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On entire functions represented by Dirichlet series

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1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{1.1}$$

$\lambda_{n+1} > \lambda_n$, $\lambda_1 > 0$, $\lim_{n\to\infty} \lambda_n = \infty$; let σ_c and σ_a be the abcissae of convergence and absolute convergence of $f(s)$. If $\sigma_c = \sigma_a = \infty$, $f(s)$ represents an entire function. Throughout this paper we assume that $\sigma_c = \sigma_a = \infty$.

Let $M(\sigma) = \max_{-\infty<t<\infty} |f(\sigma+it)|$ and $\mu(\sigma) = \max_{n\geqslant 1} |a_n e^{s\lambda_n}|$. If $\nu(\sigma)$ denotes the value of n, for which $\mu(\sigma) = |a_n| e^{\sigma\lambda_n}$, we call it the rank of the maximum term. If there are more than one such values of n, we consider as rank the greatest of them.

J. F. Ritt ([1], p. 77) has introduced the linear order of an entire function by defining it as the superior limit of $\log\log M(\sigma)/\sigma$ as $\sigma\to\infty$. C. Y. Yu ([2], p. 69) has shown subsequently that if $\overline{\lim}_{n\to\infty} \log n/\lambda_n = D < \infty$, then

$$\overline{\lim_{\sigma\to\infty}} \log\log M(\sigma)/\sigma = \overline{\lim_{n\to\infty}} \lambda_n \log\lambda_n \Big/ (\log 1/|a_n|) \tag{1.2}$$

and that

$$\log\mu(\sigma) = \log\mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)} dt. \tag{1.3}$$

Further, Yu ([2], p. 73) has proved that if

$$\overline{\lim_{n\to\infty}} \log n/\lambda_n = 0, \tag{1.4}$$

then $\log M(\sigma) \sim \log\mu(\sigma)$. From the above results it can easily be shown that if (1.4) is true,

$$\varrho = \overline{\lim_{\sigma\to\infty}} \log\log M(\sigma)/\sigma = \overline{\lim_{\sigma\to\infty}} \log\log\mu(\sigma)/\sigma$$
$$= \overline{\lim_{\sigma\to\infty}} \log\lambda\nu(\sigma)/\sigma = \overline{\lim_{n\to\infty}} \lambda_n \log\lambda_n \Big/ (\log 1/|a_n|).$$

If $\lambda = \varliminf_{\sigma\to\infty} \log\log M(\sigma)/\sigma$, then λ is the lower linear order of $f(s)$ and $\lambda = \varliminf_{\sigma\to\infty} \log\log \mu(\sigma)/\sigma = \varliminf_{\sigma\to\infty} \log \lambda\nu(\sigma)/\sigma$.

In the present paper we have obtained some properties of the derivatives of the maximum modulus on the lines parallel to the imaginary axis and certain relations between the maximum term and its rank. The results have been classified into two sections.

Section I

2. Clunie ([3], p. 175) has proved the following result:

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be an entire function and a be a positive number such that $(n+1/n)^a |a_n/a_{n+1}|$ is ultimately a steadily increasing function of n; then

$$M(r) < \big(1+o(1)\big)\Gamma(1+a)\,a^{-a-1}e^a\mu(r)\nu(r)\,. \tag{2.1}$$

We extend this result to entire functions of the kind (1.1).

THEOREM 1. *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ $(\lambda_{n+1} > \lambda_n,\ \lambda_n \to \infty)$ and*

$$\varlimsup_{n\to\infty} \log n/\log \lambda_n = \frac{D}{d}\,;$$

et a be a positive number such that $\{(\lambda_{n+1}/\lambda_n)^a |a_n/a_{n+1}|\}$ is ultimately a steadily increasing function of n; then

$$M(\sigma) < [1+o(1)]e^a a^{-a}[a^{-D}\Gamma(a+D+1)\lambda_\nu^D - a^{-d+1}\Gamma(d+a)\lambda_\nu^d]\mu(\sigma)\,. \tag{2.2}$$

Proof. Following Clunie, we consider a function

$$\varphi_a(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-a} e^{s\lambda_n}\,. \tag{2.3}$$

