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An integral formula for submanifolds and its application

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1. Introduction. Let R^m be an m-dimensional Riemannian manifold with covariant derivative D, and $x\colon M^n \to R^m$ an immersion of an n-dimensional manifold M^n into R^m. A vector field $\boldsymbol{X}$ in R^m over M^n is called a *concurrent vector field* if we have $dx + DX = 0$, where dx denotes the differential of the immersion x [3].

The main purpose of this note is to derive an integral formula for the submanifold M^n and to show that if a closed manifold M^n immersed in R^m admits a concurrent vector field, then M^n is not a minimal submanifold of R^m. The latter result generalizes a result of Chern and Hsiung on minimal submanifolds in euclidean space [2].

2. Preliminaries. Let M^n be an n-dimensional manifold immersed in an m-dimensional Riemannian manifold R^m with covariant derivative D. We choose a local field of orthonormal frames $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ in R^m such that, restricted to M^n, the vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are tangent to M^n (and, consequently, the remaining vectors $\boldsymbol{e}_{n+1}, \ldots, \boldsymbol{e}_m$ are normal to M^n). We shall make use of the following convention on the ranges of indices:

$$1 \leqslant A, B, C, \ldots, \leqslant m; \quad 1 \leqslant i, j, k, \ldots, \leqslant n; \quad n+1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant m,$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of R^m chosen above, let $\omega^1, \ldots, \omega^m$ be the field of dual frames. Then the structure equations of R^m are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \quad \omega^A_B + \omega^B_A = 0, \tag{1}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \varphi^A_B, \quad \varphi^A_B = \tfrac{1}{2} \sum K^A_{BCD}\, \omega^C \wedge \omega^D, \tag{2}$$

$$K^A_{BCD} + K^A_{BDC} = 0.$$

The covariant derivative DY of a vector field $\boldsymbol{Y} = \sum Y_A \boldsymbol{e}_A$ is given by

$$DY = \sum DY_A \otimes \boldsymbol{e}_A, \tag{3}$$

where

(4) $$DY_A = dY_A + \sum Y_B \omega_{BA}.$$

For the vectors e_A themselves equation (3) gives

(5) $$De_A = \sum \omega_{AB} \otimes e_B.$$

We restrict these forms to M^n. Then

(6) $$\omega^a = 0.$$

Since $0 = d\omega^a = -\sum \omega_i^a \wedge \omega^i$, by Cartan's lemma we may write

(7) $$\omega_i^a = \sum h_{ij}^a \omega^j, \quad h_{ij}^a = h_{ji}^a.$$

From these formulas, we obtain

(8) $$d\omega^i = -\sum \omega_j^i \wedge \omega^j, \quad \omega_j^i + \omega_i^j = 0,$$

(9) $$d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad \Omega_j^i = \tfrac{1}{2} \sum R_{jkl}^i \omega^k \wedge \omega^l,$$

$$R_{jkl}^i = K_{jkl}^i + \sum_a (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a).$$

We call $H = \frac{1}{n} \sum_a \left(\sum_i h_{ii}^a\right) e_a$ the *mean curvature vector* of M^n in R^m. It is a well-defined normal vector field over M^n. An immersion is said to be *minimal* if its mean curvature vector vanishes identically, i.e., if $\sum_i h_{ii}^a = 0$ for all a.

3. An integral formula and its applications. Let f be a smooth function on M^n. By grad f or ∇f we mean $\nabla f = \sum f_i e_i$, where f_i are given by $df = \sum f_i \omega^i$. Let X be a concurrent vector field in R^m over M^n. Then X can be decomposed into two components

(10) $$X = X_t + X_n,$$

where X_t is tangent to M^n and X_n normal to M^n. In the following, let $\langle , \rangle$ denote the scalar product in R^m. For simplicy, we define

(11) $$\varrho = \langle X, X \rangle, \quad \text{and} \quad \varrho_n = \langle X_n, X_n \rangle.$$

THEOREM 1. *Let x: $M^n \to R^m$ be an immersion of an oriented closed manifold of dimension n into a Riemannian manifold R^m of dimension m, and X be a concurrent vector field in R^m over M^n. Then for any smooth function f, we have*

(12) $$\int_{M^n} \varrho^{c-1}\{\langle \boldsymbol{X}, \nabla f\rangle \varrho - 2cf\varrho_n + (2c+n)f\varrho + n\langle X, H\rangle f\varrho\} dV = 0, \; c \geqslant 0,$$

where

$$dV = \omega^1 \wedge \ldots \wedge \omega^n$$

is the volume element of M^n.

Proof. Let $\boldsymbol{X}$ be a concurrent vector field in R^m over M^n. Put

(13) $$\Xi = \sum (-1)^{j-1} \langle \boldsymbol{X}, \boldsymbol{e}_j\rangle \omega^1 \wedge \ldots \wedge \hat{\omega}^j \wedge \ldots \wedge \omega^n,$$

where $\wedge$ denotes the omitted term. Then it is easy to see that the form Ξ is a well-defined $(n-1)$-form on M^n. Using $dx + D\boldsymbol{X} = 0$, and a straightforward calculation we can verify easily that

(14) $$d\Xi = n(1 + \langle \boldsymbol{X}, \boldsymbol{H}\rangle) dV,$$

(15) $$df \wedge \Xi = \langle \boldsymbol{X}, \nabla f\rangle dV,$$

(16) $$d\varrho \wedge \Xi = 2(\varrho - \varrho_n) dV.$$

Therefore, by (14), (15) and (16), we obtain

(17) $$\begin{aligned} d(f\varrho^c \Xi) &= \varrho^c df \wedge \Xi + cf\varrho^{c-1} d\varrho \wedge \Xi + f\varrho^c d\Xi \\ &= \varrho^{c-1}\{\langle X, \nabla f\rangle \varrho - 2cf\varrho_n + (2c+n)f\varrho + n\langle X, H\rangle f\varrho\} dV. \end{aligned}$$

Hence, by integrating both sides of (17) and applying Stokes' theorem, we obtain (12). This completes the proof of the theorem.

Remark 1. If M^n is a hypersurface of Euclidean $(n+1)$-space E^{n+1}, $c = 0$, and f is an i-th mean curvature K_i, then Theorem 1 was proved by Amur [1].

THEOREM 2. *Let M^n $(n > 0)$ be a closed submanifold of a Riemannian manifold R^m. If there exists a concurrent vector field in R^m over M^n, then M^n is not a minimal submanifold of R^m.*

Proof. If there exists a concurrent vector field $\boldsymbol{X}$ in R^m over M^n, then by (14) and a straightforward, simple calculation, we see that the Laplacian $\Delta\varrho$ of ϱ is given by

(18) $$(\Delta\varrho) dV = d\Xi = n(1 + \langle X, H\rangle) dV.$$

Hence, if M^n is a minimal submanifold of R^m, then, by (18) we obtain $\Delta\varrho = n$. Therefore, by Hopf's lemma, we see that $n = 0$. This is a contradiction.

References

[1] K. Amur, *Vector forms and integral formulas for hypersurfaces in euclidean space,* J. Dif. Geometry 3 (1969), p. 111 - 123.

[2] S. S. Chern and C. C. Hsiung, *On the isometry of compact submanifolds in Euclidean space,* Math. Ann. 149 (1963), p. 278 - 285.

[3] K. Yano and B.-Y. Chen, *On the concurrent vector fields of immersed manifolds,* Kodai Math. Sem. Rep. 23 (1971), p 343-350.

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