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On a mixed-type interpolation problem for real polynomials

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Abstract. Based on a result about weighted integral interpolation an existence and uniqueness theorem is established for real polynomials of degree $(n+k+1)$ given k extremal points and n extremal values.

Introduction. Along with the classical interpolation problem for real polynomials passing through a given number of points on the real line a different kind of interpolation was studied by several authors namely the existence and uniqueness of real polynomials of degree n determined by $(n-1)$ extremal values. This kind of interpolation was applied by Paszkowski [6] to determine disjoint intervals containing the Tchebysheff alternant points of polynomials of best uniform approximation to a continuous function on an interval.

This type of interpolation was also studied by Mycielski and Paszkowski in [5] and by Kuhn [3]. They obtained the following results.

THEOREM A [5]. *Given $(n+1)$ positive real numbers $w_0, w_1, \ldots, w_n$ there exists one and only one polynomial p of degree n with real coefficients for which there are numbers $-1 = v_0 < v_1 < \ldots < v_n = 1$ such that $p(v_k) = (-1)^{n-k} w_k$ $(k = 0, 1, \ldots, n)$ and $p'(v_l) = 0$ $(l = 1, \ldots, n-1)$.*

THEOREM B [3]. *Given $(n+1)$ real numbers $y_0, y_1, \ldots, y_n$ such that $(-1)^{n-j+1}(y_j - y_{j-1}) > 0$ $(j = 1, 2, \ldots, n)$ there exists a unique polynomial of degree $(n+2)$ with leading coefficient 1, which takes (in order of increasing x) the extremal values $y_0, y_1, \ldots, y_n$ starting with y_0 at the origin.*

In proving Theorem A the authors apply a topological method based on the properties of covering spaces studied by Browder ([1], Theorems 4–7) and Lelek and Mycielski [4]. We quote the result of [4] convenient for application in the interpolation problem discussed here.

THEOREM C. *Let F be a continuous mapping of the n-dimensional sphere S^n into itself such that $F(S^n - \{p\}) \subset S^n - \{p\}$, $F(p) = p$ for some $p \in S^n$. If the*

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mapping $F|_{S^n - \{p\}}$ *is a local homeomorphism, then* F *is a homeomorphism of* S^n *onto* S^n.

On the other hand to establish Theorem B the author uses some properties from the theory of existence, uniqueness and continuability of solutions of normal differential systems of equations.

In this note we shall refer only to Theorem B since a straightforward argument shows that this theorem is stronger than Theorem A. In turn Theorem B is equivalent to the following ([3], Theorem 2)

THEOREM B′. *Given* n *positive numbers* $F_1, F_2, \ldots, F_n$ *there exists a unique polynomial of the form* $p(x) = \prod_{k=0}^{n} (x-x_k)$, *where* $0 = x_0 < x_1 < \ldots < x_n$, *such that*

$$\int_{x_{j-1}}^{x_j} p(x)\,dx = (-1)^{n-j+1} F_j, \quad j = 1, 2, \ldots, n.$$

In this note we generalize Theorem B′ by considering positive continuous weight functions. In particular we establish an existence and uniqueness result concerning a mixed-type interpolation problem whereby a fixed number of extremal points and of extreme values are a priori assigned. The method of proof will be based on a topological argument as in [5] which simplifies the proof as compared with the method used in [3] to prove Theorems B and B′.

Finally we remark that the complex plane counterpart of the problems cited above was studied in a series of papers by Charzyński and Kozłowski [2].

2. Two lemmas.

LEMMA 1. *Let*

$$a_{jk} = \int_{x_{j-1}}^{x_j} f(x)(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)\,dx,$$

where $1 \leqslant j, k \leqslant n$, $x_0 < x_1 < \ldots < x_n$ *and* $f(x)$ *is a continuous function of constant sign defined on* (x_0, x_n). *Then*

$$\Delta = \det(a_{jk}) \neq 0.$$

Proof. Adding row 1 to row 2, row 2 to row 3, ..., row $(n-1)$ to row n, we may assume that the determinant in question has elements

$$b_{jk} = \int_{x_0}^{x_j} f(x)(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)\,dx, \quad 1 \leqslant j, k \leqslant n.$$

