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Fundamental solution E_n of the operator $\partial^2/\partial t^2 - \Delta_n$ for $n > 3$

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Abstract. In this paper a natural construction of a fundamental solution of the wave operator $\partial^2/\partial t^2 - \Delta_n$ is presented. This is done in a completely elementary way and without using the methods of Bessel functions as it is commonly the case. These methods are replaced by an application of such natural and important features of the wave operator as homogeneity and invariance with respect to the group of Lorentz transformations. Taking these features into account and using the techniques of the Fourier transformation we prove in Section 1 the main theorem which gives an explicit formula (up to a constant factor) of the unique Lorentz invariant fundamental solution E_n with support in the upper half-plane.

The following sections are devoted to the evaluation of the above-mentioned constants. This is done in terms of the recursion relations obtained by an application of the method of descent to the ealier obtained formulae for E_n with an unknown constant factor.

Lorentz invariance is one of the most important features of the wave equation.

Therefore in the study of the wave equation the approach presented by Methée [1], [2] seems to be the most natural. Methée gave, first of all, a characterization of distributions invariant with respect to Lorentz transformations, and then applied it to finding fundamental solutions of the operator $\Box_n$.

The proof presented there was not simple. Owing to the importance of the wave operator, we regard it useful to present a natural, intelligible to beginers, way of arriving at a fundamental solution E_n of the operator $\Box_n$, $n > 3$.

In Section 1 of the present paper we determine, up to a constant factor C_n, the form of a solution E_n for n even, $n \geqslant 4$. It is the most significant part of the paper, in which we apply the theorems on distributions invariant with respect to the proper group of Lorentz transformations, which were proved in [5]. These are: Theorem 1 and 3 from [5]. The Methée theorem(1) itself is not necessary for the proof. Theorem 1 from

(1) See [1] or [5], Theorem 2.

[5] would do instead; in fact, also the weaker version of it, which is easier to prove and was published in [6], would suffice.

In Section 2 dealing with the case of n odd, $n \geqslant 3$, we determine, using the results of Section 1 and applying the method of descent, the form of E_n up to a constant factor C_n.

In Section 3 we find the constants C_n applying the method of descent this time from n odd.

The present paper is strongly connected with the preceding paper [5]. We retain here, as we also did in [5], all the notation introduced in [4], which is generally used in the theory of distributions.

1. Explicit formulae for the fundamental solution E_{2m}, $m \in N$, $m \geqslant 2$, determined up to a constant factor C_{2m}. Suppose that there is a distribution $E \in S'(E^{n+1})$ (²) satisfying:

$$\square_n E = \delta, \qquad \delta \in D'(E^{n+1}). \tag{1}$$

Then the distribution(³) $\tilde{E} = F_x E$ satisfies the equation

$$\frac{\partial^2}{\partial t^2}\tilde{E} + |y|^2 \tilde{E} = (2\pi)^{-\frac{1}{2}n}\,\delta_t \otimes 1_x. \tag{2}$$

We know that the tempered distribution given by

$$\tilde{E}(t, x) = (2\pi)^{-\frac{1}{2}n}\, Y(t)\frac{\sin t|x|}{|x|} \qquad \text{for } (t, x) \in E^{n+1} \tag{3}$$

is a solution of (2). Hence

$$E = F_x^{-1}\tilde{E}, \qquad \text{where } \tilde{E} \text{ is defined by (3)}, \tag{4}$$

is a fundamental solution of $\square_n$.

In our further considerations E will always stand for the distribution defined by (4). Retaining the notation of [5] we have:

THEOREM 1. *Let E be the distribution* (4). *Then the distribution $\hat{E}$ is G-invariant, where $\hat{E} = FE = F_t\tilde{E}$.*

Proof. Notice that $\hat{E} \in D'(E^{n+1})$ is invariant with respect to rotations in E^n, because $\tilde{E}$ possesses this property and $\hat{E} = F_t\tilde{E}$. By Theorem 3 in [5] we only need prove that $\left(\tau\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial \tau}\right)\hat{E} = 0$ on E^{n+1}. On account of the following two relations:

$$F_\tau^{-1}\left(\tau\frac{\partial}{\partial y_1}w\right) = -i\frac{\partial}{\partial y_1}\frac{\partial}{\partial t}F_\tau^{-1}w, \qquad F_\tau^{-1}\left(y_1\frac{\partial}{\partial \tau}w\right) = -iy_1 t F_\tau^{-1}w,$$

(²) For simplicity we now write E instead of E_n.

