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# Extremal problems in some classes of univalent functions in a half-plane

by JADWIGA PIOTROWSKA (Łódź)

**Abstract.** The present paper aims at investigating variability regions of some functionals defined on the class of univalent functions in the upper half-plane. The method used is identical with the one employed by I. A. Aleksandrov and W. W. Sobolev [1] and by P. P. Kufariov, W. W. Sobolev, Ł. W. Sporysheva [5] for examining the variability regions on the class mentioned above.

**1. Basic notions.** Denote by $H^1$ the class of all functions which are holomorphic and univalent in the upper half-plane $P_z^+ = \{z\colon \operatorname{Im} z > 0\}$ and which map this half-plane on regions contained in the half-plane $P_w^+ = \{w\colon \operatorname{Im} w > 0\}$.

Moreover, those functions are normalized by the condition

$$\lim_{z\to\infty} [f(z)-z] = 0, \quad z \in P_z^+. \tag{1.1}$$

It follows from the above definitions that the class is non-empty, since the identity function certainly belongs to it.

Assume now that we are given $m$ functions

$$I_s(f) = I(f, \bar{f}, f', \ldots, f^{(n)}, \bar{f}^{(n)}), \quad s = 1, \ldots, m,$$

analytical with respect to their arguments in a sufficiently large region.

For an arbitrary $z_0$, $z_0 \in P_z^+$, consider functionals

$$I_s(f) = I_s\big(f(z_0), \bar{f}(z_0), \ldots, f^{(n)}(z_0)\big), \quad s = 1, \ldots, m,$$

where $f$ is an arbitrary function of the class $H^1$.

Let us consider the system of functionals

$$I(f) = \big(I_1(f), \ldots, I_p(f)\big), \quad p \leqslant m,\ p - \text{a natural number}, \tag{1.2}$$

the functionals $I_p(f)$ being defined above.

The system of functionals (1.2) is called *continuous on the class $H^1$* if the almost uniform convergence of the sequence $\{f_n\}$ in $P_z^+$, $f_n \in H^1$, to the function $f \in H^1$ results in the convergence of the sequence $\{I(f_n)\}$ to $I(f)$.

Let $f_1$ and $f_2$ be two arbitrary functions of some class $K$ of functions defined on the region $B$.

The class $W \subset K$ of functions $f(z, \varrho)$, where $\varrho \in \langle \alpha, \beta \rangle$, $z \in B$, being almost uniformly continuous on $B$ with respect to $\varrho$ and almost uniformly convergent on $B$ to $f_1$ and $f_2$ as $\varrho \to \alpha$ and $\varrho \to \beta$, respectively, will be called the *class of functions connecting* $f_1$ *and* $f_2$.

The class $K$ is said to be *connected in a region* $B$ if for arbitrary functions $f_1, f_2 \in K$ there exists a subclass $W \subset K$ connecting those functions.

LEMMA 1.1 *Class* $H^1$ *is a connected class in* $P_z^+$.

Proof. Notice first that, without losing generality, we can take an arbitrary function $f \in H^1$ and an identity function as the two arbitrary functions of class $H^1$. Notice as well that for an arbitrary $\varrho \in \langle 0, 1 \rangle$ if $f$ belongs to class $H^1$, then the function defined by the formula $f(z, \varrho) = \varrho f(z/\varrho)$ also belongs to that class. We shall show that the class $W = \{f(z, \varrho) = \varrho f(z/\varrho) \colon f \in H^1, \ \varrho \in \langle 0, 1 \rangle\}$ is the connecting class. Indeed, functions of that class are almost uniformly continuous with respect to $\varrho \in \langle 0, 1 \rangle$ in $P_z^+$. From the normalizing condition for functions of class $H^1$ we have

$$\lim[\varrho f(z/\varrho) - z] = 0 \quad \text{as } \varrho \to 0$$

and

$$\lim[\varrho f(z/\varrho) - f(z)] = 0 \quad \text{as } \varrho \to 1.$$

Therefore class $H^1$ is connected in $P_z^+$.

A class $K$ is called a *compact class* if every infinite sequence of functions of that class contains a subsequence convergent to some function of that class.

LEMMA 1.2. *Class* $H^1$ *is not compact.*

Proof. To show this it suffices to find a subset of functions of class $H^1$ tending almost uniformly to infinity in $P_z^+$. Consider a family of functions of the form $w = f(z) = a + \sqrt{(z-a)^2 - h^2}$, where the parameters $a$ and $h$ are real, and take that branch of the root for which $f(a) = a + i|h|$. Those functions map $P_z^+$ into $P_w^+$ minus a segment of the length $h$ beginning at the point $a$ and perpendicular to the real axis.

The remark that $f \equiv \infty \notin H^1$ ends the proof.

The set of all points $I(f)$, where $f$ ranges over the class $H^1$ will be called the *set of the values of system* (1.2) and denoted by $D$ while its boundary will be denoted by $L$.

The function $f_0 \in H^1$ will be called a *boundary function with respect to system* (1.2) if $I(f_0) \in L$.

The aim of the present paper is to investigate the general properties of the set $D$ and to determine it effectively in some special cases.

LEMMA 1.3. *The set of values of system* (1.2) *of continuous real-valued functionals defined on a connected class of holomorphic functions is a compact set*, [3].

It is known [3] that this set is closed if the class on which system (1.2) has been defined is compact and may turn out to be a non-closed set if the class is non-compact.

COROLLARY 1.1. *The set $D$ of values of system* (1.2) *does not have to be a closed set.*

LEMMA 1.4. *The set $D$ of values of system* (1.2) *of continuous functionals is a connected set* (cf. [3]).

To determine the boundary of a closed set it suffices to find the set of all boundary points which are non-singular (in the sense of N. A. Liebiediev [6]) and to close that set. In the case of non-compact classes the knowledge of non-sigular points does not always suffice to find the boundary.

The boundary points of the set $D$ of values has the following simple property which will be useful in our further considerations. Assume that the set $D$ possesses exterior points and let $\dot{I}$ be one of them. Then there exists a point $I_0$, $I_0 \in L$, such that

$$|I_0 - \dot{I}| \leqslant |I - \dot{I}|, \tag{1.3}$$

for all $I \in U \cap D$, where $U$ is a sufficiently small neighbourhood of the point $I_0$. The set $L'$ of points $I_0$ is dense in $L$.

Closing the former, we obtain the latter, [4].

It follows from what has been said that, in the case of a compact class, in order to characterize the set of values of a set of functionals defined on that class it suffices to find the boundary of that set. When the class is non-compact, it is sometimes possible to find the region $D$ or the region $D'$ majorizing it, i.e., the smallest region containing $D$.

The following theorems used in the sequel can be found in [5].