If $f(s)$ satisfies the conditions stated in Theorem 1, then $\varphi_a(s)$ is an entire function for which all integers beyond some definite integers are central indices. Let $\nu = \nu(\sigma, \varphi_a)$ be the rank of the maximum term of (2.3); then

$$\frac{|a_n| e^{\sigma\lambda_n}}{\lambda_n^a} \leqslant \frac{|a_\nu| e^{\sigma\lambda_\nu}}{\lambda_\nu^a} \quad \text{for all } n \text{ and } a$$

or

$$\frac{|a_n| e^{\sigma\lambda_n}}{|a_\nu| e^{\sigma\lambda_\nu}} \leqslant \left(\frac{\lambda_n}{\lambda_\nu}\right)^a.$$

For any value of ν, R can be so chosen that for all n

$$\lambda_n^a e^{-\lambda_n R} \leqslant \lambda_\nu^a e^{-\lambda_\nu R}, \quad \text{i.e.} \quad \left(\frac{\lambda_n}{\lambda_\nu}\right)^a \leqslant e^{-(\lambda_\nu - \lambda_n)R},$$

and hence

$$\frac{|a_n|\, e^{\sigma\lambda_n} e^{-\lambda_n R}}{|a_\nu|\, e^{\sigma\lambda_\nu} e^{-\lambda_\nu R}} \leqslant \left(\frac{\lambda_n}{\lambda_\nu}\right)^a e^{-(\lambda_n-\lambda_\nu)R} \leqslant 1\,.$$

Hence $\nu(\sigma, \varphi_a) = \nu(\sigma - R, f) = \nu(-R, F_a)$, where

$$F_a(s) = \sum_{n=1}^{\infty} \lambda_n^a e^{s\lambda_n}\,.$$

Also,

$$\mu(\sigma - R, f) = |a_\nu|\, e^{\lambda_\nu(\sigma - R)}$$

$$= \frac{|a_\nu|}{\lambda_\nu^a} \lambda_\nu^a e^{\sigma\lambda_\nu} e^{-\lambda_\nu R} = \mu(\sigma, \varphi_a)\, \mu(-R, F_a)\,.$$

Now

$$\frac{\sum_n |a_n|\, e^{\lambda_n(\sigma - R)}}{|a_\nu|\, e^{\lambda_\nu(\sigma-R)}} \leqslant \frac{\sum_n \lambda_n^a e^{-\lambda_n R}}{\lambda_\nu^a e^{-\lambda_\nu R}}$$

$$= \frac{-\int_0^\infty \frac{d}{dt}(t^a e^{-tR})\, n(t)\, dt}{\lambda_\nu^a e^{-\lambda_\nu R}}$$

where $n(t)$ denotes the number of λ's $\leqslant t$. Since

$$\varlimsup_{n\to\infty} \frac{\log n}{\log \lambda_n} = \frac{D}{d} = \varlimsup_{t\to\infty} \frac{\log^+ n(t)}{\log t}\,,$$

we have for an arbitrary $\varepsilon > 0$ and sufficiently large $T > 0$

$$-\int_0^\infty \frac{d}{dt}(t^a e^{-tR})\, n(t)\, dt$$

$$= -\left[\int_0^\infty n(t)\, \{a t^{a-1} e^{-tR} - R t^a e^{-tR}\}\, dt\right]$$

$$= R \int_0^\infty n(t)\, t^a e^{-tR} dt - a \int_0^\infty n(t)\, t^{a-1} e^{-tR} dt$$

$$\leqslant R \int_0^T e^{-tR} t^a [n(t) - t^{D+\varepsilon}]\, dt + R \int_0^\infty t^{D+a+\varepsilon} e^{-tR} dt -$$

$$- a \int_0^T e^{-tR} t^{a-1} [n(t) - t^{d-\varepsilon}]\, dt - a \int_0^\infty t^{d+a-1-\varepsilon} e^{-tR} dt$$

$$\leqslant O(1) + R\Gamma(D + a + 1 + \varepsilon) R^{-D-a-1-\varepsilon} - a\Gamma(d + a - \varepsilon) R^{-d-a+\varepsilon}\,.$$

Hence

$$\frac{\sum_{n=1}^{\infty}|a_n|e^{\lambda_n(\sigma-R)}}{|a_\nu|e^{\lambda_\nu(\sigma-R)}} \leqslant \frac{R^{-a-D-\varepsilon}\Gamma(a+D+\varepsilon+1)-aR^{-d-a+\varepsilon}\Gamma(d+a-\varepsilon)}{\lambda_\nu^a e^{-\lambda_\nu R}}.$$