(³) $F_x(F_t)$ denotes the Fourier transform with respect to the variable system $x = (x_1, \dots, x_n)$ (to t). $F = F_xF_t = F_tF_x$ see also [4], footnote (¹⁰).

which hold for an arbitrary $w \in S'(E^{n+1})$, it is sufficient to show that

$$\frac{\partial}{\partial y_1}\frac{\partial}{\partial t}\tilde{E} = -y_1 t\tilde{E}.$$

The proof of this formula is easily obtained from the form (3) of the distribution $\tilde{E}$.

THEOREM 2. *The fundamental solution E is G-invariant and the support of E is contained in $\overline{V}_+$.*

Proof. Let $g \in G$. Applying consecutively Proposition 3 from [4] (on the Fourier transform of the substitution), Proposition 3 in [5] and Theorem 1 we obtain(4)

$$E \circ g = (F^{-1}\hat{E})\circ g = F^{-1}\big(\hat{E}\circ(g^{-1})^t\big) = F^{-1}\hat{E} = E.$$

Hence the distribution E is G-invariant. From (3) it follows at once that $\operatorname{supp}\tilde{E} \subset \overline{E^{n+1}_+}$. Hence by (4) also $\operatorname{supp} E \subset \overline{E^{n+1}_+}$. A direct application of Proposition 6 from [5] ends the proof.

It follows by Theorem 1 and Property 2 from [5] and by Theorem 2 (above) that there exists a unique distribution $E^* \in D'(E^1)$ equal to zero on E^1_- and such that(5)

$$E[\varphi] = E^*[J(\varphi)] \quad \text{for } \varphi \in C_0^\infty(\Omega_1). \tag{5}$$

We know that E satisfies equation (1). In order to find the differential equation which is satisfied by E^*, we shall first consider the case of a G-invariant function f. If $f \in C^2(\Omega_1)$, then

$$\Box_n f(t,x) = \left(4s\frac{d^2h}{ds^2} + 2(n+1)\frac{dh}{ds}\right)\bigg|_{s=t^2-|x|^2} \quad \text{for } (t,x) \in E^{n+1}_+, \tag{6}$$

where $f(t,x) = h(t^2-|x|^2)$ on Ω_1.

Formula (6) can be extended to the case of distributions and we have the following

LEMMA 1. *Let $v \in D'(E^{n+1}_+)$. Suppose there exists a distribution $v^* \in D'(E^1)$ such that $v[\varphi] = v^*[\tilde{J}(\varphi)]$ for $\varphi \in C_0^\infty(E^{n+1}_+)$. Then*

$$\Box_n v[\varphi] = \left(4s\frac{d^2}{ds^2} + 2(n+1)\frac{d}{ds}\right)v^*[\tilde{J}(\varphi)] \quad \textit{for } \varphi \in C_0^\infty(E^{n+1}_+). \tag{7}$$

Proof. Let $C_0^\infty(E^1) \ni h_\nu \xrightarrow[\nu\to\infty]{} v^* \in D'(E^1)$ and let $f_\nu(t,x) = h_\nu(t^2-|x|^2)$ $(\nu = 1, 2, \ldots)$. We know that $f_\nu[\varphi] = h_\nu[\tilde{J}(\varphi)]$ for $\varphi \in C_0^\infty(E^{n+1}_+)$ $(\nu = 1, 2, \ldots)$, and

$$\lim_{\nu\to\infty} f_\nu[\varphi] = \lim_{\nu\to\infty} h_\nu[\tilde{J}(\varphi)] = v^*[\tilde{J}(\varphi)] = v[\varphi] \quad \text{for } \varphi \in C_0^\infty(E^{n+1}_+).$$

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(4) The symbol g^t was defined in Definition 2 in [4].

(5) We retain the notation of [5].

Applying formula (6) to the functions f_ν we obtain the relations:

$$\Box_n f_\nu(t,x) = \left(4s\frac{d^2h_\nu}{ds^2} + 2(n+1)\frac{dh_\nu}{ds}\right)\Bigg|_{s=t^2-|x|^2} \qquad (\nu = 1, 2, \ldots).$$

Hence

$$\Box_n f_\nu[\varphi] = \left(4s\frac{d^2}{ds^2} + 2(n+1)\frac{d}{ds}\right)h_\nu[\tilde{J}(\varphi)] \quad \text{for } \varphi \in C_0^\infty(E_+^{n+1})$$

$$(\nu = 1, 2, \ldots).$$

Passing to the limit as $\nu \to \infty$, we get (7).