THEOREM 1.1. *Let $f$ be an arbitrary function of class $H^1$. Let $q$ be a function such that for every $t$, $0 \leqslant t \leqslant T$, the function $f(z) + tq(z)$ is holomorphic and univalent in the set*

$$\{z\colon\ 0 < \operatorname{Im} z \leqslant h,\ h > 0\}, \tag{1.4}$$

*and maps the set* (1.4) *on a set bounded by the continua $D_0(t)$ and $D_h(t)$ corresponding to the real axis* $\operatorname{Im} z = 0$ *and the line* $\operatorname{Im} z = h$, *respectively.*

We assume, moreover, that for a sufficiently great $z$ with respect to its absolute value, in the set (1.4) we have the inequality

$$|q(z)| \leqslant \frac{c}{|z|^{a}},$$

where $c$ is a positive constant and $a$ is a positive real parameter. Then in the half-plane $P_z^+$ the function $w = \Phi(z, t)$, $z \in P_z^+$, $0 \leqslant t \leqslant T$, satisfying

$$\lim[\Phi(z, t) - z] = 0 \quad \text{as } z \to \infty,\ z \in P_z^+,$$

and mapping the half-plane $P_z^+$ on a region $B(t)$ for which $D_0(t)$ is a boundary, has the form

$$\Phi(z, t) = f(z) + tf'(z)P(z) + o(t), \tag{1.5}$$

where

$$P(z) = \lim \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Im}\left[\frac{q'(\zeta)}{f'(\zeta)}\right] \frac{dx}{\zeta - z}, \quad \zeta = x + i\beta \text{ as } \beta \to \infty. \tag{1.6}$$

In particular, let us put, analogously to what was done in [5],

$$g(z) = \sum_{k=1}^{m} \left\{\frac{A_k}{f(z) - w_k} + \frac{\bar{A}_k}{f(z) - \bar{w}_k}\right\}, \tag{1.7}$$

where $f$ is an arbitrary fixed function of class $H^1$, $w_k$, $k = 1, \ldots, m$, are arbitrary points belonging to $P_w^+$ and $A_k$, $k = 1, \ldots, m$, are arbitrary constant complex numbers.

It turns out that the function $f(z) + tq(z)$, where $q(z)$ is defined by (1.7), satisfies, for a sufficiently small $t$, all the assumptions of Theorem 1.1 and the image of the real axis $\operatorname{Im} z = 0$ under $w = f(z) + tq(z)$ lies in $P_w^+$.

THEOREM 1.2. a) *If the points $z_k$, $k = 1, \ldots, m$, $z_k \in P_z^+$, are such that $f(z_k) = w_k$, then, together with $f$, a function of the form*

$$f_*(z) = f(z) + t \sum_{k=1}^{m} \left\{\frac{A_k}{f(z) - f(z_k)} + \frac{\bar{A}_k}{f(z) - \bar{f}(z_k)} + \right. \\ \left. + \frac{A_k}{[f'(z_k)]^2} \frac{f'(z)}{z_k - z} + \frac{\bar{A}_k}{[\bar{f}'(z_k)]^2} \frac{f'(z)}{\bar{z}_k - z}\right\} + o(t) \tag{1.8}$$

*belongs to class $H^1$ for sufficiently small $t$.*

b) *If the points of the half-plane $P_z^+$ $w_1, \ldots, w_m$ are exterior for the region which is the image of $P_z^+$ under $f$, then, together with $f$, a function of the form*

$$f_*(z) = f(z) + t \sum_{k=1}^{m} \left\{\frac{A_k}{f(z) - w_k} + \frac{\bar{A}_k}{f(z) - \bar{w}_k}\right\} \tag{1.9}$$

*belongs to class $H^1$ for sufficiently small $t$,* [3].

**2. Set of values of the system $I(f) = (I_1, I_2, I_3)$.** In this chapter we shall find the set of values of the functional system

$$I(f) = (I_1, I_2, I_3), \tag{2.1}$$

where

$$I_1 = \log\left|\frac{f'(z)}{f(z)}\right|, \quad I_2 = \arg\frac{f'(z)}{f(z)}, \quad I_3 = \mathrm{Re}\{e^{i\theta}f'(z)\}, \tag{2.2}$$

$z$ being a fixed point of the half-plane $P_z^+$, $\theta$ being an arbitrary real number and functions $f$ being those functions of the class $H^1$ for which

$$\mathrm{Re}\{e^{i\theta}f(z)\} = K, \quad K \text{ — fixed}. \tag{2.3}$$

LEMMA 2.1. *Every function $f(\mathfrak{z})$ which is boundary with respect to the functional system* (2.1) *satisfies the following differential-functional equation:*

$$M(f)f'^2(\mathfrak{z}) = L(\mathfrak{z}), \tag{2.4}$$

*where*

$$M(f) = \frac{p_2}{[f(\mathfrak{z})-f(z)]^2} + \frac{\overline{p}_2}{[f(\mathfrak{z})-\overline{f}(z)]^2} + \frac{p_1}{f(\mathfrak{z})-f(z)} + \frac{\overline{p}_1}{f(\mathfrak{z})-\overline{f}(z)}, \tag{2.5}$$

$$L(\mathfrak{z}) = \frac{p_2}{(\mathfrak{z}-z)^2} + \frac{\overline{p}_2}{(\mathfrak{z}-\overline{z})^2} + \frac{q_1}{(\mathfrak{z}-z)} + \frac{\overline{q}_1}{(\mathfrak{z}-\overline{z})}, \tag{2.6}$$

*and*

$$\begin{aligned} p_0 &= -a_1 + ia_2, \\ p_1 &= p_0\frac{1}{f(z)}, \\ p_2 &= -p_0 + a_3e^{i\theta}f'(z), \\ q_1 &= p_0\frac{f'(z)}{f(z)} - p_0\frac{f''(z)}{f'(z)} + a_3f''(z)e^{i\theta}, \end{aligned} \tag{2.7}$$

$a_i$, $i = 1, 2, 3$, *are directional cosines of the vector $\overrightarrow{I_0I_e}$, $I_0$ is a regular boundary point of the set of values of the functional system* (2.1), *and $I_e$ is an exterior point of that set.*

The proof of the lemma is analogous to that given in [5], § 7.