The denominator on the right-hand side is maximum for $a/\lambda_\nu - R = 0$ and hence, majorizing the right-hand side by putting $R = a/\lambda_\nu$, we get

$$\text{R.H.S.} \leqslant \frac{(a/\lambda_\nu)^{-a-D-\varepsilon}\Gamma(a+D+\varepsilon+1)-a(a/\lambda_\nu)^{-d-a+\varepsilon}\Gamma(d+a-\varepsilon)}{\lambda_\nu^a e^{-a}}$$

$$= e^a a^{-a}[a^{-D-\varepsilon}\Gamma(a+D+\varepsilon+1)\lambda_\nu^{D+\varepsilon} - a^{-d+1+\varepsilon}\Gamma(d+a-\varepsilon)\lambda_\nu^{d-\varepsilon}].$$

Thus, making $\varepsilon \to 0$, we get the result.

COROLLARY. *If $D = d$, then*

$$M(\sigma) < [1+o(1)]e^a a^{-a} a^{-D} D\Gamma(a+D)\mu(\sigma)\lambda_\nu^D .$$

THEOREM 2. *Let $f(s)$ be an integral function as defined in* (1.1) *and let $\mu(\sigma)$ and $\nu(\sigma)$ denote the maximum term and its rank. If $\sigma_1 < \sigma_2$, then*

$$e^{\lambda_{\nu(\sigma_1)}(\sigma_2-\sigma_1)} \leqslant \frac{\mu(\sigma_2)}{\mu(\sigma_1)} \leqslant e^{\lambda_{\nu(\sigma_2)}(\sigma_2-\sigma_1)}.$$

Proof. We have from (1.3)

$$\lambda_{\nu(\sigma_1)}(\sigma_2-\sigma_1) \leqslant \log\mu(\sigma_2) - \log\mu(\sigma_1) \leqslant \lambda_{\nu(\sigma_2)}(\sigma_2-\sigma_1),$$

and hence the result.

COROLLARY. *If $f(s)$ is not an exponential polynomial and $k > 1$, then* $\lim_{\sigma\to\infty} \mu(\sigma)/\mu(k\sigma) = 0$.

3. It is known [4] that

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \leqslant 1 - \lambda/\varrho,$$

where λ and ϱ are respectively the linear lower order and the linear order of $f(s)$. For $\lambda = 0$ the inequality reduces to

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \leqslant 1.$$

This inequality cannot be further improved because, *if $f(s)$ is an entire function as defined in* (1.1) *and* $\log\lambda_{\nu(\sigma)} = o(\log\sigma)$ *for an infinite sequence of values of σ, then*

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} = 1.$$

The result follows on putting $\chi(\sigma) = \sigma\lambda_{\nu(\sigma)}$ in the following lemma of Shah ([5], p. 80).

LEMMA. *If $\chi(t)$ is a positive monotone increasing function, continuous almost everywhere and $\chi(t) \geqslant 1$ for $t \geqslant t_0$ and if $\varliminf_{\sigma\to\infty} \log\chi(\sigma)/\log\sigma = a$, then for $\sigma_0 \leqslant \sigma$*

$$\varlimsup_{\sigma\to\infty} [\chi(\sigma)]^{-1} \int_{\sigma_0}^{\sigma} \frac{\chi(t)}{t} dt \geqslant 1/a .$$

4. We have proved in an earlier paper [4] the following result: For $0 \leqslant \lambda,\ \varrho \leqslant \infty$,

$$\varlimsup_{\sigma\to\infty} \frac{\log \dfrac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)}}{\sigma} = \frac{\varrho}{\lambda} .$$

Here we give a few applications of the above result. In what follows k denotes a positive integer and A and A' two positive constants:

(1) If $\mu(\sigma, f^{(k)}) \leqslant A\mu(\sigma, f)e^{k\sigma}$ for all $\sigma > \sigma_0$, then either $\varrho < 1$ or $\varrho = 1$ and $\varlimsup e^{-\sigma}\lambda_{\nu(\sigma)} \leqslant A^{1/k}$.

(2) If $\mu(\sigma, f^{(k)}) \geqslant A'\mu(\sigma, f)e^{k\sigma}$, for $\sigma > \sigma_0$, then either $\lambda > 1$ or $\lambda = 1$ and $\varliminf e^{-\sigma}\lambda_{\nu(\sigma, f^k)} \geqslant A'^{1/k}$.