PROPOSITION 1. *Suppose that a distribution $u \in D'(\Omega_1)$ is G-invariant and satisfies the equation $\Box_n u = 0$ on Ω_1. Let u^* be a distribution from $D'(E^1)$ such that*(6) *$u[\varphi] = u^*[J(\varphi)]$ for $\varphi \in C_0^\infty(\Omega_1)$. Then*

$$4s\left(\frac{d^2}{ds^2} + 2(n+1)\frac{d}{ds}\right)u^* = 0 \qquad \text{on } E^1.$$

Proof. Let

$$w^* = \left(4s\frac{d^2}{ds^2} + 2(n+1)\frac{d}{ds}\right)u^*.$$

Since $\Box_n u = 0$, it follows by Lemma 1 that $w^*[\tilde{J}(\varphi)] = 0$ for $\varphi \in C_0^\infty(E_+^{n+1})$. By Property 2 in [5] we get $w^* = 0$.

Remark 1. As an immediate consequence of Proposition 1 we obtain that the distribution E^* defined by (5), equal to zero on E_-^1 (7), satisfies the equation

$$4s\frac{d^2T}{ds^2} + 2(n+1)\frac{dT}{ds} = 0 \qquad \text{on } E^1. \tag{8}$$

All distributional solutions on E_+^1 of (8) are classical solutions. They are of the form $a\cdot s^{(1-n)/2} + b$, where a, b are arbitrary constants. Suppose(8)

$$n = 2m, \qquad m \in N, \qquad m \geqslant 2. \tag{9}$$

Then the general solution of (8) on E_+^1 is the function:

$$c_1\frac{d^{m-1}}{ds^{m-1}}\frac{1}{\sqrt{s}} + c_2, \qquad c_1, c_2 - \text{arbitrary constants}.$$

(6) The existence of u^* follows from Theorem 1 in [5].

(7) E^* is equal to zero on E_-^1 by Corollary 1 in [5] and on account of Theorem 2.

(8) Commencing from now, (9) will be assumed in all the considerations to follow in this section.

Note that a function h defined on E^1 by the formulae

$$h(s) = \begin{cases} \dfrac{1}{\sqrt{s}} & \text{for } s > 0, \\ 0 & \text{for } s \leqslant 0, \end{cases} \tag{10}$$

is a distribution on E^1, equal to zero on E^1_-. Hence

$$T^* = \frac{d^{m-1}}{ds^{m-1}} h \tag{11}$$

is a distribution from $D'(E^1)$ of order $m-1$, equal to zero on E^1_-. What is more, for any function $a \in C_0^\infty(E^1)$, applying (10), (11), the Leibniz formula and the integration by parts, we obtain:

$$\left(4s \frac{d^2 T^*}{ds^2} + 2(n+1) \frac{dT^*}{ds}\right)[a]$$
$$= 2(-1)^{m-1} \int_0^\infty \frac{1}{\sqrt{s}} \left(2 \frac{d^{m+1}}{ds^{m+1}} \big(sa(s)\big) - (n+1) \frac{d^m}{ds^m} a(s)\right) ds = 0.$$

From the above considerations follows immediately

PROPOSITION 2. *Suppose* (9). *Then every distribution* $T \in D'(E^1)$ *equal to zero on* E^1_- *and satisfying* (8) *on* E^1 *is of the form* $T = c_1 T^* + c_2 Y$ *on* $E^1 \setminus \{0\}$, *where* c_1, c_2 *are arbitrary constants,* Y *is the Heaviside function, and* T^* *is defined by* (10), (11):

$$T^*[a] = \int_0^\infty \frac{1}{\sqrt{s}} \frac{d^{m-1}}{ds^{m-1}} a(s)\, ds \qquad \textit{for } a \in C_0^{m-1}(E^1). \tag{12}$$

By Remark 1 and Proposition 2 there exist two constants c, $\tilde{c}$ such that

$$E^* = cT^* + \tilde{c} Y \qquad \text{on } E^1 \setminus \{0\}. \tag{13}$$

Let $E^\cdot$ be defined by the formal definition

$$E^\cdot[\varphi] = \int_0^\infty \frac{1}{\sqrt{s}} \frac{d^{m-1}}{ds^{m-1}} \big(P^*(\varphi)\big)(s)\, ds \ (^9) \qquad \text{for } \varphi \in C_0^\infty(E^{n+1}). \tag{14}$$