Keeping the last column fixed substract from the columns $1, \ldots, (n-1)$ the n-th column. This enables to factor $(x_1-x_n)\,(x_2-x_n)\,\ldots\,(x_{n-1}-x_n)$. Then keeping the last two columns, $(n-1)$ and n fixed, substract the $(n-1)$-th column from the columns $1, \ldots, (n-2)$. This enables to factor $(x_1-x_{n-1}) \ldots \ldots (x_{n-2}-x_{n-1})$. Then keeping the last three columns fixed continue the process until the $(n-1)$ last columns are kept fixed and column 2 is substracted from column 1, thus factoring (x_1-x_2). This leads to consideration of the determinant whose elements are

$$c_{jk} = \int_{x_0}^{x_j} f(x)(x-x_0)(x-x_1)\ldots(x-x_{k-1})\,dx, \tag{1}$$

$1 \leqslant j, k \leqslant n$. Denote the determinant of the matrix (1) by $\Delta_n(x_n)$ to emphasize the dependence on n and x_n.

Obviously by hypothesis

$$\Delta_1(x_1) = \int_{x_0}^{x_1} f(x)(x-x_0)\,dx \neq 0$$

and

$$\Delta_2(x_2) = \begin{vmatrix} \int_{x_0}^{x_1} f(x)(x-x_0)\,dx & \int_{x_0}^{x_1} f(x)(x-x_0)(x-x_1)\,dx \\ \int_{x_0}^{x_2} f(x)(x-x_0)\,dx & \int_{x_0}^{x_2} f(x)(x-x_0)(x-x_1)\,dx \end{vmatrix}.$$

If $\Delta_2(x_2) = 0$, then since $\Delta_2(x_1) = 0$ it would follow that $\Delta_2'(\bar{x}_2) = 0$ for some $\bar{x}_2 \in (x_1, x_2)$. A short calculation shows that

$$\Delta_2'(\bar{x}_2) = f(\bar{x}_2)(\bar{x}_2-x_0) \int_{x_0}^{x_1} f(x)(x-x_0)(\bar{x}_2-x)\,dx$$

and we obtain a contradiction. Now assume by induction that Lemma 1 holds for determinants of the form (1) up to order $(m-1)$, $1 \leqslant m-1 \leqslant n-1$. In particular $\Delta_{m-1}(x_{m-1}) \neq 0$. Denote by $\Delta_m(t)$ the expression $\Delta_m(x_m)$ in which t substitutes for x_m. If $\Delta_m(x_m) = 0$, then since $\Delta_m(x_{m-1}) = 0$ we would have $\Delta_m'(\bar{x}_m) = 0$ for some $\bar{x}_m \in (x_{m-1}, x_m)$. Differentiating $\Delta_m(t)$ with respect to t and in the resulting determinant multiplying column $(m-1)$ by $(t-x_{m-1})$ and substracting from column m, column $(m-2)$ multiplied by $(t-x_{m-2})$ substracting from column $(m-1), \ldots,$ multiplying column 1 by $(t-x_1)$ and substracting from column 2, we obtain the relation

$$\Delta_m'(t) = (-1)^{m-1} f(t)(t-x_0)\det(d_{kl}),$$

where

$$d_{kl} = \int_{x_0}^{x_k} f(x)(x-t)(x-x_0)(x-x_1)\ldots(x-x_{l-1})\,dx,$$

$1 \leqslant k, l \leqslant m-1$. It remains to apply the induction hypothesis with the function $f(x)$ substituted by $f(x)(x-\bar{x}_m)$ and the interval (x_0, x_{m-1}) to obtained a contradiction to the relation $\Delta'_m(\bar{x}_m) = 0$.

In the case where $f(x)$ is a constant the determinant Δ was evaluated in closed form in [3].

