then $P := E + V$ satisfies conditions (i) and (ii).

Hence Theorem 2 is completely verified.

In the last step we turn to prove Theorem 3.

Proof of Theorem 3. Let $X = \log D$ and fix $f \in L^2(D) \cap O(D)$ with its series expansion $f(z) = \sum_{v \in (\boldsymbol{Z}_+)^n} a_v z^v$.

Assume $E_D \neq \{0\}$ and note that

(a) $\sum_{v \in (\boldsymbol{Z}_+)^n} |a_v|^2 \int_D |z^v|^2 \, d\lambda(z) \leqslant \int_D |f|^2 \, d\lambda(z) < \infty$,

(b) $\int_D |z^v|^2 \, d\lambda(z) = (2\pi)^n \int_X e^{\langle x, 2(v+\mathbf{1})\rangle} \, dx$,

where $\mathbf{1} = (1, \ldots, 1)$.

In view of Lemma 3 we may assume $X = \boldsymbol{R} \times Y$, where $Y \subset \boldsymbol{R}^{n-1}$. Then, for $\alpha \in \boldsymbol{R}^n$,

$$\int_X e^{\langle x,\alpha\rangle} dx = \Big(\int_{-\infty}^{+\infty} e^{x_1\alpha_1} dx_1\Big) \int_Y e^{x_2\alpha_2 + \ldots + x_n\alpha_n} dx_2 \ldots dx_n = +\infty.$$

This implies that, necessarily, $a_\nu = 0$ for all $\nu \in (\boldsymbol{Z}_+)^n$ which shows that $f \equiv 0$.

To argue into the converse direction assume $E_D = \{0\}$ and fix $x^0 \leftarrow \bar{X}$. In view of the proof of the second part of Theorem 2 one can construct $\alpha^1, \ldots, \alpha^n \in (\boldsymbol{Z}_+)^n_*$ such that

$$X \subset \{x \in \boldsymbol{R}^n \colon \langle x - x^0, \alpha^j\rangle < 0 \text{ for } 1 \leqslant j \leqslant n\} =: X_0$$

with rank $\begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} = n.$

Define $\nu_j = \alpha_j^1 + \ldots + \alpha_j^n - 1$ for $1 \leqslant j \leqslant n$ and put $\nu = (\nu_1, \ldots, \nu_n)$.

We shall show that

$$\int_{X_0} e^{\langle x, 2(\nu+1)\rangle} dx < \infty$$

which will give a non-zero L^2-holomorphic function.

Let

$$A := \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \quad \text{and} \quad \Delta := \{\xi \in \boldsymbol{R}^n \colon \xi_j < \xi_j^0 := \langle x^0, \alpha^j\rangle\};$$

then it turn out that

$$\int_{X_0} e^{\langle x, 2(\nu+1)\rangle} dx = \frac{1}{|\det A|} \int_\Delta e^{2(\xi_1 + \ldots + \xi_n)} d\xi = \frac{e^{2(\xi_1^0 + \ldots + \xi_n^0)}}{2^n |\det A|} < \infty.$$

Thus the proof of Theorem 3 is completed.

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Reçu par la Rédaction le 27.02.1984