The correctness of (14) follows from Property 3 from [5]. We have $E^\cdot \in D'(E^{n+1})$ and

$$E^\cdot[\varphi] = T^*[a], \tag{15}$$

(9) The operation P^* was defined in [5].

whenever $\varphi \in C_0^\infty(E^{n+1})$, $a \in C_0^{m-1}(E^1)$, $a(s) = (P^*(\varphi))(s)$ for $s \geqslant 0$. Since $P^*(\varphi) = 0$ for $\varphi \in C_0^\infty(E^{n+1} \setminus \bar{V}_+)$, we derive from (14) that

$$E^{\cdot} = 0 \quad \text{on } E^{n+1} \setminus \bar{V}_+. \tag{16}$$

Applying (5), (13), (15) and Property 1 from [5], we obtain for $\varphi \in C_0^\infty(V_+)$ the relations:

$$E[\varphi] = E^*[\tilde{J}(\varphi)] = cT^*[\tilde{J}(\varphi)] + \tilde{c}[\tilde{J}(\varphi)] = cE^{\cdot}[\varphi] + \tilde{c}w[\varphi], \tag{17}$$

where w denotes the distribution:

$$w[\varphi] = 1[\tilde{J}(\varphi)] \quad \text{for } \varphi \in C_0^\infty(V_+). \tag{18}$$

To prove $\tilde{c} = 0$ we need the following lemmas:

LEMMA 2. (a) *Suppose* (9). *Then the distribution* $E^{\cdot}$ *is homogeneous of order* $1-n$ *on* E^{n+1}, *and the distribution* w *is homogeneous of order zero on* V_+.

(b) *The distribution* E *is homogeneous of order* $1-n$ *on* E^{n+1}.

Proof. Let $A_h(t, x) = (ht, hx)$, $h > 0$. To prove (a), note that for $\varphi \in C_0^\infty(E^{n+1})$ we have

$$P^*(\varphi \circ A_h)(s) = \frac{1}{h^{n-1}} (P^*(\varphi))(h^2 s) \quad \text{for } s \geqslant 0. \tag{19}$$

Set $a(s) = (P^*(\varphi))(s)$ for $s \geqslant 0$. By (14) and (19) we get

$$(E^{\cdot} \circ A_h)[\varphi] = \frac{1}{h^{n+1}} E^{\cdot}[\varphi \circ A_{1/h}] = \frac{1}{h^2} \int_0^\infty \frac{1}{\sqrt{s}} \frac{d^{m-1}}{ds^{m-1}} a\left(\frac{s}{h^2}\right) ds$$
$$= \frac{1}{h^{2m-1}} \int_0^\infty \frac{1}{\sqrt{t}} \frac{d^{m-1}}{dt^{m-1}} a(t)\,dt = \frac{1}{h^{n-1}} E^{\cdot}[\varphi].$$

Similarly we have

$$(w \circ A_h)[\varphi] = w[\varphi] \quad \text{for } \varphi \in C_0^\infty(V_+).$$

To prove (b) observe that $\tilde{E} = F_x E$ implies the relation $FE = F_t \tilde{E}$. Applying (3) and introducing a convergence factor $e^{-\varepsilon t}$ we have $FE = \lim_{\varepsilon \to 0} v_\varepsilon$ in $D'(E^{n+1})$, where distributions v_ε are functions:

$$v_\varepsilon(t, y) = (2\pi)^{-\frac{1}{2}(n+1)} \cdot \frac{1}{|y|} \int_0^\infty e^{-it\tau - \varepsilon\tau} \sin \tau |y|\, d\tau = -(2\pi)^{-\frac{1}{2}(n+1)} \frac{1}{(t - i\varepsilon)^2 - |y|^2}.$$

Hence, by the continuity of the operation of superposition, we obtain

$$(FE) \circ A_h = (\lim_{\varepsilon \to 0} v_\varepsilon) \circ A_h = \lim_{\varepsilon \to 0} (v_\varepsilon \circ A_h) = \lim_{\varepsilon \to 0} \frac{1}{h^2} v_{\varepsilon/h} = \frac{1}{h^2} FE.$$

Therefore the distribution FE is homogeneous of order -2. Then by Corollary 2 in [4] we see that E is homogeneous of order $1-n$.