Let $\Phi(\sigma)$ be a function, non-decreasing and positive for $\sigma > \sigma_0$ and such that $\log\Phi(\sigma) = o(\sigma)$. Then:

(3) If $\mu(\sigma, f^{(1)}) \geqslant 1/\Phi(\sigma)\cdot\mu(\sigma, f)e^{\sigma}$, or in general if $\mu(\sigma, f^{(k+1)}) \geqslant 1/\Phi(\sigma)\cdot\mu(\sigma, f^{(k)})e^{\sigma}$, for a sequence of values of $\sigma\to\infty$, then $\varrho \geqslant 1$. If the hypothesis holds for all values of $\sigma > \sigma_0$, then $\lambda \geqslant 1$.

(4) If $\mu(\sigma, f^{(1)}) \leqslant \Phi(\sigma)\mu(\sigma, f)e^{\sigma}$, or in general if $\mu(\sigma, f^{(k+1)}) \leqslant \leqslant \Phi(\sigma)\mu(\sigma, f^{(k)})e^{\sigma}$, for a sequence of values of $\sigma\to\infty$, then $\lambda \leqslant 1$. If the hypothesis is true for all values of $\sigma > \sigma_0$, then $\varrho \leqslant 1$.

(5) If $e^{\sigma}/\Phi(\sigma) \leqslant \dfrac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)} \leqslant \Phi(\sigma)e^{\sigma}$ or in general if

$$\frac{1}{\Phi(\sigma)} e^{\sigma} \leqslant \frac{\mu(\sigma, f^{(k+1)})}{\mu(\sigma, f^{(k)})} \leqslant \Phi(\sigma)e^{\sigma} ,$$

for $\sigma > \sigma_0$, then $\varrho = \lambda = 1$.

(6) If $\varrho < 1$, then for $\sigma > \sigma_0$,

$$\mu(\sigma, f) > e^{-\sigma}\Phi(\sigma)\mu(\sigma, f^{(1)}) > \ldots > [\Phi(\sigma)e^{-\sigma}]^k\mu(\sigma, f^{(k)}) .$$

(7) If $\lambda > 1$, then for $\sigma > \sigma_0$,

$$\mu(\sigma, f) < \frac{1}{\Phi(\sigma)e^{\sigma}}\mu(\sigma, f^{(1)}) < \ldots < \left[\frac{1}{\Phi(\sigma)e^{\sigma}}\right]^k \mu(\sigma, f^{(k)}) .$$

(8) If $\lambda = 1$ and $\varliminf_{\sigma\to\infty} \lambda_{\nu(\sigma)} e^{-\sigma} > 1$, then for $\sigma > \sigma_0$,

$$\mu(\sigma, f) < \Phi(\sigma)\mu(\sigma, f^{(1)})e^{-\sigma} < \dots < [\Phi(\sigma)e^{-\sigma}]^k \mu(\sigma, f^{(k)}) .$$

(9) If $\varrho > 1$, there is a sequence of numbers σ tending to infinity for which

$$\mu(\sigma, f) < \frac{1}{\Phi(\sigma)e^{\sigma}} \mu(\sigma, f^{(1)}) < \dots < \left[\frac{1}{\Phi(\sigma)e^{\sigma}}\right]^k \mu(\sigma, f^{(k)}) .$$

(10) If $\lambda < 1$, there is a sequence of numbers σ tending to infinity for which

$$\mu(\sigma, f) > \Phi(\sigma)\mu(\sigma, f^{(1)})e^{-\sigma} > \dots > [\Phi(\sigma)e^{-\sigma}]^k \mu(\sigma, f^{(k)}) .$$

Section II

5. Doetsch has shown that $\log M(\sigma)$ is an increasing convex function of σ. The result implies that $\log M(\sigma)$ is differentiable almost everywhere with an increasing derivative; the set of points where the left-hand derivative is less than the right-hand derivative is of measure zero. This enables us to write $\log M(\sigma)$ in the following form:

$$\log M(\sigma) = \log M(\sigma_0) + \int_{\sigma_0}^{\sigma} M'(t)/M(t)\, dt \tag{5.1}$$

for an arbitrary σ_0.