It can easily be shown that the function $F(w, \tau)$ corresponding (cf. [5], § 2) to a boundary function with respect to the functional system (2.1) satisfies the following differential-functional equation:

$$B(F, \tau)F_w'^2(w, \tau) = A(w), \tag{2.8}$$

where

$$B(F,\tau) = \frac{q_1}{(F-a)^2} + \frac{\bar{q}_1}{(F-\bar{a})^2} + \frac{Q_1}{F-a} + \frac{\bar{Q}_1}{F-\bar{a}}, \tag{2.9}$$

$$A(w) = \frac{p_2}{(w-\xi)^2} + \frac{\bar{p}_2}{(w-\bar{\xi})^2} + \frac{p_1}{(w-\xi)} + \frac{\bar{p}_1}{(w-\bar{\xi})}, \tag{2.10}$$

and

$$a(\tau) = F\big(f(z),\tau\big), \quad f(z) = \xi,$$

$$Q_1 = p_0 \frac{[F'_w(f(z),\tau)]^{-1}}{f(z)} - p_0 \frac{d[F'_w(f(z),\tau)]^{-1}}{dw} + \\ + a_3 e^{i\theta} f'(z) \frac{d[F'_w(f(z),\tau)]^{-1}}{dw}, \tag{2.11}$$

$p_0, p_1, p_2, q_1$ — defined in (2.7).

From the normalizing condition for the class $H^1$ it follows that $\lim \frac{F(w,\tau)}{w} = 1$ as $w \to \infty$ and $\lim F'_w(w,\tau) = 1$ as $w \to \infty$. This allows us to state that polynomials in the denominators of the rational functions $B(F,\tau)$ and $A(w)$ are of the third degree and, moreover, the coefficients at the greatest powers are equal.

Now, our way of reasoning being analogous to that in [5], § 10, we get

$$A(w) = A_0 \frac{(w-w_0)^2(w-w_1)}{(w-\xi)^2(w-\bar{\xi})^2}, \tag{2.12}$$

$$B(F,\tau) = B_0 \frac{(F-\mu)^2(F-\nu)}{(F-a)^2(F-\bar{a})^2}, \tag{2.13}$$

and $A_0 = B_0 = \text{const}$.

Moreover, the functions $\mu(\tau)$, $\nu(\tau)$, $a(\tau)$ satisfy

$$2\mu - 2(a+\bar{a}) + \nu = \text{const}, \tag{2.14}$$

$$\frac{da}{d\tau} = \frac{1}{\mu - a}, \tag{2.15}$$

$$\frac{d\nu}{d\tau} = \frac{1}{\mu - \nu}. \tag{2.16}$$

Notice first that $B_0$ is a real number. Indeed, $B_0 = Q_1 + \bar{Q}_1 = 2\,\mathrm{Re}\,Q_1$.

Let $w$ from equation (2.8) tend to infinity. Then

$$\frac{(a-\mu)^2(a-\nu)}{(a-\bar{a})^2} = \frac{q_1}{B_0} = \text{const}. \tag{2.17}$$

Comparing in (2.17) the main arguments of the two sides, we obtain

$$2\operatorname{Arg}(a-\mu)+\operatorname{Arg}(a-\nu) = \theta_0 = \text{const}, \tag{2.18}$$

where

$$\theta_0 = \operatorname{Arg} q_1 + \pi k. \tag{2.19}$$

Depending on whether the coefficient $B_0$ is positive or negative, the number $k$ takes the values 0 or 1, respectively.

Let us introduce the following notations:

$$a(\tau) = u(\tau)+iv(\tau), \tag{2.20}$$

$$a-\mu = |a-\mu|e^{i\arg(a-\mu)} = r_1 e^{it_1}, \tag{2.21}$$

$$a-\nu = |a-\nu|e^{i\arg(a-\nu)} = r_2 e^{it_2}. \tag{2.22}$$

With the use of the above notations, relation (2.18) may be rewritten in the form

$$t_2 = \theta_0 - 2t_1. \tag{2.23}$$

We shall now show that the function $\nu(\tau)$ is a real function of the variable $\tau$. Indeed, since

$$2\mu - 2(a+\bar{a}) = -\nu + \text{const},$$

we infer that $\operatorname{Im}\nu = \text{const}$. On the other hand, we find that

$$\frac{d\operatorname{Im}\nu}{d\tau} = \frac{\operatorname{Im}\nu}{|\mu-\nu|^2} = 0$$

and hence $\nu(\tau) = \bar{\nu}(\tau)$.

Moreover, from the definition of the function $F(w,\tau)$ it follows that the function $v(\tau)$ is positive and hence the following constraints must be imposed on the numbers $t_1$ and $t_2$:

$$0 < t_1 < \pi, \quad 0 < t_2 < \pi.$$

From these inequalities and from the fact that the functions $\mu(\tau)$ and $\nu(\tau)$ are real we find

$$r_1 = \frac{v(\tau)}{\sin t_1}, \quad r_2 = \frac{v(\tau)}{\sin(\theta_0 - 2t_1)}, \tag{2.24}$$

where $a(\tau) = u(\tau)+iv(\tau)$.

Consider again condition (2.17) and use relations (2.24). We have

$$\left|\frac{q_1}{B_0}\right| = \frac{r_1^2 r_2}{4v^2} = \text{const}.$$

Write

$$\varrho = \left|\frac{q_1}{B_0}\right| = \text{const}.$$

On the other hand, we have

$$\frac{q_1}{B_0} = \frac{r_1^2 e^{2it_1} r_2 e^{it_2}}{-4v^2} = \frac{v^2 e^{2it_1} v e^{it_2}}{-4v^2 \sin^2 t_1 \sin(\theta_0 - 2t_1)}$$
$$= \frac{v e^{i\theta_0}}{-4\sin^2 t_1 \sin(\theta_0 - 2t_1)} = \text{const}$$

and thus

$$\varrho = \left|\frac{q_1}{B_0}\right| = \frac{v}{\sin^2 t_1 \sin(\theta_0 - 2t_1)}.$$

Hence we obtain

(2.25) $$v = \varrho \sin^2 t_1 \sin(\theta_0 - 2t_1).$$

We shall now write the derivative with respect to $\tau$ of the function $t_1(\tau)$ as the function of $t_1$.

From (2.15) we have

(2.26) $$\frac{dv}{d\tau} = \frac{-v}{(\mu - a)^2},$$

and from (2.25)

$$\frac{dv}{dt_1} = 2\varrho \sin t_1 \cos t_1 \sin(\theta_0 - 2t_1) - 2\varrho \sin^2 t_1 \cos(\theta_0 - 2t_1).$$

Thus we obtain

(2.26′) $$\frac{dv}{dt_1} = 2\varrho \sin t_1 \sin(\theta_0 - 3t_1).$$

Note that

$$\frac{dt_1}{d\tau} = \frac{dv}{d\tau} : \frac{dv}{dt_1} = \frac{v}{|\mu - a|^2}[2\varrho \sin t_1 \sin(\theta_0 - 3t_1)]^{-1}.$$

Hence, after some calculations, we have

(2.27) $$\frac{dt_1}{d\tau} = [2\varrho^2 \sin t_1 \sin(\theta_0 - 2t_1)\sin(\theta_0 - 3t_1)]^{-1}.$$

Since $\sin t_1 > 0$ and $\sin(\theta_0 - 2t_1) > 0$, the sign of the derivative depends on the sign of $\sin(\theta_0 - 3t_1)$, which in turn is the same as the sign of the number $\nu - \mu$. Thus the sign is constant for every value of the parameter $\tau \in (0, \tau_0)$.