LEMMA 3. *If n is even, $n \geqslant 4$, then* $V_+ \subset \operatorname{supp} E$.

Proof. Suppose, conversely, that there exists an open set $U \subset V_+$ such that $E[\varphi] = 0$ for $\varphi \in C_0^\infty(U)$. Since obviously $V_+ \subset E^{\cdot}$ and $V_+ \subset \operatorname{supp} w$, it follows by Lemma 2(a) that for the constants c, $\tilde{c}$ different from zero, the functional $cE^{\cdot} + \tilde{c}w$, where $E^{\cdot}$, w are defined by (14), (18), respectively, is not equal to zero on $C_0^\infty(U)$. Therefore from (17) it follows that $c = \tilde{c} = 0$. Hence $E = 0$ on V_+ and by Theorem 2 $\operatorname{supp} E \subset \partial \overline{V}_+$. So, from (5) and Property 2 of [5], we derive the equality $\operatorname{supp} E^* = \{0\}$. This implies the existence of a finite number of constants a_k such that $E^* = \sum_k a_k \delta^{(k)}$. Because E^* must satisfy (8), we have $a_k = 0$ for $k \neq (n-3)/2 = m - \frac{3}{2}$, so $E^* = 0$ on E^1, and on account of (5) $E = 0$ on E_+^{n+1}. Applying once again Theorem 2, we obtain $\operatorname{supp} E = \{0\}$. Hence $E = 0$ on E^{n+1} by Proposition 2 from [4]. This contradiction ends the proof of Lemma 3.

We known that $V_+ \subset \operatorname{supp} E^{\cdot}$, $V_+ \subset \operatorname{supp} w$. By Lemma 3 also $V_+ \subset \operatorname{supp} E$. Therefore the orders of the distributions $E, E^{\cdot}$, w are determined uniquely on V_+. Applying (17) and Lemma 2 we obtain $\tilde{c} = 0$. Therefore, from (13), $E^* = cT^*$ on $E^1 \setminus \{0\}$ and by the same argument as in the proof of Lemma 3, we get $E^* = cT^*$ on E^1. Now, from (5) and (15) follows: $E[\varphi] = cT^*[J(\varphi)] = cE^{\cdot}[\varphi]$ for $\varphi \in C_0^\infty(E_+^{n+1})$. Hence, in virtue of Theorem 2 and formula (16), the distributions E and $E^{\cdot}$ are equal to zero on $E^{n+1} \setminus \overline{V}_+$ and, again by Proposition 2 from [4], we have $E = cE^{\cdot}$ on E^{n+1}.

Returning to the notation $E = E_n$ [10] we rewrite the last equality in the form

$$E_n = C_n E^{\cdot}. \tag{20}$$

Applying (14), (20), Lemma 3 and Theorem 2 we get the following

THEOREM 3. *Suppose* (9). *Then there exists a constant C_n such that the distribution E_n given by*:

$$E_n[\varphi] = C_n \int_0^\infty \frac{1}{\sqrt{s}} \frac{d^{m-1}}{ds^{m-1}} \left(\int_{E^n} \frac{\varphi(\sqrt{s+|x|^2}, x)}{2\sqrt{s+|x|^2}} \, dx \right) ds \tag{21}$$

$$\text{for } \varphi \in C_0^\infty(E^{n+1})$$

is a fundamental solution of the wave operator $\partial^2/\partial t^2 - \Delta_n$.

This fundamental solution is unique [11] *in the class of distributions having support in E_+^{n+1}. Moreover, E_n is G-invariant, $E_n \in S'(E^{n+1})$,*

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[10] See footnote [2].

[11] See Theorem 24.3 in [3].

$\operatorname{supp} E_n = \overline{V}_+$. *Inside* V_+, E_n *is a function* [12]:

$$E_n(t, x) = C_n(\tfrac{1}{2})^{m-1} \frac{1\cdot 3\cdot \ldots \cdot(2m-3)}{(t^2-|x|^2)^{\frac{1}{2}(n-1)}} \quad \textit{for } (t, x) \in V_+.$$

