ANNALES
POLONICI MATHEMATICI
XIX (1967)

On the approximate solutions of an abstract equation

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T. Ważewski established in [4] a theorem on the existence and uniqueness of solutions and the convergence of the successive approximations for equations considered in an abstract space.

In the present paper we discuss such equations and introduce the notion of approximate solutions. We give some estimations for those solutions. Applications to a special space and to some equations are also considered.

1. We introduce the following

ASSUMPTION H_1 (see [4]).

1° G is a partially ordered set, i.e. for some pairs of elements $u, v \in G$ a relation $u < v$ is defined in such a way that:

(a) $u < v$ exclude $u = v$,

(b) if $u < v$ and $v < w$, then $u < w$;

2° in G there exists a minimal element $0 \in G$, i.e. for any $u \in G$, $0 \leqslant u$ (we write $u \leqslant v$ if $u < v$ or $u = v$);

3° for any $u, v \in G$ a relation $u+v$ is defined and has the following properties:

(a) if $u, v \in G$, then $u+v \in G$, $u+v = v+u$, $u+0 = u$,

(b) if $u, v, w \in G$ and $u \leqslant v$, then $u+w \leqslant v+w$,

(c) if $u, v, w \in G$ and $u+v \leqslant w$, then $u \leqslant w$;

4° for any non-increasing sequence $\{u_n\}$, $u_n \in G$, $u_{n+1} \leqslant u_n$, $n = 1, 2, \ldots$ there exists a unique element $u \in G$ called the limit of the sequence $\{u_n\}$ (we write $u = \lim\limits_{n\to\infty} u_n$ or $u_n \searrow u$).

The limit has the following properties:

(a) $\lim\limits_{n\to\infty} u_n$ is invariant with respect to the change of the finite elements of the sequence $\{u_n\}$,

(b) if $u_n = u$, $n = 1, 2, \ldots$, then $\lim\limits_{n\to\infty} u_n = u$,

(c) if $u_n \searrow u$, $v_n \searrow v$ and $u_n \leqslant v_n$, then $u \leqslant v$,

(d) if $u_n \searrow u$, $v_n \searrow v$, then $u_n + v_n \searrow u + v$.

ASSUMPTION H_2. The function $a(u)$ is defined for $u \in \Delta \subset G$ and has the properties:

1° $0 \in \Delta$ and if $k \in \Delta$, then for any $u \leqslant k$, $u \in \Delta$;

2° $a(\Delta) \subset G$ ($a(\Delta)$ is the set of the values of the function $a(u)$ for $u \in \Delta$);

3° if $u, v \in \Delta$ and $u \leqslant v$, then $a(u) \leqslant a(v)$;

4° if $u_n \in \Delta$, $n = 1, 2, \ldots$, and $u_n \searrow u$, then $a(u_n) \searrow a(u)$;

5° $u = 0$ is only the solution in Δ of the equation $u = a(u)$.

DEFINITION. For any $u \in \Delta$ we define the sequence $\{a_n(u)\}$ of the iterations of the element u by the recurrent formula

$$a_0(u) = u\,, \quad a_{n+1}(u) = a\big(a_n(u)\big)\,, \quad \text{if} \quad a_n(u) \in \Delta\,,\ n = 1, 2, \ldots$$

2. We can write the following

LEMMA 1 (see [4]). *If Assumption* H_2 *is fulfilled and there exists a* $c \in \Delta$ *such that* $a(c) \leqslant c$, *then all iterations* $a_n(c)$, $n = 0, 1, \ldots$, *of the element* c *exist and*

$$a_{n+1}(c) \leqslant a_n(c) \leqslant c\,, \quad n = 0, 1, \ldots, \quad \text{and} \quad a_n(c) \searrow 0\,.$$

LEMMA 2 (see [4]). *If Assumption* H_2 *is fulfilled and there exist* $q \in \Delta$ *and* $b \in \Delta$ *such that*

$$q + a(b) \leqslant b\,,$$

then the equation

$$u = a(u) + q \tag{1}$$

has the solution $u = m(b, q) \leqslant b$, *which has the properties:*

1° $m(b, q) = \lim_{n\to\infty} b_n(b, q)$, *where* $b_0(b, q) = b$, $b_{n+1}(b, q) = q + a\big(b_n(b, q)\big)$, $n = 0, 1, \ldots,$

2° *if* $p \leqslant b$ *and* $p \leqslant q + a(p)$, *then* $p \leqslant m(b, q)$.