This integral representation of $\log M(\sigma)$ helps us in deriving interesting properties of $\log M(\sigma)$. Here we prove the existence of a linear proximate order under certain conditions. We first prove the following lemmas:

LEMMA 1. *If $\sigma > \sigma_0$, $\varepsilon > 0$ and $f(s)$ is not an exponential polynomial, then*

$$M'(\sigma) > \frac{M(\sigma)\log M(\sigma)}{(1+\varepsilon)\sigma}, \tag{5.2}$$

where $\varepsilon = \varepsilon(\sigma) \to 0$ as $\sigma \to \infty$.

Proof. Writing for the left-hand derivative of $\log M(\sigma)$, we have

$$M'(\sigma)/M(\sigma) = d/d\sigma \log M(\sigma) \geqslant \frac{\log M(\sigma) - \log M(\sigma_1)}{\sigma - \sigma_1}$$
$$> \log M(\sigma)/(1+\varepsilon)\sigma ,$$

where $\sigma_1 < \sigma$ and $\varepsilon = \varepsilon(\sigma) \to 0$ as $\sigma \to \infty$.

LEMMA 2.

$$\varlimsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \varlimsup_{\sigma\to\infty} \frac{\log\{M'(\sigma)/M(\sigma)\}}{\sigma} . \tag{5.3}$$

Proof. From (5.1) we have

$$\log M(\sigma) \leqslant \log M(\sigma_0) + (\sigma - \sigma_0) M'(\sigma)/M(\sigma) ,$$

and therefore

$$\varlimsup_{\sigma\to\infty} \log\log M(\sigma)/\sigma \leqslant \varlimsup_{\sigma\to\infty} 1/\sigma \cdot \{\log M'(\sigma)/M(\sigma)\} . \tag{5.4}$$

Moreover, for an arbitrary fixed $k > 0$

$$\log M(\sigma + k) = \log M(\sigma) + \int_{\sigma}^{\sigma+k} \frac{M'(t)}{M(t)}\, dt \geqslant k \frac{M'(\sigma)}{M(\sigma)} ,$$

and therefore

$$\varlimsup_{\sigma\to\infty} \frac{\log\log M(\sigma + k)}{\sigma} \geqslant \varlimsup_{\sigma\to\infty} \frac{\log\{M'(\sigma)/M(\sigma)\}}{\sigma} . \tag{5.5}$$

Combining (5.4) and (5.5) we get the result.

Lemma 3.

$$\varliminf_{\sigma\to\infty} \frac{M'(\sigma)}{M(\sigma)\log M(\sigma)} \leqslant \lambda \leqslant \varrho \leqslant \varlimsup_{\sigma\to\infty} \frac{M'(\sigma)}{M(\sigma)\log M(\sigma)} .$$

This follows from the lemma of Shah already mentioned.

We now show the existence of linear proximate order for the class of functions for which

$$\varliminf_{\sigma\to\infty} \frac{M'(\sigma)}{M(\sigma)\log M(\sigma)} = \varlimsup_{\sigma\to\infty} \frac{M'(\sigma)}{M(\sigma)\log M(\sigma)} . \tag{A}$$

Theorem. *Corresponding to every entire function of the kind* (1.1) *satisfying* (A), *there exists a function* $\varrho(x)$ *which satisfies the following conditions*:

(i) $\varrho(x)$ *is continuous for* $x > x_0$ *and* $\varlimsup_{x\to\infty} \varrho(x) = \varrho$.

(ii) $\varrho(x)$ *is differentiable almost everywhere except at the end-points of adjacent intervals, where it possesses left-hand and right-hand derivatives.*

(iii) $\varlimsup_{x\to\infty} e^{-x\varrho(x)} \log M(x) = 1$.

(iv) $\varlimsup_{x\to\infty} x\varrho'(x) = 0$.

Proof. Consider $F(x) = \log\log M(x)/x$. Properties (i), (ii) and (iii) are obviously satisfied. Differentiating with respect to x, we get

$$F'(x) = -\frac{1}{x^2} \log\log M(x) + \frac{M'(x)}{M(x)\log M(x)} \cdot \frac{1}{x}$$

or

$$xF'(x) = \left[\frac{M'(x)}{M(x)\log M(x)} - \frac{\log\log M(x)}{x}\right]$$

and since $M'(x)/M(x)\log M(x)$ and $F(x)$ have the same limit as x tends to infinity the result follows.

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References

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