Indeed, the inverse assumption would result in $\nu(\tau_1) = \mu(\tau_1)$ for a certain argument $\tau_1$, which is obviously impossible.

Hence we can state that the function $t_1(\tau)$ is monotonous, which results in the possibility of introducing an inverse function of the variable $t_1$ and $t_1 \in \langle t_1(0), t_1(\tau_0)\rangle$. Thus $\tau = \tau(t_1)$.

From Löwner's equation for a half-plane (cf. [5], § 2, equation (2.13)) the following equation can easily be obtained:

$$\frac{F''_{w\tau}}{F'_\tau} = \frac{1}{[\mu(\tau)-F(w,\tau)]^2}.$$

Let us put $f(z) = \xi$ in this equation. Then $F(\xi,\tau) = a(\tau)$, and, by (2.21), we get

$$d\log F'_w(\xi,\tau) = \frac{d\tau}{[\mu(\tau)-F(w,\tau)]^2} = e^{-2it_1}\frac{d\tau}{|\mu(\tau)-a(\tau)|^2}.$$

Further, using (2.26) we have

$$d\log F'_w(\xi,\tau) = e^{-2it_1}\frac{dv}{v},$$

which can be rewritten in an equivalent form as:

$$d\log F'_w(\xi,\tau) = e^{-2it_1}d\log v. \tag{2.28}$$

Integrating both sides of equation (2.28) in the interval $\langle t_1(0), t_1(\tau_0)\rangle \equiv \langle t'_1, t''_1\rangle$, we get

$$\log f'(z) = \int_{t'_1}^{t''_1} e^{-2it_1}d\log v(t_1). \tag{2.29}$$

Notice that

$$\int_{t'_1}^{t''_1} e^{-2it_1}d\log v(t_1) = \int_{t'_1}^{t''_1} e^{-2it_1}d\log\big(\varrho\sin^2 t_1\sin(\theta_0-2t_1)\big).$$

We shall write

$$I = \int e^{-2it_1}d\log\big(\varrho\sin^2 t_1\sin(\theta_0-2t_1)\big).$$

Then

$$\begin{aligned} I &= 2\big((\cos 2t_1 - i\sin 2t_1)\big(\cot t_1 - \cot(\theta_0-2t_1)\big)\big)dt_1 \\ &= 2\log\sin t_1 - 2\sin^2 t_1 - 2it_1 - 2i\sin t_1\cos t_1 + \\ &\qquad +2\frac{\cos\theta_0}{4}\log\left[-\cot^2\left(t_1-\frac{\theta_0}{2}\right)\right] + \cos 2t_1 + \\ &\qquad\qquad +i\frac{\sin\theta_0}{2}\log\left[-\cot^2\left(t_1-\frac{\theta_0}{2}\right)\right] - \sin 2t_1. \end{aligned}$$

Hence relation (2.29) takes the form

$$\begin{aligned}(2.30)\quad \log f'(z) &= 2\log\sin t_1'' - 2\log\sin t_1' - 2\sin^2 t_1'' + 2\sin^2 t_1' - \\ &\quad -2i(t_1''-t_1') - 2\sin 2t_1'' + 2\sin t_1' - \frac{\cos\theta_0}{2}\log\left[-\cot^2\left(t_1''-\frac{\theta_0}{2}\right)\right] + \\ &\quad + \frac{\cos\theta_0}{2}\log\left[-\cot^2\left(t_1'-\frac{\theta_0}{2}\right)\right] + \cos 2t_1'' - \cos 2t_1' + \\ &\quad + i\frac{\sin\theta_0}{2}\log\left[-\cot^2\left(t_1''-\frac{\theta_0}{2}\right)\right] - i\frac{\sin\theta_0}{2}\log\left[-\cot^2\left(t_1'-\frac{\theta_0}{2}\right)\right].\end{aligned}$$

In expression (2.30) the numbers $t_1''$ and $t_1'$ are unkown and should be calculated.

From equation (2.15) we have

$$(2.31)\qquad \frac{du}{d\tau} = \frac{\mu-u}{|\mu-a|^2},$$

and thus, by (2.20) and (2.21), we get

$$\mu - u = -|a-\mu|\cos t_1.$$

Hence formula (2.31) takes the form

$$\frac{du}{d\tau} = \frac{-\cos t_1}{\varrho\sin t_1\sin(\theta_0-2t_1)}.$$

Next, by (2.27), we obtain

$$(2.32)\qquad \frac{du}{dt_1} = 2\varrho\cos t_1\sin(\theta_0-3t_1).$$

Consider now relation (2.25). Putting successively $\tau = 0$ and $\tau = \tau_0$ in (2.25), we get the following relation:

$$(2.33)\qquad \varrho = \frac{v_0}{\sin^2 t_1'\sin(\theta_0-2t_1')} = \frac{v_1}{\sin^2 t_1''\sin(\theta_0-2t_1'')},$$

where $v_0 = \operatorname{Im} z$ is given together with a point $z\in P_z^+$ but $v_1 = v(\tau_0)$ is unknown.

Integrating (2.32) in the interval $\langle t_1', t_1''\rangle$, we have

$$\begin{aligned}u(t_1'') - u(t_1') = -\tfrac{1}{2}\varrho[&\cos(\theta_0-2t_1'') - \cos(\theta_0-2t_1') - \\ &-\tfrac{1}{2}\cos(4t_1''-\theta_0) + \tfrac{1}{2}\cos(4t_1'-\theta_0)].\end{aligned}$$

Notice that $u(t_1') = u_0 = \operatorname{Re} z$ is given together with a point $z\in P_z^+$ and the number $u(t_1'') = u_1$ is unknown and thus we have

$$\begin{aligned}u_1 = u_0 + \tfrac{1}{2}\varrho[&\cos(\theta_0-2t_1') - \cos(\theta_0-2t_1'') + \\ &+\tfrac{1}{2}\cos(4t_1''-\theta_0) - \tfrac{1}{2}\cos(4t_1'-\theta_0)]\end{aligned}$$

or in an equivalent form, using formula (2.33),

$$(2.34)\qquad u_1 = u_0+v_0\left[\cot^2 t_1'\cot(\theta_0-2t_1')-\frac{\cos^2 t_1''\cos(\theta_0-2t_1'')}{\sin^2 t_1'\sin(\theta_0-2t_1')}\right].$$

It can be seen that, using the accepted notations, we can write (2.14) in the form

$$(2.14')\qquad 2(\mu-u)-(u-\nu)-u = \text{const}.$$

We shall now put into (2.34) the quantities given by formulas (2.21), (2.22), (2.24) and the quantities obtained from (2.14′) by putting successively $\tau = \tau_0$ and $\tau = 0$.