LEMMA 3. *If Assumption* H_2 *is fulfilled and* $q_n \in \Delta$, $q_{n+1} \leqslant q_n$, $n = 0, 1, 2, \ldots$, $b \in \Delta$ *and*

$$q_n + a(b) \leqslant b\,,$$

then the equation

$$u = a(u) + q_n \tag{2}$$

has a solution $u = m(b, q_n) \leqslant b$ *such that*

$$m(b, q_{n+1}) \leqslant m(b, q_n)\,, \quad n = 0, 1, 2, \ldots \tag{3}$$

Moreover, if $q_n \searrow q$, *then* $m(b, q_n) \searrow m(b, q)$, *and consequently if* $q_n \searrow 0$, *then* $m(b, q_n) \searrow 0$.

Proof. By Lemma 2

$$m(b, q_n) = \lim_{k\to\infty} b_k(b, q_n),$$

where

$$b_0(b, q_n) = b, \quad b_{k+1}(b, q_n) = q_n + a(b_k(b, q_n)), \quad n = 0, 1, 2, \ldots$$

Further, we obtain by induction

$$b_k(b, q_{n+1}) \leqslant b_k(b, q_n), \quad n, k = 0, 1, \ldots$$

Hence, if $k\to\infty$, we obtain relation (3). But relation (3) implies that there exists a $\lim_{n\to\infty} m(b, q_n) = \bar{u}$.

Now in view of the fact that $m(b, q_n)$ is the solution of equation (2), we obtain

$$\bar{u} = a(\bar{u}) + q,$$

and therefore, according to Lemma 2, $\bar{u} \leqslant m(b, q)$. But $q \leqslant q_n$, whence we have

$$m(b, q) \leqslant m(b, q_n) \quad \text{and} \quad m(b, q) \leqslant \bar{u}.$$

Finally we obtain $\bar{u} = m(b, q)$.

The last part of Lemma 3 follows immediately from Lemma 1.

LEMMA 4. *If Assumption* H_2 *is fulfilled,* $q \in \Delta$, $b \leqslant b'$, $b' \in \Delta$ *and*

$$q + a(b) \leqslant b, \quad q + a(b') \leqslant b',$$

then

$$m(b, q) \leqslant m(b', q).$$

Proof. From Lemma 2 we obtain

$$m(b, q) = \lim_{n\to\infty} b_n(b, q), \quad m(b', q) = \lim_{n\to\infty} b_n(b', q),$$

but it follows by induction that

$$b_n(b, q) \leqslant b_n(b', q),$$

whence we obtain the assertion of Lemma 4.

DEFINITION. $m(q)$ is called the *maximal solution of equation* (1) iff it satisfies this equation and for any solution $u(q)$ of (1) the inequality $u(q) \leqslant m(q)$ holds true.

LEMMA 5. *If Assumption* H_2 *is fulfilled,* $\Delta = G$, *and for any* $q \in G$ *the equation*

$$u = a(u) + q$$

has a maximal solution $m(q)$, $p \in G$, *and* $p \leqslant a(p) + q$, *then*

$$p \leqslant m(q).$$

Moreover, if $q \leqslant q'$, *then* $m(q) \leqslant m(q')$.

Proof. Now for any $b, b' \in G$ such that, $b \leqslant b'$ and $q+a(b) \leqslant b$, $q+a(b') \leqslant b'$, we have

$$m(b, q) \leqslant m(b', q) \leqslant m(q) .$$

Take $\bar{q} \in G$, $\bar{q} > p$ and $\bar{b} = m(\bar{q})$, where $m(\bar{q})$ is the maximal solution of the equation

$$u = a(u)+q+\bar{q} .$$

We see that $\bar{b} > p$ and $\bar{b} \geqslant a(\bar{b})+q$. By Lemma 2 we obtain

$$p \leqslant m(b, q) \leqslant m(q) .$$

Consider $m(q)$, $m(q')$; we now have

$$m(q) = a\big(m(q)\big)+q \leqslant a\big(m(q)\big)+q' .$$

Hence, by the first part of our lemma, we infer the relation $m(q) \leqslant m(q')$; thus Lemma 5 is proved completely.

