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The functional equation $f^2(x) = g(x)$

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I. Introduction and notation. In this paper we are interested in functional equations of the form

$$f^n(x) = g(x) \tag{1.1}$$

on a closed interval $[a, b]$ of the real line. Here f^n denotes the nth iterate of the function f:

$$f^0(x) = x, \quad f^{k+1}(x) = f(f^k(x)), \quad k = 0, 1, 2, \ldots$$

In particular we will consider the problem of the existence and construction of continuous solutions of the equation

$$f^2(x) = g(x) \tag{1.2}$$

on the closed interval $[a, b]$ of the real line.

We denote by $R[a, b]$ the set of functions defined on $[a, b]$ with values in $[a, b]$ and by $D[a, b] \subset R[a, b]$ all functions which possess the Darboux property on $[a, b]$. We denote by $C[a, b] \subset D[a, b]$ the functions which are continuous on $[a, b]$ and by $M[a, b] \subset C[a, b]$ the functions which are piecewise monotone (written p.m.). Here a function g is p.m. on $[a, b]$ if there exists a finite partition $P = [p_0, \ldots, p_n]$ of $[a, b]$ such that on every subinterval $[p_i, p_{i+1}]$ the function g is strictly monotone (written s.m.). If every partition P^* which has this property with respect to g is a refinement of P, then P is said to be the partition associated with g and will be denoted by $P(g)$. Let $S(n, g)$ denote the set of all solutions $f \in R[a, b]$ of equation (1.1). In an earlier paper [3] the author has shown that there exists $g \in C[a, b]$ such that $S(2, g) \cap C[a, b]$ is empty while $S(2, g) \cap D[a, b]$ is not empty. However, if $g \in M[a, b]$ then $S(n, g) \cap D[a, b] = S(n, g) \cap M[a, b]$. In light of this fact, it would seem natural in studying continuous solutions of (1.1) to restrict oneself to the case in which $g \in M[a, b]$. Thus in this paper we shall study the set $S(2, g) \cap D[a, b] = S(2, g) \cap M[a, b]$ for $g \in M[a, b]$. We shall further

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restrict ourselves to functions g which satisfy conditions A1 and A2 given below. We first have the following definitions.

For any $g \in R[a, b]$ we define $\gamma(g) = \{x|\ x \in [a, b]$ and $g(x) = x\}$. $\gamma(g)$ is called the *set of fixed points of* g. Let $P(g) = [p_0, \ldots, p_n]$ denote the partition associated with $g \in M[a, b]$, then the set of points $X(g) = \{x_i|\ x_i = p_i,\ i = 1, \ldots, n-1\}$ will be called the *break points of* g. Assume $\gamma(g)$ is finite, then if q is the first point of $P(g) \cup \gamma(g)$ preceding [following] the point $\gamma_j \in \gamma(g)$ we shall denote the open interval (q, γ_j) $[(\gamma_j, q)]$ by $I(j, -)$ $[I(j, +)]$. We say that $I = I(j, +)$ or $I = I(j, -)$ is a *terminal segment* if $g(I) \subset I$ and $\lim_{i\to\infty} g_i(x) = \gamma_j$ for any $x \in I$. Since $g \in M[a, b]$ it is easily seen that $I(j, +) = (\gamma_j, q)$ $[I(j, -) = (q, \gamma_j)]$ is a terminal segment if and only if $\gamma_j < g(x) < x$ for all $x \in (\gamma_j, q)$ $[x < g(x) < \gamma_j$ for all $x \in (q, \gamma_j)]$.

We now state conditions A1 and A2.

A1. *The set* $\gamma(g)$ *is finite and the set of terminal segments of* g *is not empty.*

A2. *For every* $x \in [a, b]$ *there exists an integer* M, *depending on* x, *such that* $g^M(x)$ *is contained in* $\gamma(g)$ *or in a terminal segment of* g.

For $g \in M[a, b]$ and satisfying conditions A1 and A2 we shall show in section II that if $f \in S(2, g) \cap D[a, b]$ and $f(\gamma_j) = \gamma_j$ for all $\gamma_j \in \gamma(g)$ then f restricted to the set $\gamma(g) \cup \bigcup_{i=0}^{\infty} f^i(P(g))$ satisfies conditions B1, B2, and B3 as stated in section II. Conversely if a function $\bar{f}$ defined on $P(g) \cup \gamma(g)$ is such that its extension by equation (1.2) to the set $\gamma(g) \cup \bigcup_{i=0}^{\infty} \bar{f}^i(P(g))$ is single valued and satisfies conditions B1, B2, and B3, then $\bar{f}$ may be extended to a function $f \in S(2, g) \cap M[a, b]$. In general if S is any set and $\bar{f}$ is defined on S with values in $[a, b]$ we may extend the definition of $\bar{f}$ to the set $\bigcup_{i=0}^{\infty} f^i(S)$ as follows:

$$f^{i+1}(x) = f^2(f^{i-1}(x)) = g(f^{i-1}(x)) \quad \text{for} \quad i \geqslant 1 \text{ and } x \in S\,.$$

We shall refer to this extension of $\bar{f}$ as the extension of $\bar{f}$ by iteration. It should be noted that for arbitrary $\bar{f}$ and S this extension may not be single valued. Thus if $p_1, p_2 \in S$, $\bar{f}(p_1) = \bar{f}(p_2)$, and $g(p_1) \neq g(p_2)$ then $\bar{f}$ defined by iteration is not single valued on $\bar{f}(p_1)$. We shall avoid this difficulty by restricting ourselves to sets S and functions $\bar{f}$ for which the extension $\bar{f}$ is single valued on $\bigcup_{i=0}^{\infty} f^i(S)$.