LEMMA 2. *Let $f(x)$ be a function defined for $x > 0$, integrable on every finite interval and such that $f(x) \geqslant m > 0$ for some positive constant m and $x > 0$. For given n distinct positive reals $x_1 < x_2 < \ldots < x_n$ define n numbers $A_1, A_2, \ldots, A_n$ by the formulas*

$$(2) \qquad A_j = (-1)^{n-j+1} \int_{x_{j-1}}^{x_j} f(x)p(x)\,dx, \quad j = 1, 2, \ldots, n,$$

where $x_0 = 0$ and $p(x) = \prod_{k=0}^{n} (x-x_k)$. Then if

$$\alpha_n = \operatorname*{Min}_{1 \leqslant j \leqslant n} (x_j - x_{j-1}, x_n^{-1}) \to 0 \quad \textit{also} \quad \beta_n = \operatorname*{Min}_{1 \leqslant j \leqslant n} (A_j, A_j^{-1}) \to 0.$$

Proof. One verifies that $\beta_n > 0$. We show that there exists a positive function $h(d)$ defined for $d > 0$, $h(d) \to 0$ as $d \to 0$, such that $\beta_n \geqslant h$ implies $\alpha_n \geqslant d^{2n+4}$ for all sufficiently small positive d. Indeed in this case we have $h \leqslant A_j \leqslant h^{-1}$. Therefore by (2) choosing an interval $(x_{\mu-1}, x_\mu)$ such that $x_\mu - x_{\mu-1} \geqslant x_n/n$ we have

$$h^{-1} \geqslant m \int_{x_{\mu-1}}^{x_\mu} (x-x_{\mu-1})^\mu (x_\mu - x)^{n-\mu+1}\,dx$$

$$= m(x_\mu - x_{\mu-1})^{n+2} \frac{\mu!\,(n-\mu+1)!}{(n+2)!} \geqslant m \frac{x_n^{n+2}}{n^{n+2}(n+2)!}.$$

Hence

$$(3) \qquad x_n \leqslant ch^{-1/(n+2)},$$

where c is a positive constant which depends only on n and m. Furthermore for $1 \leqslant j \leqslant n$

$$(4) \qquad h \leqslant \int_{x_{j-1}}^{x_j} f(x)|p(x)|\,dx \leqslant x_n^n(x_j - x_{j-1}) \int_0^{x_n} f(x)\,dx.$$

For $t > 0$ let $\varphi(t) = \int_0^t f(x)\,dx$ and let $\psi = \varphi^{-1}$. For $d > 0$ define $h(d) = c^{n+2}(\psi(d^{-1}))^{-(n+2)}$ or $d = (\varphi(ch^{-1/(n+2)}))^{-1}$. By hypothesis on f, $\varphi(t) \to \infty$ as $t \to \infty$ and thus $\psi(s) \to \infty$ as $s \to \infty$ and hence $h(d) \to 0$ as $d \to 0$.

We now have by (4) and (3)

(5) $$x_j - x_{j-1} \geq h x_n^{-n} \Big(\int_0^{x_n} f(x)\,dx\Big)^{-1}$$
$$\geq c^{-n} h^{(2n+2)/(n+2)} \big(\varphi(ch^{-1/(n+2)})\big)^{-1} = c^{-n} h^{(2n+2)/(n+2)} d.$$

Now since $\varphi(t) \geq mt$, $\psi(t) \leq t/m$ for $t > 0$.

Therefore (3) implies

(6) $$x_n^{-1} \geq c^{-1} h^{1/(n+2)} = \big(\psi(d^{-1})\big)^{-1} \geq md \geq d^{2n+4}$$

for all sufficiently small positive d.

Also since $\psi(d^{-1}) \leq d^{-1}/m$ we have

(7) $$h(d) = c^{n+2}\big(\psi(d^{-1})\big)^{-(n+2)} \geq c^{n+2} m^{n+2} d^{n+2}$$

By (5) and (7)

(8) $$x_j - x_{j-1} \geq c^{n+2} m^{2n+2} d^{2n+3} \geq d^{2n+4}$$

for all sufficiently small positive d.

By (6) and (8) $\alpha_n \geq d^{2n+4}$ for all sufficiently small positive d. As a result we have established that $\alpha_n < d^{2n+4}$ implies $\beta_n < h(d)$ for sufficiently small positive d and $h(d) \to 0$ as $d \to 0$. This completes the proof of Lemma 2.