We shall get

$$(2.35)\quad u_1+v_1[2\cot t_1''+\cot(\theta_0-2t_1'')] = u_0+v_0[2\cot t_1'+\cot(\theta_0-2t_1')].$$

On the other hand, by (2.2), we have

$$(2.35')\qquad u_1\cos\theta-v_1\sin\theta = K,$$

and thus equation (2.35) enables us to find the required $u_1$ and $v_1$. Notice that the numbers $t_1'$ and $t_1''$ will be determined as well.

We shall now find the value of the function $a(\tau)$ for $\tau = \tau_0$ because we have $a(\tau_0) = u(\tau_0)+iv(\tau_0) = f(z)$.

To do this we shall integrate equations (2.26) and (2.32) in the interval $\langle t_1', t_1''\rangle$.

We obtain

$$(2.36)\qquad \operatorname{Im} f(z) = \operatorname{Im} z+\tfrac{1}{2}\varrho[\sin(\theta_0-2t_1'')-\sin(\theta_0-2t_1')+ \\ +\tfrac{1}{2}\sin(4t_1''-\theta_0)-\tfrac{1}{2}\sin(4t_1'-\theta_0)],$$

and

$$(2.37)\qquad \operatorname{Re} fz = \operatorname{Re} z-\tfrac{1}{2}\varrho[\cos(\theta_0-2t_1'')-\cos(\theta_0-2t_1')+ \\ +\tfrac{1}{2}\cos(4t_1''-\theta_0)-\tfrac{1}{2}\cos(4t_1'-\theta_0)],$$

respectively.

From (2.36) and (2.37) we get

$$(2.38)\qquad \log f(z) = \log[z-\tfrac{1}{2}\varrho(e^{-i(\theta_0-2t_1'')}-e^{-i(\theta_0-2t_1')}+ \\ +\tfrac{1}{2}e^{-i(4t_1''-\theta_0)}-\tfrac{1}{2}e^{-i(4t_1'-\theta_0)})].$$

Moreover, from (2.30) we find

$$(2.39)\qquad \operatorname{Re} e^{i\theta}f'(z) = \pm\cos\theta_0\left(\frac{\sin t_1''}{\sin t_1'}\right)^2\left[\frac{\tan\left(\dfrac{\theta_0}{2}-t_1''\right)}{\tan\left(\dfrac{\theta_0}{2}-t_1'\right)}\right]^{\cos\theta_0}\times \\ \times\exp(\cos 2t_1''-\cos 2t_1'-2\sin^2 t_1''+2\sin^2 t_1').$$

Basing ourselves on the results obtained, we can formulate the following theorem:

THEOREM 2.1. *The points* $I_{10}$, $I_{20}$, $I_{30}$, *where*

$$I_{10}+iI_{20} = \log\frac{f'(z)}{f(z)}, \quad I_{30} = \operatorname{Re} e^{i\theta}f'(z),$$

*belong to the boundary of the region of the values of the functional system* (2.1). *The quantities* $\log f'(z)$, $\log f(z)$ *and* $\operatorname{Re} e^{i\theta}f'(z)$ *are determined by formulas* (2.30), (2.38) *and* (2.39), *respectively, while the parameters* $t_1'$, $t_2''$ *may be calculated from equations* (2.35), (2.35′) *and* $\theta_0$ *is an arbitrary real number.*

**3. Extrema of the functional** $I(f) = \arg\frac{f'(z_0)}{f(z_0)}$. Let us consider a problem concerning an extremum of the functional

$$I(f) = \arg\frac{f'(z_0)}{f(z_0)} \tag{3.1}$$

defined on the class $H^1$ of functions $f$, where $z_0$ is a fixed point of the half-plane $P_z^+$, under the condition

$$K(f) = \operatorname{Re}\{e^{i\theta}f(z_0)\} = K, \tag{3.2}$$

assuming $K$ to be fixed and $K > \operatorname{Im} z_0$.

Assume that in the class $H^1$ the extremal value of fuctional (3.1) under condition (3.2) is attained by a function $f \neq \infty$. This function will be called an *extremal function.*

Suppose now that the image $B$ of the half-plane $P_z^+$ under the mapping $w = f(z)$ has an exterior point $w_1 \in P_w^+$ and hence another exterior point $w_2 \in P_w^+$. Then for a sufficiently small $t$ and for arbitrary $A_k$ the function

$$f_*(z) = f(z) + t\sum_{k=1}^{2}\left\{\frac{A_k}{f(z)-w_k} + \frac{\overline{A}_k}{f(z)-\overline{w}_k}\right\}$$

belongs to the class $H^1$ (Theorem 1.2).

We shall calculate $I(f_*)-I(f)$ and $K(f_*)-K(f)$ for this function. We obtain

$$K(f_*)-K(f) = t\operatorname{Re}\left\{A_1\left[\frac{e^{i\theta}}{f(z_0)-w_1} + \frac{e^{-i\theta}}{\overline{f(z_0)}-w_1}\right] + \right. \tag{3.3}$$

$$\left. + A_2\left[\frac{e^{i\theta}}{f(z_0)-w_2} + \frac{e^{-i\theta}}{\overline{f(z_0)}-w_2}\right]\right\}$$

and

$$(3.4)\qquad I(f_*)-I(f) = -t\operatorname{Re}\left\{\sum_{k=1}^{2}\left[\frac{A_k i}{f(z_0)\big(f(z_0)-w_k\big)}-\frac{A_k i}{\bar f(z_0)\big(\bar f(z_0)-w_k\big)}\right]-\right.$$
$$\left.-\sum_{k=1}^{2}\left[\frac{A_k i}{\big(f(z_0)-w_k\big)^2}-\frac{A_k i}{\big(\bar f(z_0)-w_k{}^2\big)}\right]\right\}.$$

It follows from condition (3.2) that

$$(3.5)\qquad K(f_*)-K(f) = 0.$$

It can be seen that, by (3.3), relation (3.5) will be satisfied provided that the constant numbers $A_1$ and $A_2$ are chosen as follows:

$$A_2 = -A_1\frac{e^{i\theta}[f(z_0)-w_1]^{-1}+e^{-i\theta}[\bar f(z_0)-w_1]^{-1}}{e^{i\theta}[f(z_0)-w_2]^{-1}+e^{-i\theta}[\bar f(z_0)-w_2]^{-1}} = A,$$

where $A$ is a complex constant. Therefore, by (3.4) the quantity $I(f_*)-I(f)$ takes the form

$$I(f_*)-I(f) = -t\operatorname{Re}\{A\cdot C(w_1,w_2)\},\qquad \text{where } C(w_1,w_2)\neq 0,$$

the points $w_1$ and $w_2$ being suitably chosen.