3. We introduce

Assumption H_3 (see [4]). R is an abstract space such that

1° for some sequences $\{x_n\}$, $x_n \in R$, $n = 1, 2, \ldots$, there exist uniquely determined limits $\lim\limits_{n\to\infty} x_n = x$, $x \in R$; $\lim\limits_{n\to\infty} x_n$ is invariant with respect to the change of the finite elements of $\{x_n\}$ (the relation $\lim\limits_{n\to\infty} x_n = x$ will also be written as $x_n \to x$);

2° if $x_n = s \in R$, $n = 1, 2, \ldots$, then $\lim\limits_{n\to\infty} x_n = s$;

3° the function $r(x, y)$ is defined on the product $R \times R$ and has the following properties:

(a) $r(x, y) \in G$,

(b) $r(x, y) = 0$ iff $x = y$,

(c) for any $x, y, z \in R$

$$r(x, y) \leqslant r(x, z)+r(y, z);$$

4° for any $x^* \in R$ and $b \in G$ the sphere

$$S(x^*, b) = [x\colon\ x \in R,\ r(x, x^*) \leqslant b]$$

is a closed set;

5° the space R is complete in the following sense: if $c_n \in G$, $n = 1, 2, \ldots$, $c_n \searrow 0$ and for $\{x_n\}$, $x_n \in R$, $n = 1, 2, \ldots$, the Cauchy condition

$$r(x_n, x_{n+m}) \leqslant c_n , \qquad n, m = 1, 2, \ldots$$

is satisfied, then there exists a limit y of sequence $\{x_n\}$,

$$\lim_{n\to\infty} x_n = y , \qquad y \in R .$$

ASSUMPTION H_4. The function $f(x)$ is defined on the sphere $S(x^*, b) \subset R$, $x^* \in R$, $b \in \Delta$, and has the properties:

1° $f(x) \in R$;

2° for any $x, y \in S(x^*, b)$

$$r\big(f(x), f(y)\big) \leqslant a\big(r(x, y)\big) ,$$

where the function $a(u)$ satisfies Assumption H_2:

3° there exists a $q \in \Delta$ such that

$$r\big(x^*, f(x^*)\big) \leqslant q \quad \text{and} \quad q + a(b) \leqslant b .$$

Now we can formulate the following

THEOREM 1 ([4], [2]). *If Assumption* H_4 *is satisfied, then in the sphere* $S(x^*, b)$ *there exists a unique solution* $\bar{x}$ *of the equation*

$$x = f(x) . \tag{4}$$

ASSUMPTION H_5. Suppose that

1° the function $f(x)$ is defined for $x \in R$, $f(x) \in R$;

2° for any $x, y \in R$

$$r\big(f(x), f(y)\big) \leqslant a\big(r(x, y)\big) ,$$

where the function $a(u)$ satisfies Assumption H_2 with $\Delta = G$;

3° for any $q \in G$ the equation

$$u = a(u) + q \tag{5}$$

has a maximal solution $m(q)$.

THEOREM 2. *If Assumption* H_5 *is satisfied, then equation* (4) *has in* R *a unique solution* $\bar{x}$.

Proof. Let x^* be an arbitrarily fixed element of the space R. For any solution x of equation (4) we have

$$r(x, x^*) \leqslant r\big(f(x), f(x^*)\big) + r\big(x^*, f(x^*)\big) \leqslant a\big(r(x, x^*)\big) + r\big(x^*, f(x^*)\big) ;$$

hence by Lemma 5 we obtain

$$r(x, x^*) \leqslant m(q') \leqslant m(q) ,$$

where $q' = r\big(x^*, f(x^*)\big)$ and $m(q)$ is the maximal solution of equation (5). This means that all solutions of equation (4) are in the sphere $S(x^*, b)$, where $b = m(q)$. But in this sphere the assumptions of Theorem 1 are fulfilled, and therefore there exists only one solution of equation (4) in the whole space R.

4. Now let us consider two equations:

$$x = f(x) \quad \text{and} \quad y = g(y) . \tag{6}$$

We want to give the estimation of $r(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are solutions of equations (6).