In section III we show for $g \in M[a, b]$ satisfying conditions A1 and A2 the existence or non existence of a function $f \in S(2, g) \cap M[a, b]$, such that $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$, may be established by checking a finite number of finite sets $A(k)$, $k = 1, 2, \ldots, r$. Here the number r of sets $A(k)$ and the number of points in each set $A(k)$ are completely determined by the given function g. In fact given g one may always construct the sets $A(k)$ and thus the question of the existence, or non existence, of a solution $f \in S(2, g) \cap D[a, b]$ which satisfies the condition $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$ can be completely answered in a finite number of steps.

II. The equation $f^2(x) = g(x)$. We shall assume that $g \in M[a, b]$ and satisfies conditions A1 and A2. The sets $P(g) = \{p_0, \ldots, p_n\}$ and $X(g)$ are as defined in section I. Let $I = I(\gamma_j, \pm)$ denote a terminal segment of g and S be any finite set contained in I. Let $\bar{f}$ be defined on $S \cap I$ into I. Since $g(I) \subset I$, using the relation $\bar{f}^2(x) = g(x)$, the function $\bar{f}$ may formally be defined on the set $S^* = \bigcup_{i=0}^{\infty} \bar{f}^i(S) \subset I$ by iteration. We assume this extension is single valued. Since g is s.m. on I and I is a terminal segment it follows that S^* is finite on $I \cap CN(\gamma_j)$, where $CN(\gamma_j)$ denotes the complement of any open interval $N(\gamma_j)$ containing γ_j. We say that $\bar{f}$ is *compatible with respect to I and S* if for every $r, s \in S^*$, $\bar{f}$ is single valued and $\bar{f}(r) < \bar{f}(s)$ if and only if $r < s$. Next, consider the case of two adjoining terminal segments $I_1 = I(i, +)$ and $I_2 = I(i, -)$ and a finite set $S \subset I_1 \cup I_2$. Assume $\bar{f}$ is defined on S and that $\bar{f}(S \cap I_1) \subset I_2$, $\bar{f}(S \cap I_2) \subset I_1$. Again we assume that $\bar{f}$ has a single valued extension, by iteration, to the set $S^* = \bigcup_{i=0}^{\infty} \bar{f}^i(S)$ which is finite on $(I_1 \cup I_2) \cap CN(\gamma_i)$ for any $N(\gamma_i)$. Then $\bar{f}$ is said to be *compatible with respect to $I_1 \cup I_2$ and S* if it is single valued on S^* and if for every $r, s \in S^*$ one has $\bar{f}(r) > \bar{f}(s)$ if and only if $r < s$. We now have the following two theorems.

THEOREM 2.1. *Let S be a finite set contained in a terminal segment $I = I(i, +)$ and let $\bar{f}$ be defined on S into I. Then a necessary and sufficient condition that there exists a continuous s.m. function f on $I \cup \gamma_i$ into $I \cup \gamma_i$ satisfying $f^2(x) = g(x)$ and $f|S^* = \bar{f}$ is that $\bar{f}$ be compatible with respect to I and S.*

THEOREM 2.2. *Given two terminal segments $I_1 = I(i, +)$, $I_2 = I(i, -)$, a finite set $S \subset I_1 \cup I_2$, a function $\bar{f}$ defined on S into $I_1 \cup I_2 \cup \gamma_i$, then a necessary and sufficient condition that there exist a continuous s.m. function f on $I_1 \cup I_2 \cup \gamma_1$ into itself satisfying $f^2(x) = g(x)$ and $f|S^* = \bar{f}$ is that either* (1) *or* (2) *below is satisfied.*

(1) *$\bar{f}|S \cap I_i$ is compatible with I_i and $S \cap I_i$ for $i = 1, 2$.*

(2) *$\bar{f}$ is compatible with $I_1 \cup I_2$ and S.*

These two theorems are direct consequences of the work of Kuczma [2]. In that paper he gives a complete description of the general solution of equation (1.1) for the case in which g is a s.m. function on the interval E

of the real line. In Theorems 2.1 and 2.2 the sets $I \cup \gamma_1$ and $I_1 \cup I_2 \cup \gamma_i$ play the role of the domain E in the work of Kuczma. It then follows for the case where $E = I \cup \gamma_i$ that any continuous solution f of equation (1.1) must be s.m. increasing on E and satisfy $f(\gamma_i) = \gamma_i$. Thus for any finite set $S \subset I$ if $\bar{f} = f|S$ then $\bar{f}$ will be compatible with respect to I and S. For the case where $E = I_1 \cup I_2 \cup \gamma_i$ one has that $f(\gamma_i) = \gamma_i$ and f is either s.m. increasing on E or s.m. decreasing on E. In the first case if $S \subset I_1 \cup I_2$ is any finite set and $\bar{f} = f|S$ then $\bar{f}$ on $I_1 \cap S$ $[I_2 \cap S]$ will be compatible with respect to I_1 and $I_1 \cap S$ $[I_2$ and $I_2 \cap S]$. In the second case $\bar{f}$ will be compatible with respect to $I_1 \cup I_2$ and S.