Since $A$ is an arbitrary constant number, the sign of the difference can be made positive or negative and hence $I(f)$ cannot attain either the maximum or the minimum in this case.

LEMMA 3.1. *The set $f(P_z^+)$, where $f$ is an extremal function with respect to functional* (3.1), *does not possess any exterior points in the half-plane $P_w^+$.*

We shall show that $f(P_z^+)$, where $f$ is an extremal function, is obtained from the half-plane $P_w^+$ by deleting from it an analytical arc begininng at a finite point on the real axis.

In order to do that we shall use the second of the variational formulas in the class $H^1$, i.e., (1.8), and find variations of the functionals $K(f)$ and $I(f)$. We have

$$(3.6)\qquad K(f_*)-K(f) = t\operatorname{Re}\{A_1U(z_1)+A_2U(z_2)\}+o(t),$$

where

$$(3.7)\qquad U(s) = \frac{e^{i\theta}}{f(z_0)-f(s)}+\frac{e^{-i\theta}}{\bar f(z_0)-f(s)}+\frac{1}{[f'(s)]^2}\left[\frac{e^{i\theta}f'(z_0)}{s-z_0}+\frac{e^{-i\theta}\bar f'(z_0)}{s-\bar z_0}\right],$$

and

$$(3.8)\qquad I(f_*)-I(f) = -t\operatorname{Re}\{iA_1V(z_1)-iA_2V(z_2)\}+o(t),$$

where

$$(3.9)\qquad V(s) = \frac{[f(z_0)]^{-1}}{f(z_0)-f(s)} - \frac{[\bar{f}(z_0)]^{-1}}{\bar{f}(z_0)-f(s)} - \frac{1}{[f(z_0)-f(s)]^2} + \\ + \frac{1}{[\bar{f}(z_0)-f(s)]^2} + \frac{1}{[f'(s)]^2}\left[\frac{f'(z_0)+f''(z_0)[f'(z_0)]^{-1}}{s-z_0} + \right. \\ \left. + \frac{f'(z_0)+f''(z_0)[\bar{f}'(z_0)]^{-1}}{s-\bar{z}_0} + \frac{1}{(s-z_0)^2} + \frac{1}{(s-\bar{z}_0)^2}\right].$$

Functional (3.1) will attain its minimum if

$$-t\operatorname{Re}\{iA_1V(z_1)+iA_2V(z_2)\}+o(t) > 0,$$
$$t\operatorname{Re}\{A_1U(z_1)+A_2U(z_2)\}+o(t) = 0.$$

Thus, by the option of the constant numbers $A_k$, $k = 1, 2$, we have

$$(3.10)\qquad iA_1V(z_1)+iA_2V(z_2) = 0, \qquad A_1U(z_1)+A_2U(z_2) = 0.$$

Since the determinant

$$\begin{vmatrix} V(z_1) & V(z_2) \\ U(z_1) & U(z_2) \end{vmatrix}$$

of the homogeneous system (3.10) should be equal to zero, denoting $\dfrac{V(z_2)}{U(z_2)} = \lambda$ and using the fact that $z_2$ is an arbitrary point from the half-plane $P_z^+$, we shall obtain, after replacing $z_2$ by $z$, the following equality:

$$V(z)-\lambda U(z) = 0.$$

This condition together with (3.7) and (3.8) will give the following differential-functional equation, which ought to be satisfied by the extremal function $f(z)$:

$$(3.11)\qquad \frac{\dfrac{i}{f(z_0)}-\lambda e^{i\theta}}{f(z_0)-f(z)} + \frac{\dfrac{-i}{\bar{f}(z_0)}-\lambda e^{-i\theta}}{\bar{f}(z_0)-f(z)} - \frac{i}{[\bar{f}(z_0)-f(z)]^2} + \frac{i}{[f(z_0)-f(z)]^2} \\ = \frac{1}{[f'(z)]^2}\left[\frac{if'(z_0)+if''(z_0)[f'(z_0)]^{-1}-\lambda e^{i\theta}f'(z_0)}{z-z_0} - \right. \\ \left. - \frac{i\bar{f}'(z_0)-i\bar{f}''(z_0)[\bar{f}'(z_0)]^{-1}+\lambda e^{-i\theta}\bar{f}'(z_0)}{z-\bar{z}_0} + \frac{i}{(z-z_0)^2} - \frac{i}{(z-\bar{z}_0)^2}\right].$$

Equation (3.11) affirms that the boundary of the extremal region consists of the finite number of analytical arcs (dissections), which, together with the fact that a double zero of the right-hand side of equation (3.11) corresponds to each of the ends of a dissection, results in the statement that the dissection can have only one end.

Moreover, that dissection should be finite in the half-plane $P_w^+$ since if it were not, the condition that an extremal function belongs to class $H^1$, would not be fulfilled.

Let $\Gamma$ be that dissection along the Jordan arc $w = \Psi(t)$, $0 \leqslant t \leqslant \tau_0$ and let $\Psi(0)$ be an end of the arc $\Gamma$ lying in $P_w^+$.

Consider a region $B_\tau = \{w\colon \operatorname{Im} w > 0,\ w \neq \Psi(t),\ t \in \langle \tau, \tau_0 \rangle\}$ and a function $z = F(w, \tau)$ mapping the region $B_\tau$ onto $P_z^+$.

This function will be called a *function associated with a function f* if the inverse function of $F(w, \tau)$ with respect to the first argument, denoted here by $\Phi(z, \tau)$, is a function of class $H^1$ for an arbitrary $\tau \in \langle 0, \tau_0 \rangle$.

Notice that the following relations hold:

$$F(w, \tau_0) = w \quad \text{and} \quad \Phi(z, 0) = f(z).$$

Let us denote by $\mu(\tau)$ a point on the real axis $\operatorname{Im} z = 0$ which is mapped onto the end of the dissection $w = \Psi(t)$, under $w = \Phi(z, \tau)$.

In [5], § 2, it is stated that such a point always exists and

$$\frac{\partial F(w, \tau)}{\partial \tau} = \frac{1}{\mu(\tau) - P(w, \tau)}, \quad F(w, \tau_0) = w, \ \tau \in \langle 0, \tau_0 \rangle, \tag{3.12}$$

$$\frac{\partial \Phi(z, \tau)}{\partial \tau} + \frac{1}{\mu - z} \frac{\partial \Phi(z, \tau)}{\partial z} = 0. \tag{3.13}$$

We shall now obtain one more variational formula in the class $H^1$. In order to do that let us replace $f(z)$ by $\Phi(z, \tau)$ in formula (7.2) of [5] and put $z = F(w, \tau)$, $z_k = F(w_k, \tau)$, where $w_k = \Phi(z_k, \tau)$. Then clearly the function

$$f_*(z) = f(z) + t \sum_{k=1}^{m} \left\{ \frac{A_k}{f(z) - \Phi(z_k, \tau)} + \frac{\bar{A}_k}{f(z) - \bar{\Phi}(z_k, \tau)} + \right. \tag{3.14}$$
$$\left. + \frac{A_k}{[\Phi_z(z_k, \tau)]^2} \frac{[F'_w(w, \tau)]^{-1}}{z_k - F(w, \tau)} + \frac{\bar{A}_k}{[\bar{\Phi}_z(z_k, \tau)]^2} \frac{[F'_w(w, \tau)]^{-1}}{\bar{z}_k - F(w, \tau)} \right\} + O(t)$$

belongs to the class $H^1$ together with $f(z)$ and, for a small $t$, maps $P_z^+$ on a region in $P_z^+$ close to the region $B$.