We can formulate

THEOREM 3. *If Assumption* H_4 *is satisfied and*

1° $\bar{x}, \bar{y}$ *are solutions of equations* (6) *such that* $r(\bar{x}, \bar{y}) \leqslant b$, $\bar{x} \in S(x^*, b)$, $\bar{y} \in S(x^*, b)$;

2° $r\big(f(\bar{y}), \bar{y}\big) \leqslant q$,

then

$$r(\bar{x}, \bar{y}) \leqslant m\Big(b, r\big(f(\bar{y}), g(\bar{y})\big)\Big) \leqslant m(b, q).$$

Proof. From the relations

$$\bar{x} = f(\bar{x}) \quad \text{and} \quad \bar{y} = g(\bar{y})$$

we obtain

$$r(\bar{x}, \bar{y}) = r\big(f(\bar{x}), g(\bar{y})\big) \leqslant r\big(f(\bar{x}), f(\bar{y})\big) + r\big(f(\bar{y}), g(\bar{y})\big)$$
$$\leqslant a\big(r(\bar{x}, \bar{y})\big) + r\big(f(\bar{y}), g(\bar{y})\big).$$

Now by Lemma 2 we obtain the assertion of Theorem 3.

THEOREM 4. *If Assumption* H_5 *is satisfied and*

1° $\bar{x}, \bar{y}$ *are solutions of equation* (6),

2° $r\big(f(\bar{y}), \bar{y}\big) \leqslant q$,

then

$$r(\bar{x}, \bar{y}) \leqslant m\Big(r\big(f(\bar{y}), g(\bar{y})\big)\Big) \leqslant m(q).$$

Proof. In view of the relations

$$\bar{x} = f(\bar{x}), \quad \bar{y} = g(\bar{y})$$

we obtain

$$r(\bar{x}, \bar{y}) \leqslant a\big(r(\bar{x}, \bar{y})\big) + r\big(f(\bar{y}), g(\bar{y})\big);$$

hence, making use of Lemma 5, we find the assertion of Theorem 4.

Theorems 3, 4 and Lemma 3 imply

THEOREM 5. *If the sequence of the equations*

$$y = g_n(y), \quad n = 1, 2, \ldots,$$

is such that for $n = 1, 2, \ldots$ *the assumptions of Theorem* 3 (*or Theorem* 4) *are satisfied,* $r\big(f(\bar{y}_n), \bar{y}_n\big) \leqslant q_n$ *and* $q_n \searrow 0$, *then*

$$\lim_{n\to\infty} \bar{y}_n = \bar{x}.$$

5. Now we shall again consider the equation

$$x = f(x). \tag{7}$$

We shall suppose that the function $f(x)$ satisfies Assumption H_4 (or H_5).

We introduce the following

DEFINITION. An element $x \in R$ is called an *ε-approximate solution* (shortly an *ε-solution*) of equation (7) iff the relation

$$r\big(x, f(x)\big) \leqslant \varepsilon, \quad \varepsilon \in G$$

holds true.

THEOREM 6. *If Assumption* H_4 *is satisfied and*

1° *x and y are, respectively, the ε_1-solution and the ε_2-solution of equation* (7),

2° $r(x, y) \leqslant b$, $x, y \in S(x^*, b)$, $\varepsilon_1 + \varepsilon_2 \leqslant q$,

then

$$r(x, y) \leqslant m(b, \varepsilon_1 + \varepsilon_2) \leqslant m(b, q).$$

Proof. By the definition of the ε-solution we get

$$r\big(x, f(x)\big) \leqslant \varepsilon_1, \quad r\big(y, f(y)\big) \leqslant \varepsilon_2,$$

and further

$$r(x, y) \leqslant r\big(x, f(x)\big) + r\big(f(x), f(y)\big) + r\big(f(y), y\big) \leqslant a\big(r(x, y)\big) + \varepsilon_1 + \varepsilon_2,$$

hence and by Lemma 2 we arrive at the assertion of Theorem 6.

Similarly we obtain the following

THEOREM 7. *If Assumption* H_5 *is satisfied and x and y are, respectively, the ε_1-solution and the ε_2-solution, then*

$$r(x, y) \leqslant m(\varepsilon_1 + \varepsilon_2).$$

Remark. From Lemma 3 it follows that if we have the sequence $\{x_n\}$ of ε_n-solutions of equation (7) and $\varepsilon_n \searrow 0$, then

$$\lim_{n\to\infty} x_n = \bar{x},$$

where $\bar{x}$ is the unique solution of equation (7).