The converse statement that if $\bar{f}$ is compatible with respect to I and S $[I_1 \cup I_2$ and $S]$ then $\bar{f}$ may be extended to a continuous solution of (1.2) on $I \cup \gamma_i$ $[I_1 \cup I_2 \cup \gamma_i]$ is also a consequence of the work of Kuczma. Consider first the case of a single terminal segment I and a finite set $S \subset I$. Let p denote the point in S such that $S \cap (\gamma_i, p)$ is empty. Then since g is s.m. on $I \cup \gamma_i$ and $\bar{f}$ is compatible with respect to I and S, one may show that for each $q \in S$ the interval $[\bar{f}(p), p)$ contains a unique iterate $\bar{f}^h(q)$ of q. Here the integer h depends on the point $q \in S$. Since $\bar{f}$ is compatible with respect to I and S, we have that $\bar{f}$ is s.m. on $[\bar{f}(p), p] \cap S^*$ and we may extend $\bar{f}$, in many ways, to a continuous s.m. function defined on the closed interval $[\bar{f}(p), p]$. Then the construction given by Kuczma [2] suffices to extend $\bar{f}$ to a continuous s.m. function on the entire set $E = I \cup \gamma_i$.

In the case where $E = I_1 \cup I_2 \cup \gamma_i$ and $\bar{f}$ is compatible with respect to $I_1 \cup I_2$ and S a similar argument is used. Let p denote the point in $[S \cup \bar{f}(S)] \cap I_1$ for which $[\gamma_i, p) \cap [S \cap \bar{f}(S)]$ is empty. Then one may again show that for every $q \in S$ the interval $[f(p), g(f(p))]$ contains a unique iterate $\bar{f}^h(q)$ of q. Since $\bar{f}$ is compatible with respect to $I_1 \cup I_2$ and S, $\bar{f}$ is s.m. on $[\bar{f}(p), g(\bar{f}(p))] \cap S^*$ and so may be extended, in many ways, to a continuous s.m. function defined on the closed interval $[\bar{f}(p), g(\bar{f}(p))]$. Again the work of Kuczma [2] assures us that $\bar{f}$ may be extended to a continuous s.m. function on the entire set $E = I_1 \cup I_2 \cup \gamma_i$. This completes the proof of Theorems 2.1 and 2.2.

We are now ready to state and prove Theorem 2.3. Since $g \in M[a, b]$, we know that any continuous solution f of (1.2) must also belong to $M[a, b]$ and $P(f) \subset P(g)$. Thus one might hope to state necessary and sufficient conditions for a function $\bar{f}$ defined on $P(g)$ to be the restriction of a continuous solution f of (1.2) to the set $P(g)$. This is the content of Theorem 2.3 for solutions f of (1.2) which satisfy the condition $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$. We note that if $\bar{f}$ is defined on the set $P(g)$ and if $\bar{f}$ is to be extended to a solution of (1.2) then its extension to the set $\bigcup_{i=0}^{\infty} f^i(P(g))$ is completely determined by iteration as described in section I. Thus the conditions of our theorem are stated in terms of the behaviour of $\bar{f}$ on the set $\gamma(g) \cup \bigcup_{i=0}^{\infty} f^i(P(g))$. To facilitate the statement of Theorem 2.3 we introduce the following conditions.

B1. *$\bar{f}$ is defined on $P(g) \cup \gamma(g)$ and the extension of $\bar{f}$ by iteration to the set $\gamma(g) \cup \bigcup_{i=0}^{\infty} f^i(P(g))$ is single valued. $\bar{f}(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$.*

B2. (a) *$\bar{f}(p_i) \neq \bar{f}(p_{i+1})$ for any i, and $\bar{f}$ is s.m. on*

$$\left[\gamma(g) \cup \bigcup_{i=0}^{\infty} \bar{f}^i(P(g))\right] \cap [p_i, p_{i+1}]$$

for $i = 0, 1, \ldots, n-1$.