Our way of reasoning being analogous to the one employed while introducing equations (3.11) and using formula (3.14), we find that the function $F(w, \tau)$ should satisfy the equation

$$\left[ \frac{\partial F(w, \tau)}{\partial w} \right]^2 B(F, \tau) = A(w), \tag{3.15}$$

where

$$B(F,\tau) = \frac{q_1}{F-a} + \frac{\bar{q}_1}{F-\bar{a}} + \frac{i}{(F-a)^2} - \frac{i}{(F-\bar{a})^2}, \tag{3.16}$$

$$A(w) = \frac{p_1}{\xi - w} + \frac{\bar{p}_1}{\bar{\xi}-w} + \frac{-i}{(\xi-w)^2} + \frac{i}{(\bar{\xi}-w)^2}, \tag{3.17}$$

$$q_1 = -i\left\{[F'_w(w,\tau)]^{-1}\frac{1}{f(z_0)} - \frac{d[F'_w(w,\tau)]^{-1}}{dw} - \lambda e^{i\theta}[F'_w]^{-1}\right\}, \tag{3.18}$$

$$p_1 = \frac{i}{f(z_0)} - \lambda e^{i\theta}, \tag{3.19}$$

$$\xi = f(z_0), \quad a = a(\tau) = F(\xi,\tau). \tag{3.20}$$

Notice that equation (3.11) for the extremal function $f(z)$ is obtained from (3.15) for $\tau = 0$.

From the normalizing condition for the class $H^1$ it follows that polynomials appearing in the denominators of the rational functions $A(w)$ and $B(F,\tau)$ are of equal degrees (in our problem of the third degree) and that the coefficients at the greatest powers of those polynomials are equal.

From equations (3.12) and (3.15) it follows that

$$A(w) = A_0\frac{(w-w_0)^2(w-w_1)}{(w-\xi)^2(w-\bar{\xi})^2}, \tag{3.21}$$

$$B(F,\tau) = B_0\frac{[F-\mu(\tau)]^2[F-\varkappa(\tau)]}{[F-a(\tau)]^2[F-\overline{a(\tau)}]^2}, \tag{3.22}$$

and $A_0 = B_0 = \text{const}$.

Moreover, the interrelations between the functions $a(\tau)$, $\mu(\tau)$ and $\varkappa(\tau)$ are the following:

$$2\mu - 2(a+\bar{a}) + \varkappa = \text{const}, \tag{3.23}$$

$$\frac{da}{d\tau} = \frac{1}{\mu - a}, \tag{3.24}$$

$$\frac{d\varkappa}{d\tau} = \frac{1}{\mu-\varkappa}. \tag{3.25}$$

From (3.19) and (3.16) we find that $B_0$ is real.

Let $w$ tend to $\xi$ in (3.15). Then $F(w,\tau)$ tends to $a(\tau)$ and

$$\frac{[a-\mu]^2(a-\varkappa)}{(a-\bar{a})^2} = \frac{i}{B_0} = \text{const}. \tag{3.26}$$

Comparing the main arguments of the two sides we obtain

$$2\operatorname{Arg}(a-\mu)+\operatorname{Arg}(a-\varkappa) = \frac{\pi}{2}(1+2k), \tag{3.27}$$

and $k$ can take the value 0 or 1 depending on whether $B_0 < 0$ or $B_0 > 0$, respectively.

Let us write

$$a = x+iy, \tag{3.28}$$

$$a-\mu = |a-\mu|e^{i\arg(a-\mu)} = r_1 e^{it_1}, \tag{3.29}$$

$$a-\varkappa = |a-\varkappa|e^{i\arg(a-\varkappa)} = r_2 e^{it_2}. \tag{3.30}$$

Notice that from (3.25) it follows that

$$\frac{d\operatorname{Im}\varkappa}{d\tau} = \frac{\operatorname{Im}\varkappa}{[\mu-\varkappa]^2}, \tag{3.31}$$

which, by (3.23), implies $\operatorname{Im}\varkappa = 0$. Thus we find that the function $\varkappa(\tau)$ is a real function.

We shall now use the fact, following from (3.28) and (3.20), that $y > 0$ and therefore $t_1 \in (0, \pi)$ and $t_2 \in (0, \pi)$.

Moreover, from (3.29) and (3.30) we have

$$r_1 = \frac{y}{\sin t_1} \quad \text{and} \quad r_2 = \frac{y}{\sin[\frac{1}{2}\pi(2k+1)-2t_1]}. \tag{3.32}$$

Thus from (3.26) we obtain

$$\left|\frac{1}{B_0}\right| = \frac{r_1^2 r_2}{y^2} = \text{const},$$

and from above and (3.32) we find

$$y = B_0^{-1}\sin^2 t_1 \cos 2t_1 \quad \text{when } k = 0, \tag{3.33}$$

and

$$y = -B^{-1}\sin^2 t_1 \cos 2t_1 \quad \text{when } k = 1. \tag{3.33'}$$

Since from (3.23) we have $\dfrac{dy}{d\tau} = \dfrac{y}{|\mu-a|^2}$ and from (3.33) and (3.33′) we find $\dfrac{dy}{dt_2} = \pm 2B_0^{-1}\sin t_1 \cos 3t_1$, where the "+" corresponds to the number $k = 0$ and "−" to the number $k = 1$, therefore

$$\frac{dt_1}{d\tau} = \frac{dy}{d\tau} : \frac{dy}{dt_1} = B_0^2[2\sin t_1 \cos 2t_1 \cos 3t_1]^{-1}. \tag{3.34}$$

Notice that the function $t_1(\tau)$ is strictly monotonous and thus it has an inverse function $\tau(t_1)$ defined in the interval $\langle t_1(0), t_1(\tau_0)\rangle = \langle t_1', t_1''\rangle$.