Remark 2. The fundamental solution (21) can be written in the form [13]:

$$(22) \qquad E_n[\varphi] = C_n \int\limits_{E^n} \left(\int\limits_{E^1} \frac{Y(t-|x|)}{\sqrt{t^2-|x|^2}} \frac{\partial}{\partial t} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{m-2} \left(\frac{\varphi(t, x)}{2\cdot t}\right) dt \right) dx$$

$$\text{for } \varphi \in C_0^\infty(E^{n+1}),\ n = 2m,\ m \geqslant 2,$$

which follows from the following computations:

$$(23) \qquad E_n[\varphi] = C_n \int\limits_{E^n} \left(\int\limits_0^\infty \frac{1}{\sqrt{s}} \frac{\partial^{m-1}}{\partial s^{m-1}} \left(\frac{\varphi(\sqrt{s+|x|^2}, x)}{2\sqrt{s+|x|^2}} \right) ds \right) dx$$

$$= C_n \int\limits_{E^n} \left(\int\limits_{E^1} \frac{Y(t-|x|)}{\sqrt{t^2-|x|^2}} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{m-1} \left(\frac{\varphi(t, x)}{2t}\right) 2t\,dt \right) dx.$$

2. Explicit formulae for the fundamental solution E_{2m+1}, $m \in N$, $m \geqslant 1$, determined up to a constant factor C_{2m+1}. Let $k \in N$, $k \geqslant 1$, $n = 2k+2$, $x = (\tilde{x}, x_n)$, $\tilde{x} = (x_1, \ldots, x_{n-1})$.

We know that $\operatorname{supp} E_{2(k+1)} = \overline{V}_+$, and so we can apply the method of descent to find a fundamental solution E_{2k+1} from the fundamental solution $E_{2(k+1)}$ of the operator $\square_{2(k+1)} = \square_{2k+1} - \partial^2/\partial x_n^2$. Let $\gamma \in C_0^\infty(E^{2k+1})$, and let $\{\eta_\nu\}$ be a sequence convergent to 1 in E^1 [14]. Taking

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[12] Notice that inside the cone V_+, E_n can be written in the form

$$E_n[\varphi] = \frac{1\cdot 3\cdot \ldots \cdot(2m-3)}{2^{m-1}} C_n \int\limits_0^\infty s^{-\frac{1}{2}(n-1)} \left(\int\limits_{E^n} \frac{\varphi(\sqrt{s+|x|^2}, x)}{2\sqrt{s+|x|^2}}\, dx \right) ds$$

$$= \frac{1\cdot 3\cdot \ldots \cdot(2m-3)}{2^{m-1}} C_n \int\limits_{E^{n+1}} \frac{\varphi(t, x)}{(t^2-|x|^2)^{\frac{1}{2}(n-1)}}\, dt\,dx \quad \text{for } \varphi \in C_0^\infty(V_+).$$

[13] By $\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^0$ we mean the identity operator, $\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^1$ stands for $\frac{1}{2t}\frac{\partial}{\partial t}$ and $\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^k = \frac{1}{2t}\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1}$ for $k = 1, 2, \ldots$

[14] $\{\eta_\nu\}$ is convergent to 1 in the sense that $\eta_\nu \in C_0^\infty(E^1)$, $\eta_\nu(x) = 1$ for $|x| < \nu$ and for every $\alpha \in N_0$ there exists a constant C_α such that $|D^\alpha \eta_\nu(x)| < C_\alpha$ for $x \in E^1$ ($\nu = 1, 2, \ldots$). The method of descent is presented in 24.4 in [3].

the fundamental solution E_n, $n = 2(k+1)$, in the form (22) we obtain

(24) $$E_{2k+1}[\gamma] = \lim_{\nu\to\infty} E_{2k+2}[\gamma(t, \tilde{x})\eta_\nu(x_n)]$$
$$= C_{2k+2} \int_{E^{2k+1}} \left(\int_{E^1} \left(\int_{E^1} \frac{Y(t-|x|)}{\sqrt{t^2-|x|^2}} \frac{\partial}{\partial t}\left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1} \frac{\gamma(t, \tilde{x})}{2t}\right) dt\right) dx_n\right) d\tilde{x}.$$

Note that

$$\int_{E^1} \frac{Y(t-|x|)}{\sqrt{t^2-|x|^2}}\, dx_n = 2Y(t-|\tilde{x}|) \int_0^{\sqrt{t^2-|\tilde{x}|^2}} \frac{dx_n}{\sqrt{t^2-|\tilde{x}|^2-x_n^2}} = \pi Y(t-|\tilde{x}|).$$