(b) *There exists a subset E of $X(g)$ such that if $x_i \in E$ then*

$$\operatorname{sign}[\bar{f}(p_{i-1}) - \bar{f}(x_i)] = \operatorname{sign}[\bar{f}(p_{i+1}) - \bar{f}(x_i)].$$

If $x_i \in X(g) - E$ then,

$$\operatorname{sign}[\bar{f}(p_{i+1}) - \bar{f}(x_i)] = \operatorname{sign}[\bar{f}(x_i) - \bar{f}(p_{i-1})].$$

(c) *$\bar{f}(x_i) \in E$ for all $x_i \in X(g) - E$.*

(d) *If $x_j \in [\bar{f}(p_\sigma), \bar{f}(p_{\sigma+k})] \cap E$ for any σ and k, then $f(p_{\sigma+i}) = x_j$ for some value of $i = 0, 1, \ldots, k$.*

B3. *Let $I = I(j, \pm)$ denote any terminal segment and $J(I)$ the subset of $P(g)$ whose iterates are eventually contained in I. For each $p_i \in J(I)$ there is a first iterate which belongs to I which we denote by $\bar{p}_i$. Define for each I the set $S = \{\bar{p}_i;\ p_i \in J(I)\}$ and let $\bar{f}$ be defined on S by iteration. We then insist that for every $I = I(i, \pm)$, $\bar{f}$ is compatible with respect to I and S or else $\bar{f}$ is compatible with $I(i, +) \cup I(i, -)$ and S. In the second case, $I_1 = I(i, +)$ and $I_2 = I(i, -)$ are both terminal segments and $S = \{\bar{p}_i;\ p_i \in J(I_1) \cup J(I_2)\}$.*

THEOREM 2.3. *Let $g \in M[a, b]$ and satisfy* A1 *and* A2. *Let $f \in S(2, g) \cap D[a, b]$ and $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$. Then if $\bar{f} = f| \bigcup_{i=0}^{\infty} f^i(P(g)) \cup \gamma(g)$ $\bar{f}$ will satisfy conditions* B1, B2, B3. *Conversely, if $\bar{f}$ is defined on $P(g) \cup \gamma(g)$, and its extension by iteration is single valued and satisfies* B1, B2, *and* B3 *then there exists $f \in S(2, g) \cap M[a, b]$ such that $f|\ P(g) \cup \gamma(g) \equiv \bar{f}$.*

Proof. We first prove that if $f \in S(2, g) \cap D[a, b]$ and $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$ then $\bar{f} = f| \bigcup_{i=0}^{\infty} f^i(P(g)) \cup \gamma(g)$ satisfies conditions B1, B2, and B3. Since $\bar{f}$ is the restriction of a solution f of (1.2) to the set $\bigcup_{i=0}^{\infty} f^i(P(g)) \cup \gamma(g)$ it is single valued on this set and satisfies equation (1.2). By hypothesis $f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$, so condition B1 is satisfied by $\bar{f}$. Since

$f \in S(2, g) \cap D[a, b]$ we know that $f \in M[a, b]$ and $P(f) \subset P(g)$ (see [3]). Thus f is s.m. on $[p_i, p_{i+1}]$ for $i = 0, 1, \ldots, n-1$, and $\bar{f}$ satisfies B2a. Since $P(f) \subset P(g)$ we have that $X(f) \subset X(g)$. Set $E = X(f)$, then $\bar{f}$ satisfies B2b. If $p_i \in X(g)$ but $p_i \notin X(f)$ then $f(p_i) \in X(f)$. For if this were not the case, since $f \in M[a, b]$, we would have that f was s.m. on an open set containing p_i and $f(p_i)$. Thus $f^2 = g$ would be s.m. on an open set containing p_i which contradicts $p_i \in X(g)$. Thus $\bar{f}$ satisfies B2c. Let $x_j \in \big(f(p_\sigma), f(p_{\sigma+k})\big) \cap E$. Then since f is continuous there exists $r \in (p_\sigma, p_{\sigma+k})$ such that $f(r) = x_j$. If $r \notin P(g)$ then f is s.m. on an open set containing r. Since $f(r) = x_j \in X(f)$ we have that $r \in P(g)$. Thus $r = p_{\sigma+i}$ for some $i = 1, \ldots, k-1$ and $x_j = f(p_{\sigma+i})$. If x_j is an endpoint of $[f(p_\sigma), f(p_{\sigma+k})]$ then either $x = f(p_\sigma)$ or $x = f(p_{\sigma+k})$. Thus $\bar{f}$ will satisfy B2d. That $\bar{f}$ satisfies B3 follows from Theorems 2.1 and 2.2.

We now prove that if $\bar{f}$ is defined on $P(g) \cup \gamma(g)$ and its extension to $\gamma(g) \cup \bigcup_{i=0}^{\infty} \bar{f}^i\big(P(g)\big)$ by iteration is single valued and satisfies B1, B2, and B3, then $\bar{f}$ may be extended to a continuous of (1.2) on $[a, b]$. Define the set

$$S_1 = \gamma(g) \cup P(g) \cup \Big[\bigcup_{i=1}^{\infty} \bar{f}^i\big(P(g)\big)\Big] \cup \Big[\bigcup_{i=1}^{k} I_i\Big]$$