Let us now put $f(z_0) = \xi$ in equation (3.12). Then, using notations (3.20) and (3.28), we obtain

$$d\log F_w'(\xi, \tau) = \frac{d\tau}{[\mu(\tau) - F(w, \tau)]^2} = e^{-2it_1}\frac{d\tau}{|\mu(\tau) - a(\tau)|^2},$$

and since $\dfrac{dy}{d\tau} = \dfrac{y}{|\mu - a|^2}$, we have

$$d\log F_w'(\xi, \tau) = e^{-2it_1} d\log y. \tag{3.35}$$

Integrating (3.35) in the interval $\langle t_1', t_1''\rangle$, we get

$$\log f'(z_0) = \int_{t_1'}^{t_1''} e^{-2it_1} d\log y(t_1) \tag{3.36}$$

and hence

$$\log f'(z_0) = 2i(\sin 2t_1'' - \sin 2t_1') - 2i(t_1'' - t_1') + \log\frac{\sin 2t_1''}{\sin 2t_1'}. \tag{3.37}$$

In (3.37) the numbers $t_1'$ and $t_1''$ are unknown and we are going to determine them. From (3.24) we obtain

$$\frac{dx}{d\tau} = \frac{\mu - x}{|\mu - a|^2} \tag{3.38}$$

and thus, by (3.28) and (3.29),

$$\mu - x = -|a - \mu|\cos t_1,$$

which gives

$$\frac{dx}{d\tau} = \frac{-\cos t_1}{\pm B_0 \sin t_1 \cos 2t_1}.$$

Basing ourselves on (3.31), we get

$$\frac{dx}{dt_1} = \frac{-2\cos t_1 \cos 3t_1}{\pm B_0^3 \cos 2t_1}. \tag{3.39}$$

Take under consideration, in turn, relations (3.33) and (3.33′). Putting successively $\tau = \tau_0$ and $\tau = 0$, we obtain

$$\pm B_0 = \frac{y_1}{\sin^2 t_1'' \cos 2t_1''} = \frac{y_0}{\sin^2 t_1' \cos 2t_1'}, \tag{3.40}$$

where $y_0 = \operatorname{Im} z_0$ is given together with the point $z_0 \in P_z^+$ but $y_1 = y(\tau_0)$ is still unknown.

Integrating equation (3.39) in the interval $\langle t_1', t_1''\rangle$, we have

$$x(t_1'')-x(t_1')=\pm B_0^{-3}\left[-\sin 2t_1''-\frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''}+\sin 2t_1'+\frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'}\right].$$

It is known that $x(t_1')=x_0=\mathrm{Re}\,z_0$ is given together with the point $z_0$ but $x(t_1'')=x_1$ is unknown. Thus

$$x_1=x_0\pm B_0^{-3}\left[-\sin 2t_1''-\frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''}+\sin 2t_1'+\frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'}\right]$$

and hence, by (3.40), we get

$$x_1=x_0+\frac{y_0^3}{\sin^6 t_1'\cos^3 2t_1'}\left[-\sin 2t_1''-\frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''}+\right.$$
$$\left.+\sin 2t_1'+\frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'}\right]. \tag{3.41}$$

Notice that (3.27) may be written in an equivalent form as

$$2(\mu-x)-(x-\varkappa)-x=\text{const}. \tag{3.27'}$$

Put in (3.27') $\tau=0$ and $\tau=\tau_0$, successively.
Then, using (3.28)–(3.30), we get

$$x_0+y_0(2\cot t_1'+\cot 2t_1')=x_1+y_1(2\cot t_1''+\cot 2t_1''). \tag{3.42}$$

On the other hand, from (3.2) we have

$$x_1\cos a-y_1\sin a=K, \tag{3.43}$$

and therefore it is possible to calculate the unknown quantities $x_1$ and $y_1$ from (3.42) and the numbers $t_1'$ and $t_1''$ will be calculated at the same time.

It is obvious that $a(\tau_0)=x(\tau_0)+iy(\tau_0)=f(z_0)$. We shall now find $f(z)$ and in order to do that we shall integrate successively the equations $\dfrac{dy}{d\tau}=\dfrac{y}{|\mu-a|^2}$ and (3.39) in the interval $\langle t_1', t_1''\rangle$.

We get

$$\mathrm{Re}f(z_0)=\mathrm{Re}\,z_0\pm B_0^{-3}\left[-\sin 2t_1''-\frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''}+\right.$$
$$\left.+\sin 2t_1'+\frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'}\right] \tag{3.44}$$

and

$$\mathrm{Im}f(z_0)=\mathrm{Im}\,z\pm B_0^{-3}\left[\tfrac{1}{2}\cos 2t_1''-\tfrac{1}{4}\cos 4t_1''-\tfrac{1}{2}\cos 2t_1'+\tfrac{1}{4}\cos 4t_1'\right]. \tag{3.45}$$

Thus

$$(3.46)\qquad f(z_0) = z_0 \pm B_0^{-3}\left[-\sin 2t_1'' - \frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''} + \sin 2t_1' + \right.$$

$$\left. + \frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'}\right] \pm B_0^{-1}\left[\tfrac{1}{2}\cos 2t_1'' - \tfrac{1}{4}\cos 4t_1'' - \tfrac{1}{2}\cos 2t_1' + \tfrac{1}{4}\cos 4t_1'\right].$$

Moreover, from (3.37) we obtain

$$(3.47)\qquad f'(z_0) = \exp\left[2i(\sin 2t_1'' - \sin 2t_1') - 2i(t_1'' - t_1')\right]\frac{\sin 2t_1''}{\sin 2t_1'}.$$

Equalities (3.46) and (3.47) enable us to determine $\operatorname{Arg}\dfrac{f'(z_0)}{f(z_0)}$ and thus to find the required extremum.

THEOREM 3.1. *The extremal values of functional* (3.1) *are determined by the following formula*:

$$J\text{extr} = \operatorname{Arg}\left\{\exp\left[2i(\sin 2t_1'' - \sin 2t_1') - 2i(t_1'' - t_1')\right]\frac{\sin 2t_1''}{\sin 2t_1'} \times \right.$$

$$\times\left[z_0 \pm B_0^{-3}\left(\sin 2t_1' + \frac{1}{2}\log\frac{1-\sin 2t_1'}{1+\sin 2t_1'} - \sin 2t_1'' - \frac{1}{2}\log\frac{1-\sin 2t_1''}{1+\sin 2t_1''}\right) \pm \right.$$

$$\left.\left. \pm iB_0^{-1}\left(\tfrac{1}{2}\cos 2t_1'' - \tfrac{1}{4}\cos 4t_1'' - \tfrac{1}{2}\cos 2t_1' - \tfrac{1}{4}\cos 4t_1'\right)\right]^{-1}\right\}$$

*and the numbers* $t_1'$, $t_1''$ *and* $B_0$ *are defined by formulas* (3.41), (3.42) *and* (3.43), *respectively.*

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