6. Now we shall give some applications of the results obtained in previous paragraphs.

Let G_0 denote the set of non-negative functions $u(\cdot)$ defined for $t \in I = \langle t_0, T)$, $-\infty < t_0 \leqslant T \leqslant +\infty$, bounded and L-integrable in $\langle t_0, t_1\rangle$, for any $t_1 \in I$.

If $u, v \in G_0$, then we write $u < v$ iff $u(t) < v(t)$ for any $t \in I$.

It is easy to see that the set G_0 thus defined has all the properties listed in Assumption H_1.

Take an arbitrarily fixed Banach space B with the norm $\|\cdot\|$. Let R_0 denote the set of the functions x: $t \to x(t) \in B$, $t \in I$, bounded and B-integrable in $\langle t_0, t_1\rangle$ for any $t_1 \in I$. The B-integral is understood as the Bohner integral. Convergence in R_0 is understood as usual convergence with respect to the norm $\|\cdot\|$ at each point $t \in I$, i.e. if $x_n(\cdot) \in R_0$, $x(\cdot) \in R_0$ then

$$x_n(\cdot) \to x(\cdot) \quad \text{if} \quad \|x_n(t) - x(t)\| \to 0 \quad \text{for} \quad t \in I .$$

The function $r(x, y)$ is defined by relation

$$r(x, y) = \|x(t) - y(t)\| , \qquad t \in I .$$

We can easily verify [1] that the set R_0 thus defined has all the properties listed in Assumption H_3.

Now we shall assume that the functions $a(u)$ and $f(x)$ which appear in Assumptions $\mathrm{H}_2, \mathrm{H}_4, \mathrm{H}_5$ depend also on $t \in I$. We write $a(t, u(\cdot))$, and $f(t, x(\cdot))$ instead of $a(u)$ and $f(x)$.

Assumption H_6. Suppose that

1° the function $a(t, u(\cdot))$ is defined for $t \in I$, $u \in G_0$;

2° $u \in G_0$ and $v(t) = a(t, u(\cdot))$, $t \in I$, imply $v \in G_0$;

3° $u, v \in G_0$ and $u \leqslant v$ imply $a(t, u(\cdot)) \leqslant a(t, v(\cdot))$, $t \in I$;

4° $u_n \in G_0$, $n = 1, 2, \ldots$, and $u_n \searrow u \in G_0$ imply $a(t, u_n(\cdot)) \searrow a(t, u(\cdot))$;

5° for any $q \in G_0$ in G_0 the maximal solution $m(t, q)$ of the equation

$$u(t) = a(t, u(\cdot)) + q(t) \tag{8}$$

exists, and if $q = 0$, then $u = 0$.

Assumption H_7. Suppose that

1° the function $f(t, x(\cdot))$ is defined for $t \in I$ and $x \in R_0$;

2° $x \in R_0$ and $y(t) = f(t, x(\cdot))$, $t \in I$, imply $y \in R_0$;

3° for any $x, y \in R_0$

$$\|f(t, x(\cdot)) - f(t, y(\cdot))\| \leqslant a(t, \|x(\cdot) - y(\cdot)\|) ,$$

where the function $a(t, u(\cdot))$ satisfies Assumption H_6.

From Lemma 5 we obtain

Conclusion 1. *If Assumption* H_6 *is satisfied and*

$$p(t) \leqslant a(t, p(\cdot)) + q(t) , \qquad t \in I , \ p \in G_0 ,$$

then

$$p(t) \leqslant m(t, q) , \qquad t \in I ,$$

where $m(t, q)$ *is the solution of equation* (8).

However, Theorem 2 implies

Conclusion 2. *If Assumption* H_7 *is satisfied, then the equation*

$$x(t) = f(t, x(\cdot)) , \tag{9}$$

has a solution $\bar{x}(t)$ *unique in* R_0, *and for any fixed* $x^* \in R_0$ *we have the inequality*

$$\|\bar{x}(t)-x^*(t)\| \leqslant m(t, q^*) ,$$

where $q^*(t) = \|x^*(t)-f(t, x^*(\cdot))\|$ *and* $m(t, q^*)$ *is the solution of equation* (8) *with* q^* *instead of* q.