where I_i, $i = 1, \ldots, k$, denote the terminal segments of g. By A2 the set $S_1 - \big(\bigcup_{i=1}^{k} I_i\big)$ is finite. As a consequence of B2, B3 the function $\bar{f}$ satisfies the conditions of Theorems 2.1 and 2.2 and so may be extended to a function f_1 defined on S_1 and possessing the following properties. On each of the sets $S_1 \cap [p_i, p_{i+1}]$, $i = 0, \ldots, n-1$, f_1 is s.m. For all $x \in S_1$, $f_1^2(x) = g(x)$ and $f_1(x) \subset S_1$. We now define S_{j+1}, $j = 1, 2, 3, \ldots$, as the union

$$S_{j+1} \cap [p_i, p_{i+1}], \qquad i = 0, \ldots, n-1 ,$$

where

$$S_{j+1} \cap [p_i, p_{i+1}] = [p_i, p_{i+1}] \cap \big\{x\colon\ g(x) \in f_j\big(S_j \cap [f_j(p_i), f_j(p_{i+1})]\big)\big\} .$$

We define f_{j+1} on $S_{j+1} \cap [p_i, p_{i+1}]$ as $f_{j+1}(x) = f_j^{-1}\big(g(x)\big)$ where f_j^{-1} denotes that branch of the inverse of f_j which has its values in the set $[f_j(p_i), f_j(p_{i+1})] \cap S_j$.

Now by B2d,

$$\big(\bar{f}(p_i), \bar{f}(p_{i+1})\big) \cap E = \emptyset ,$$

and so if f_j is s.m. on $[p_i, p_{i+1}]$ for all i, and if $f_j | P(g) \equiv \bar{f}$, then this inverse is well defined and is s.m. on this set. However, $g(x)$ is s.m. on $[p_i, p_{i+1}]$ and so the composition $f_{j+1}(x)$ is s.m. on $[p_i, p_{i+1}] \cap S_{j+1}$. Thus, it follows by induction, that for every j, f_{j+1} is s.m. on the sets $S_{j+1} \cap$

$\cap [p_i, p_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$. It is also clear from the definition of S_{j+1} that $f_j f_{j+1}(x) = g(x)$ for all $x \in S_{j+1}$. We next note that $S_1 \subset S_2$. This reduces to

$$S_1 \cap [p_i, p_{i+1}] \subset [p_i, p_{i+1}] \cap \big[x\colon\ g(x) \in f_1\big(S_1 \cap [f_1(p_i), f_1(p_{i+1})]\big)\big] .$$

If $x \in I_j \cap [p_i, p_{i+1}]$ then from Theorems 2.1 and 2.2 we have that

$$f(x) \in \bigcup_{j=1}^{k} I_j \cap [f(p_i), f(p_{i+1})]$$

and

$$g(x) = f_1^2(x) \in I_j \cap [p_i, p_{i+1}] .$$

Since $\bigcup_{j=1}^{k} I_j \subset S_1$ it follows that

$$g(x) \in f_1\big(S_1 \cap [f(p_i), f(p_{i+1})]\big)$$

and so $x \in S_2$. If $x \in \gamma(g)$, say $x = \gamma_i \in [p_j, p_{j+1}]$, then $f_1(x) = g(x) = x$ and $x \in [f_1(p_j), f_1(p_{j+1}])$. It then follows that $x \in S_2$. Finally, if

$$x \in \Big[\bigcup_{j=0}^{\infty} f_1^j\big(P(g)\big)\Big] \cap [p_i, p_{i+1}] ,$$

say $x = f_1^r(p_j)$, then

$$g(x) = f_1\big(f_1(x)\big) = f_1^{r+2}(p_j) .$$

But $f_1(x) = f_1^{r+1}(p_j) \in S_1$ and by the monotonicity of f_1 on $[p_i, p_{i+1}]$ it belongs to $[f_1(p_i), f_1(p_{i+1})]$. Thus, $x \in S_2$ and it follows that $S_1 \subset S_2$. Now if $x \in S_1$ then $g(x) = f_1\big(f_1(x)\big)$ and so $f_2(x) = f_1^{-1}\big(g(x)\big) = f_1(x)$ which is to say that $f_2|S_1 \equiv f_1$. We now assume that $S_{n-1} \subset S_n$, $f_n|S_{n-1} = f_n$ and show that $S_n \subset S_{n+1}$ and $f_{n+1}|S_n = f_n$. The assertion $S_n \subset S_{n+1}$ may be written

$$[p_i, p_{i+1}] \cap \big[x\colon\ g(x) \in f_{n-1}\big(S_{n-1} \cap [f_{n-1}(p_i), f_{n-1}(p_{i+1})]\big)\big]$$
$$\subset [p_i, p_{i+1}] \cap \big[x\colon\ g(x) \in f_n\big(S_n \cap [f_n(p_i), f_n(p_{i+1})]\big)\big] ,$$

which is an immediate consequence of the fact that $S_{n-1} \subset S_n$ and $f_n|S_{n-1} \equiv f_{n-1}$. Thus, $S_n \subset S_{n+1}$. To see that $f_{n+1}|S_n \equiv f_n$ we note that for $x \in S_n$ we have

$$f_{n-1}\big(f_n(x)\big) = f_n\big(f_n(x)\big) = g(x) .$$

But

$$f_{n+1}(x) = f_n^{-1}\big(g(x)\big) = f_n^{-1}\big(f_n(f_n(x))\big) = f_n(x) ,$$

so that

$$f_{n+1}|S_n = f_n \quad \text{and} \quad f_{n+1}^2(x) = g(x) \quad \text{for all } x \in S_{n+1}.$$