Hence (24) takes the form

$$E_{2k+1}[\gamma] = C_{2(k+1)}\pi \int_{E^{2k+1}} \left(\int_{E^1} Y(t-|\tilde{x}|) \frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1} \left(\frac{\gamma(t, \tilde{x})}{2t}\right) dt\right) d\tilde{x}$$
$$= -\pi C_{2(k+1)} \int_{E^{2k+1}} \left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1} \frac{\gamma(t, \tilde{x})}{2t}\Bigg|_{t=|\tilde{x}|}\right) d\tilde{x}.$$

Thus we have proved

THEOREM 4. *There exists a fundamental solution of the operator* $\Box_n$ *for* $n = 2k+1$, $k \in N$, $k \geq 1$, *of the form*:

(25) $$E_n[\gamma] = C_n \int_{E^n} \left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1} \frac{\gamma(t, x)}{2t}\Bigg|_{t=|x|}\right) dx \quad \text{for } \gamma \in C_0^\infty(E^{n+1}),$$

where C_n *is a suitable constant.* $E_n \in S'(E^{n+1})$ *and* $\operatorname{supp} E_n \subset \partial \overline{V}_+$. E_n *is G-invariant and it is a unique fundamental solution in the class of distributions having support in* $\overline{E_+^{n+1}}$.

Remark 3. The constants C_n occurring in formulae (21) and (25) satisfy the relation

(26) $$C_{2k+1} = -\pi C_{2(k+1)}.$$

3. An application of the method of descent to the computation of the constants C_n, $n > 3$. To find the constants C_n in formulae (21) and (25), we apply once more the method of descent, this time from E_n, n odd. Let $n = 2k+1$, $\tilde{x} = (x_1, \ldots, x_{n-1})$, $\lambda \in C_0^\infty(E^n)$ and let $\{\eta_\nu\}$ be a sequence convergent to 1. Because $\operatorname{supp} E_{2k+1} \subset \partial \overline{V}_+$, we obtain from (25):

$$E_{2k}[\lambda] = \lim_{\nu\to\infty} E_{2k+1}[\lambda(t,\tilde{x})\eta_\nu(x_n)]$$
$$= C_n \int_{E^n} \left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\lambda(t,\tilde{x})}{2t}\right)\bigg|_{t=|x|}\right) dx$$
$$= C_n \int_0^\infty \left(\left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1} \int_{|x|=r} \frac{\lambda(t,\tilde{x})}{2t}\, d\sigma_x\right)\bigg|_{t=r}\right) dr.$$

Noting that

$$\int_{|x|=r} \lambda(t,\tilde{x})\, d\sigma_x = 2r \int_{|\tilde{x}|\leqslant r} \frac{\lambda(t,\tilde{x})}{\sqrt{r^2-|\tilde{x}|^2}}\, d\tilde{x},$$

we obtain

$$E_{2k}[\lambda] = C_{2k+1} \int_0^\infty r\left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{1}{t}\int_{|\tilde{x}|\leqslant r} \frac{\lambda(t,\tilde{x})}{\sqrt{r^2-|\tilde{x}|^2}}\, d\tilde{x}\right)\bigg|_{t=r}\right) dr$$
$$= C_{2k+1} \int_0^\infty r\left(\int_{|\tilde{x}|\leqslant r} \frac{1}{\sqrt{r^2-|\tilde{x}|^2}}\left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1}\frac{\lambda(t,\tilde{x})}{t}\right)\bigg|_{t=r} d\tilde{x}\right) dr$$
$$= C_{2k+1} \int_{E^{2k+1}} \left(r\frac{Y(r-|\tilde{x}|)}{\sqrt{r^2-|\tilde{x}|^2}}\left(\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{k-1}\frac{\lambda(t,\tilde{x})}{t}\right)\bigg|_{t=r}\right) dr\, d\tilde{x} = C_{2k+1}\frac{E_{2k}[\lambda]}{C_{2k}}.$$

Therefore

$$C_{2k+1} = C_{2k} \quad \text{for } k\in N,\ k\geqslant 1. \tag{27}$$

It is well known that $C_3 = 1/2\pi$, $C_2 = 1/2\pi$.

Hence and from (26) and (27) we derive by induction the explicit formulae for C_n:

$$C_{2k} = C_{2k+1} = \tfrac{1}{2}(-1)^{k+1}\frac{1}{\pi^k} \quad (k=1,2,\ldots).$$

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[6] — Appendix to [3], Part I.

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