Now if we consider the equations

$$(10) \qquad x(t) = f(t, x(\cdot)) \quad \text{and} \quad y(t) = g(t, y(\cdot)) ,$$

then Theorem 4 implies

CONCLUSION 3. *If Assumption* H_7 *is fulfilled and*

1° $\bar{x}(\cdot), \bar{y}(\cdot)$ *are the solutions of equations* (10);

2° $\|f(t, \bar{y}(\cdot))-\bar{y}(t)\| \leqslant \bar{q}(t)$, $t \in I$, $\bar{q} \in G_0$,

then

$$\|\bar{x}(t)-\bar{y}(t)\| \leqslant m(t, q_1) \leqslant m(t, \bar{q}) ,$$

where

$$q_1(t) = \|f(t, \bar{y}(\cdot))-g(t, \bar{y}(\cdot))\| \leqslant \bar{q}(t) , \quad t \in I ,$$

and $m(t, q_1)$ *and* $m(t, \bar{q})$ *are the solutions of equation* (8) *with* q_1 *and* $\bar{q}$, *respectively, instead of* q.

At last, if we consider the ε-solutions of equation (9), then we have

CONCLUSION 4. *If Assumption* H_7 *is fulfilled and*

1° x_1 *and* $x_2 \in R_2$ *are, respectively, the* ε_1-*solution and the* ε_2-*solution, i.e.*

$$\|x_i(t)-f(t, x_i(\cdot))\| \leqslant \varepsilon_i , \quad \varepsilon_i \in G_0 , \ i = 1, 2;$$

2° $\varrho(t) = \|x_1(t)-f(t, x_1(\cdot))-x_2(t)-f(t, x_2(\cdot))\|$,

$$\varepsilon(t) = \varepsilon_1(t)+\varepsilon_2(t) , \quad t \in I ,$$

then

$$\|x_1(t)-x_2(t)\| \leqslant m(t, \varrho) \leqslant m(t, \varepsilon) ,$$

where $m(t, \varrho)$ *and* $m(t, \varepsilon)$ *are the maximal solutions of equation* (8) *with* ϱ *and* ε, *respectively, instead of* q.

Proof. We can write

$$x_1(t)-x_2(t) = f(t, x_1(\cdot))-f(t, x_2(\cdot))+\{[x_1(t)-f(t, x_1(\cdot))]-[x_2(t)-f(t, x_2(\cdot))]\}$$

whence

$$\|x_1(t)-x_2(t)\| \leqslant a(t, \|x_1(\cdot)-x_2(\cdot)\|)+\varrho(t)$$

and by Conclusion 1 we obtain

$$\|x_1(t)-x_2(t)\| \leqslant m(t, \varrho);$$

but we know that $\varrho \leqslant \varepsilon$ implies $m(t, \varrho) \leqslant m(t, \varepsilon)$, and therefore the proof of Conclusion 4 is finished.

7. The function $f(t, x(\cdot))$ appearing in equation (9) may take different forms. Let us give some examples:

$$\text{(a)} \qquad f(t, x(\cdot)) = \int_{t_0}^{t} F(t, s, x(s))\, ds + \varphi(t)\,,$$

$$\text{(b)} \qquad f(t, x(\cdot)) = \int_{t}^{T} F(t, s, x(s))\, ds + \varphi(t)\,,$$

$$\text{(c)} \qquad f(t, x(\cdot)) = F\Big(t, x(t), \int_{t_0}^{t} g(t, s, x(s))\, ds\Big)\,,$$

$$\text{(d)} \qquad f(t, x(\cdot)) = \int_{t_0}^{t} F\big(t, s, x(s), x(\tau(s))\big)\, ds + \varphi(t)\,, \qquad \tau(s) \leqslant s\,.$$

It is quite clear that the Cauchy problem for the differential equations

$$x'(t) = f(t, x(t))\,,$$

$$x'(t) = f(t, x(t), x'(t))$$

and also for the equations with a time-lag (cf. [2])

$$x'(t) = f(t, x(\cdot)_t)\,,$$

$$x'(t) = f(t, x(\cdot)_t, x'(\cdot)_t)$$

can be reduced to an equation of form (9). Thus, using conclusions 2, 3, 4, we can obtain corresponding theorems on the existence, uniqueness and estimations of solutions of suitable equations.