We now define

$$S = \bigcup_{j=1}^{\infty} S_j \quad \text{and} \quad f(x) = \lim_{j\to\infty} f_j(x) \quad \text{for any } x \in S .$$

It is clear that f is s.m. on each of the intervals $[p_i, p_{i+1}]$, $f|S_m = f_m$, and that $f(x) = f^{-1}(g(x))$ for each $x \in S \cap [p_i, p_{i+1}]$, where f^{-1} denotes the branch of the inverse of f with values in $[f(p_i), f(p_{i+1})]$. Since $(f(p_i), f(p_{i+1})) \cap E = \emptyset$ it is clear that f^{-1} is well defined and s.m. Finally, $f^2(x) = g(x)$ for all $x \in S$. It remains to show that $S = [a, b]$ and that $f(x)$ is continuous at each point of S.

We shall now show, for any x, that if $g(x) \in S_j$ then $x \in S_{j+2}$. Let $x \in [p_i, p_{i+1}]$ and denote by $x_j, x_{j+1}, \ldots, x_m$ the points of $X(g)$ which are contained in $(f(p_i), f(p_{i+1}))$. Define

$$\delta = [f(p_i), f(p_{i+1})],$$

$$\delta_0 = [f(p_i), x_j], \qquad \delta_1 = [x_j, x_{j+1}], \qquad \ldots, \qquad \delta_p = [x_m, f(p_{i+1})].$$

Since $f(x_{j+h}) \in E$, $h = 0, 1, \ldots, m-j$, we denote it by e_{j+h}. Thus,

$$(g(p_i), g(p_{i+1})) \cap E = [e_j, \ldots, e_m]$$

and we define

$$K_0 = [g(p_i), e_j], \qquad K_1 = [e_j, e_{j+1}], \qquad \ldots, \qquad K_p = [e_m, g(x_{i+1})].$$

Then, $a = g(x) \in K_i$ for some i, say $g(x) \in K_n$. Define $w = f_j(a)$ and $\gamma = g^{-1}(w)|\delta_n$. Since f_j is s.m. on $\delta_n \cap S_j$, we have that

$$w = f_j(a) \in [f(e_{j+n-1}), f(e_{j+n})] = [g(x_{j+n-1}), g(x_{j+n})]$$

and γ is a single point. It then follows that $\gamma \in S_{j+1}$ and $f_{j+1}(\gamma) = a$. From this it follows that $x \in S_{j+2}$ and $f_{j+2}(x) = \gamma$. But by A2, for every $x \in [a, b]$ there exists an M such that $g^M(x) \in S_1$ and so $x \in S_{2M+1} \subset S$. Thus, $S = [a,b]$. We have already observed that f is s.m. on $[p_i, p_{i+1}]$ and $[f(p_i), f(p_{i+1})]$ for every i. The continuity of f will follow from the following lemma.

LEMMA 2.1. *Let f be s.m. on $[a, b]$ and h be s.m. on $[f(a), f(b)]$. If $g(x) = h(f(x))$ is continuous on $[a, b]$ then f is continuous on $[a, b]$ and h is continuous on $[f(a), f(b)]$.*

Proof of lemma. Since f and h are s.m. it follows that g is s.m. Thus, g defines a homeomorphism of $[a, b]$ onto $[g(a), g(b)]$. It follows that f is a 1-1 map of $[a, b]$ onto $[f(a), f(b)]$ and h is a 1-1 map of $[f(a), f(b)]$ onto $[g(a), g(b)]$. Since f and h are s.m. it follows that they are continuous. This completes the proof of the lemma.

Since $S = [a, b]$ it follows from the above lemma that f is continuous on $[p_i, p_{i+1}]$ for each i. Thus, we have shown that if $\bar{f}$ has the properties stated in the theorem it has an extension $f \in M[a, b] \cap S(2, g)$. This completes the proof of Theorem 2.3.

In the above theorem we have restricted ourselves to solutions f which satisfied the restriction $f(x) = x$ for all $x \in \gamma(g)$. We now display a function $g \in M[a, b]$ for which there does not exist any $f \in M[a, b] \cap \cap S(2, g)$ satisfying this restriction. The construction of g, however, makes it clear that $M[a, b] \cap S(2, g)$ is not empty.

EXAMPLE 2. Define:

$$f(x) = \begin{cases} -x, & x \in [-1, 1], \\ -2+3(x+2), & x \in [-2, -1], \\ -1+3(x-1), & x \in [1, 2] \end{cases}$$

and

$$g(x) = f^2(x).$$

Now if $f \in S(2, g)$ and $f(x) = x$ for $x \in \gamma(g) = [-1, 1] \cup -2 \cup 2$ then it is easily seen that f is not continuous on $[-2, 2]$.