3. The Main Theorems.

THEOREM 1. *Let $f(x)$ be a continuous positive function defined on the positive real axis such that $xf(x)$ is integrable near the origin. Assume that $f(x)$ is bounded away from zero. For any n ($n \geq 1$) positive numbers $A_1, A_2, \ldots, A_n$ there exists a unique polynomial $p(x) = \prod_{k=0}^{n} (x - x_k)$, $0 = x_0 < x_1 < \ldots < x_n$ such that*

(9) $$\int_{x_{j-1}}^{x_j} p(x) f(x)\,dx = (-1)^{n-j+1} A_j, \quad j = 1, 2, \ldots, n.$$

Proof. Denote by P the subset of $\mathbf{R}^n$ consisting of all points $(y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$ such that $0 < y_1 < y_2 < \ldots < y_n$. Let $Q \supset P$ be the set of all points $(z_1, z_2, \ldots, z_n) \in \mathbf{R}^n$ such that $z_i > 0$, $i = 1, 2, \ldots, n$.

Define a mapping φ: $P \to Q$ as follows: For a point $x = (x_1, x_2, \ldots, x_n) \in P$ construct the polynomial of degree $(n+1)$, $p(x) = \prod_{k=0}^{n} (x - x_k)$, where $x_0 = 0$ and define numbers

$$A_j = (-1)^{n-j+1} \int_{x_{j-1}}^{x_j} p(x) f(x)\,dx, \quad j = 1, 2, \ldots, n.$$

One verifies easily that $A = (A_1, A_2, \ldots, A_n) \in Q$. Define the mapping φ by setting $\varphi(x) = A$. Since both P and Q are homeomorphic to $\mathbf{R}^n$ we can consider φ to be a mapping from S^n to S^n by adjoining points ∞_P and ∞_Q to P and Q, respectively, with the one point compactification topology and letting $\varphi(\infty_P) = \infty_Q$. By Lemma 1, φ is a local homeomorphism on $S^n - \{\infty_P\}$. By Lemma 2, φ is continuous at the point ∞_P. Therefore all conditions of Theorem C are satisfied and φ is a homeomorphism of S^n onto S^n and, in particular, of P onto Q. This implies Theorem 1.

As a consequence of Theorem 1 we obtain the following result concerning the mixed-type interpolation problem:

THEOREM 2. *Let the k points* $(k \geq 0)$

$$x_{-k} \leq x_{-k+1} \leq \ldots \leq x_{-1} < 0,$$

and the n values $y_0, y_1, \ldots, y_{n-1}$ *which satisfy* $(-1)^{n-j}(y_j - y_{j-1}) > 0$ *be given* $(j = 1, 2, \ldots, n-1)$. *There exists a unique monic polynomial of degree* $(n+k+1)$, $Q(x)$, *which has the extreme points* $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ *and has the extreme values* $y_0, y_1, \ldots, y_{n-1}$ *taken according to increasing values on the x-axis starting with* y_0 *at the origin.*

Proof. Define

$$A_j = (-1)^{n-j}\frac{y_j - y_{j-1}}{n+k+1}, \quad j = 1, 2, \ldots, n-1.$$

By Theorem 1 with $f(x) = \prod_{i=1}^{k}(x - x_{-i})$, there exists a unique polynomial $p(x) = \prod_{l=0}^{n-1}(x - x_l)$ such that

$$\int_{x_{j-1}}^{x_j} p(x) f(x)\,dx = (-1)^{n-j} A_j,$$
$$j = 1, 2, \ldots, n-1, \; 0 = x_0 < x_1 < \ldots < x_{n-1}.$$

Let

$$Q(x) = (n+k+1)\int_{x_0}^{x} p(x) f(x)\,dx + y_0.$$

Then $Q(x)$ is a monic polynomial of degree $(n+k+1)$ which satisfies $Q(x_j) = y_j$, $j = 0, 1, \ldots, (n-1)$ and $Q'(x_{-i}) = Q'(x_j) = 0$, $i = 1, 2, \ldots, k$; $j = 0, 1, \ldots, n-1$.

With regards to the uniqueness of $Q(x)$ one notices that if $Q(x)$ has the required properties by Theorem 2, then Theorem 1 implies that the points $x_0, x_1, \ldots, x_{n-1}$ are uniquely determined. Hence the zeros of $Q'(x)$ as well as its leading coefficient are uniquely determined. Therefore $Q'(x)$ and thus $Q(x)$ are uniquely determined. For $k = 0$, Theorem 2 reduces to Theorem B.

Remark. The requirement on $f(x)$ to be bounded away from zero cannot be entirely dispensed with. It is easy to give an example with $n = 1$,

$f(t) = O(t^{-4})$ for large t such that the problem discussed in Theorem 1 has no solution.

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