8. We want to consider in more detail the case of the Cauchy problem for the differential equation

$$\text{(11)} \qquad x'(t) = F(t, x(t), x'(t))$$

with the initial condition $x(t_0) = x_0$.

By substituting $x'(t) = y(t)$ we reduce equation (11) to the following one:

$$y(t) = F\Big(t, \int_{t_0}^{t} y(s)\, ds + x_0, y(t)\Big)\,.$$

Suppose that

1° the function

$$f(t, y(\cdot)) = F\Big(t, \int_{t_0}^{t} y(s)\, ds + x_0, y(t)\Big)$$

has the properties listed in Assumption H_7;

2° $\|F(t, x, y) - F(t, \bar{x}, \bar{y})\| \leqslant \omega(t, \|x - \bar{x}\|, \|y - \bar{y}\|)$, for $t \in I$ and $x, \bar{x}, y, \bar{y} \in B$;

3° the function

$$a\big(t, u(\cdot)\big) = \omega\Big(t, \int_{t_0}^{t} u(s)\,ds + u_0, u(t)\Big), \qquad u_0 \geqslant 0,$$

satisfies Assumption H_6.

Under these assumptions we infer that

1° there exists a unique solution $\bar{x}(t)$ of the equation (11) and

$$\|\bar{x}(t)\| \leqslant \int_{t_0}^{t} \overline{m}(s)\,ds + \|x_0\|,$$

where $\overline{m}(t)$ is the maximal solution of the equation

$$u(t) = \omega\Big(t, \int_{t_0}^{t} u(s)\,ds + \|x_0\|, u(t)\Big) + \|f(t, 0, 0)\|.$$

2° if $\bar{z}(\cdot)$ is the solution of the equation

$$z'(t) = G\big(t, z(t), z'(t)\big)$$

with the initial condition $z(t_0) = z_0$, then

$$\|\bar{z}'(t) - \bar{x}'(t)\| \leqslant m_1(t),$$

$$\|\bar{z}(t) - \bar{x}(t)\| \leqslant \int_{t_0}^{t} m_1(s)\,ds + \|x_0 - z_0\|$$

where $m_1(t)$ is the maximal solution of the equation

$$u(t) = \omega\Big(t, \int_{t_0}^{t} u(s)\,ds, u(t)\Big) + q_1(t)$$

and

$$q_1(t) = \Big\|F\big(t, \bar{z}(t) + x_0 - z_0, \bar{z}'(t)\big) - G\big(t, \bar{z}(t), \bar{z}'(t)\big)\Big\|.$$

3° if $x_1(\cdot)$ and $x_2(\cdot)$ are, respectively, the ε_1-solution and the ε_2-solution of equation (11), i.e.

$$\Big\|x_i'(t) - F\big(t, x_i(t), x_i'(t)\big)\Big\| \leqslant \varepsilon_i(t), \qquad i = 1, 2,$$

and

$$\varrho(t) = \Big\|x_1'(t) - F\big(t, x_1(t), x_1'(t)\big) - \Big[x_2'(t) - F\big(t, x_2(t), x_2'(t)\big)\Big]\Big\|$$

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t),$$

then

$$\|x_1'(t)-x_2'(t)\| \leqslant m_2(t) \leqslant m_3(t) ,$$

$$\|x_1(t)-x_2(t)\| \leqslant \int_{t_0}^{t} m_2(s)\,ds+\|x_{10}-x_{20}\| \leqslant \int_{t_0}^{t} m_3(s)\,ds+\|x_{10}-x_{20}\| ,$$

where $m_2(t)$ and $m_3(t)$ are the solutions of the equation

$$u(t) = \omega\Big(t, \int_{t_0}^{t} u(s)\,ds+\|x_{10}-x_{20}\|, u(t)\Big)+q(t)$$

with $\varrho(t)$ and $\varepsilon(t)$, respectively, instead of $q(t)$.