III. Application of Theorem 2.3. In this section, when referreing to a solution of $f^2(x) = g(x)$, we shall always assume that we are restricting ourselves to solutions as described in Theorem 2.3. Theorem 2.3 assures one that, under suitable restrictions on f and g the existence of a solution for $f^2(x) = g(x)$ may be reduced to the existence of a function f defined on $P(g)$ and possessing certain properties. Let f_1, f_2 be defined on $P(g)$ and assume that f_1 may be extended to a solution of $f^2(x) = g(x)$. Then one might expect that the existence of a one to one order preserving map of $f_1(P(g)) \cup P(g) \cup \gamma(g)$ onto $f_2(P(g)) \cup P(g) \cup \gamma(g)$, which is the identity on $P(g) \cup \gamma(g)$, could imply that f_2 would also be extended to a solution of $f^2(x) = g(x)$. If this were true then the problem of solving the equation $f^2(x) = g(x)$ under the given restrictions on f and g would reduce to making a finite set of choices for $f(P(g))$ which define all possible distinct orderings of $f(P(g)) \cup P(g) \cup \gamma(g)$. Unfortunately, this is not the case. However, we shall define the sets $A(f)$, where the number of points in $A(f)$ is finite and essentially independent of f, and B^M, which is finite and depends only on g for which we have the following result. Let f_1 and f_2 be defined on $P(g)$ and assume that there is a one to one order preserving map of $A(f_1) \cup B^M \cup \gamma(g)$ onto $A(f_2) \cup B^M \cup \gamma(g)$ which is the identity on $\gamma(g) \cup P(g)$. We then say that $A(f_1)$ and $A(f_2)$ are equivalent. In Theorem 3.1 we show that if $A(f_1)$ and $A(f_2)$ are equivalent and if f_1 can be extended to a continuous solution of (1.2) on $[a, b]$ then f_2 may also be extended to a continuous solution of (1.2) on $[a, b]$. Corollary 3.2 then asserts that the existence or non existence of a continuous solution of (1.2) as described in Theorem 2.3 may be determined by examining a finite number of finite sets. To facilitate the statement and proof of Theorem 3.1 we first make several observations and definitions.

Consider any terminal segment I, and let $J(I)$ be as defined in B3. Denote the elements of $J(I)$ by $x_1, \dots, x_r$. Then we may assume that M has been chosen so large that the set

$$[g^{M-1}(x_1), g^M(x_1)) \cap \bigcup_{l=1}^{\infty} g^l(x_j)$$

is not empty and contains for each j exactly one point of the form $g^k(x_j)$, where k depends on j but $k \leqslant M$ for all j. If $I = I(j, \pm)$ is a terminal segment, but $I(j, \mp)$ is not, then for any solution f of $f^2(x) = g(x)$ the set

$$\{[g^{M-1}(x_1), g^M(x_1)] \cap [\bigcup_{l=1}^{\infty} g^l(x_j) \cup \bigcup_{k=1}^{\infty} g^k(f(x_j))]\}$$

contains for each j exactly two points of the form $g^l(x_j)$ and $g^h(f(x_j))$ where h, l depend on j, $h \leqslant M$, and $h = l$ or $h = l \pm 1$. If both $I_1 = I(j, +)$ and $I_2 = I(j, -)$ are terminal segments and f is any solution of $f^2(x) = g(x)$ such that $f(I_1) \subset I_1$, $f(I_2) \subset I_2$, then the analogous results hold in I_1 and I_2. If $f(I_1) \subset I_2$, $f(I_2) \subset I_1$, $J(I_1) = \{x_1, \dots, x_r\}$, and $J(I_2) = \{y_1, \dots, y_r\}$ then for every j and n the set

$$\{[g^{M-1}(x_1), g^M(x_1)] \cap [\bigcup_{l=1}^{\infty} g^l(x_j) \cup \bigcup_{i=1}^{\infty} g^i(f(y_n))]\}$$

contains two points $g^l(x_j)$ and $g^h(f(y_n))$, where l depends on j, h depends on n, $l \leqslant M$, $h+1 \leqslant M$. The set

$$\{[g^{M-1}(f(x_1)), g^M(f(x_1))] \cap [\bigcup_{k=1}^{\infty} g^k(y_m) \cup \bigcup_{m=1}^{\infty} g^m(f(x_j))]\}$$

also contains two points $g^k(y_n)$, $g^m(f(x_j))$, where $m = l$, $k = h+1$.

The above statements are a direct consequence of Theorem 2.1, Theorem 2.2, and the following lemma.

LEMMA 3.1. *Let $I_1 = (\gamma_i, p_{i+1})$, $I_2 = (p_i, \gamma_i)$ be terminal segments and f be as given in* (2) *of Theorem* 2.2. *Assume also that f satisfies the conditions of Theorem* 2.3. *Then either*

(1) $f(p_i) \in (p_{i+1}, g(p_{i+1}))$ *and* $f(p_{i+1}) \in (p_i, g(p_i))$,

(2) $f(p_i) = p_{i+1}$ *and* $f(p_{i+1}) = g(p_i)$, *or*

(3) $f(p_{i+1}) = p_i$ *and* $f(p_i) = g(p_{i+1})$.