Remark. If the function $\omega(t, u, v)$ has the form

$$\omega(t, u, v) = \varphi_0(t)u+\varphi_1(t)v ,$$

$\varphi_i(t) \geqslant 0$, $i = 0, 1$, $\varphi_1(t) \leqslant a_1 < 1$, $a_1 = \text{const}$, then we have the case considered in [3].

In our considerations the function $\omega(t, u, v)$ may take more general forms, for instance

$$\omega(t, u, v) = \gamma(t, u)+kv , \quad 0 \leqslant k < 1 ,$$

where the function $\frac{1}{1-k}\gamma(t, u)$ is non-decreasing with respect to u and for any u_0 there exists a maximal solution of the equation

$$u'(t) = \frac{1}{1-k}\gamma\big(t, u(t)\big)$$

with the initial condition $u(t_0) = u_0$ and if $u_0 = 0$, then $u(t) \equiv 0$ is the only solution of that equation.

9. Now we shall consider an implicit equation of the form

$$h\big(t, x(\cdot)\big) = 0 . \tag{12}$$

We introduce

ASSUMPTION H_8. For any $t \in I$ there exists an operator $C(t)$ from B into B such that

1° no equation $C(t)x = 0$, $t \in I$, has a non-trivial solution in B;

2° for any $y \in R_0$, $z(t) = C(t)y(t)$, $t \in I$, implies $z \in R_0$;

3° the function

$$f(t, x(\cdot)) = C(t)h\big(t, x(\cdot)\big)+x(t) , \quad t \in I , \ x \in R_0 ,$$

satisfies Assumption H_7.

Now we can formulate

CONCLUSION 5. *If Assumption* H_8 *is satisfied, then equation* (12) *has a unique solution* $\bar{x} \in R_0$.

Proof. To prove this conclusion it is sufficient to point that under Assumption H_8 equation (12) is equivalent to the equation

$$x(t) = x(t) + C(t)h\big(t, x(\cdot)\big) = f\big(t, x(\cdot)\big) .$$

Further, by Conclusion 2 we obtain the assertion of Conclusion 5.

Definition. An element $x \in R_0$ is called an *ε-solution of equation* (12) iff the relation

$$\big\|h\big(t, x(\cdot)\big)\big\| \leqslant \varepsilon(t)$$

holds for $\varepsilon \in G_0$.

Conclusion 4 implies

Conclusion 6. *If Assumption* H_8 *is satisfied and*

1° *there exists a constant* $d > 0$ *such that*

$$\|C(t)x\| \leqslant d\|x\| \quad \textit{for} \quad x \in B\,, \; t \in I\,,$$

2° x_1 *and* x_2 *are, respectively, the* ε_1*-solution and the* ε_2*-solution of equation* (12),

3° $$\bar{\varrho}(t) = \big\|C(t)h\big(t, x_1(\cdot)\big) - C(t)h\big(t, x_2(\cdot)\big)\big\| ,$$
$$\bar{\varepsilon}(t) = [\varepsilon_1(t) + \varepsilon_2(t)]d\,, \quad t \in I\,,$$

then

$$\|x_1(t) - x_2(t)\| \leqslant \overline{m}(t, \bar{\varrho}) \leqslant \overline{m}(t, \bar{\varepsilon})$$

where $\overline{m}(t, \bar{\varrho})$, $\overline{m}(t, \bar{\varepsilon})$ *are the maximal solutions of equation* (8) *with* $\bar{\varrho}$ *and* $\bar{\varepsilon}$, *respectively, instead of* q.

Remark. The function $h\big(t, x(\cdot)\big)$ may be of different forms, for instance:

(a) $$h\big(t, x(\cdot)\big) = k\Big(t, x(t), \int_{t_0}^{t} g\big(t, s, x(s)\big)\,ds\Big) ,$$

(b) $$h\big(t, x(\cdot)\big) = k\big(t, x(t)\big) ,$$

(c) $$h\big(t, x(\cdot)\big) = k\Big(t, x(t), \int_{t_0}^{T} g\big(t, s, x(s)\big)\,ds\Big) .$$

However, the differential equations

$$k\big(t, x(t), x'(t)\big) = 0 ,$$

$$k\big(t, x(\cdot)_t, x'(\cdot)_t, x'(t)\big) = 0$$

can easily be reduced to equations of form (12).

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Reçu par la Rédaction le 5. 11. 1965