Proof. The restriction, $f(\gamma_j) = \gamma_j$ for $\gamma_j \in \gamma(g)$, justifies the assumption that the endpoints of I_1 and I_2, other than γ_i, are not in $\gamma(g)$. If $[p_i, p_{i+1}] = [a, b]$ we have either (1), (2) or (3). Thus, we will assume that $[p_i, p_{i+1}] \neq [a, b]$. Then either p_i or p_{i+1} must belong to E. If neither $p_i = a$ nor $p_{i+1} = b$ then both p_i, p_{i+1} must belong to $X(g)$. Assume p_i and p_{i+1} do not belong to E, then since $f(X-E) \subset E$ it is clear that $f(p_i) \neq p_{i+1}$ and $f(p_{i+1}) \neq p_i$. But if $f(p_{i+1}) < p_i$ and $f(p_i) \geqslant p_{i+1}$, there is a $y \in [p_i, \gamma_i]$

such that $f(y) = p_{i+1}$ and $g(y) = f^2(y) < p_i$ which is impossible. If $f(p_i) < p_{i+1}$ and $f(p_{i+1}) < p_i$, then $f(p_i) \notin E$ which is impossible. If $f(p_{i+1}) \geqslant p_i$ we have $f(p_{i+1}) \notin E$ which is impossible. Thus, either p_i or p_{i+1} belongs to E. If both p_i, p_{i+1} belong to $E \cup a \cup b$ then f satisfies (1), (2), or (3). If only one of p_i and p_{i+1} belongs to $E \cup a \cup b$ then f satisfies either (2) or (3). This completes the proof of Lemma 3.1.

Denote the terminal segments of g by $I_1, \ldots, I_m$. Then for each I_j there is an $x_1^j \in J(I_j)$ which possesses the same properties with respect to I_j as x_1 does with respect to I, in the above discussion. Define

$$\sigma_j = [g^{M-1}(x_1^j), g^M(x_1^j)] \quad \text{for} \quad j = 1, \ldots, m.$$

If f is a solution of $f^2(x) = g(x)$, we define

$$S(f) = \gamma(g) \cup P(g) \cup \bigcup_{l=1}^{M} \Big[g^l\big(P(g)\big) \cup g^l\Big(f\big(P(g)\big)\Big)\Big].$$

We also define the sets: $B^0 = P(g)$, $B^k = B^{k-1} \cup g^k\big(P(g)\big)$ for $k = 1, 2, \ldots$; $A_i(f) = S(f) \cap \sigma_i$ for $i = 1, \ldots, m$; and $A(f) = \bigcup_{i=1}^{m} A_i(f)$.

THEOREM 3.1. *Let f be a solution of $f^2(x) = g(x)$ and A be a finite set of points such that $A \subset \bigcup_{i=1}^{m} \sigma_i$ and $A \cap B^M = \emptyset$. Let ζ be a one to one order preserving map of*

$$[(A \cup B^M) \cap \bigcup_{i=1}^{m} \sigma_i] \cup \gamma(g) \quad \textit{into} \quad \bigcup_{i=1}^{m} \sigma_i \cup \gamma(g).$$

Assume that $\zeta\big((A \cup B^M) \cap \sigma_i\big) = A_i(f)$ and $\zeta|[(B^M \cap \sigma_i) \cup \gamma(g)]$ is the identity map for $i = 1, \ldots, n$. Then there exists a solution f_1 of $f^2(x) = g(x)$ such that

$$A_i(f_1) = (A \cup B^M) \cap \sigma_i \quad \textit{for} \quad i = 1, \ldots, m.$$

To aid in the proof of Theorem 3.1 we establish the following lemma.

LEMMA 3.2. *Let $[h_1, h_2] \cap B^k$, $k \geqslant 1$, be empty and $y = g^{-1}(h_1)$ be given. Then there exists a $\sigma \in g^{-1}(h_2)$ such that $[\sigma, y] \cap B^{k-1}$ is empty.*

Proof. Since $h_1 \notin B^k$ we may assume that $y \in (p_j, p_{j+1})$ for some j. Since $[h_1, h_2] \cap B^k$ is empty, it follows that $g^{-1}(h_2) \cap (p_j, p_{j+1})$ contains a unique point σ. If $[\sigma, y] \cap B^{k-1}$ is not empty, since $g(x)$ is monotone and continuous on (x_j, x_{j+1}), it follows that $[h_1, h_2] = [g(\sigma), g(y)] \cap B^k$ is not empty. Thus, $[\sigma, y] \cap B^{k-1}$ is empty and our result follows.

Proof of Theorem 3.1. For every $p_j \in P(g)$ there is an integer $i(j)$, $i(j)+1 \leqslant 2M$, such that $f^{i(j)}(p_j)$ and $f^{i(j)+1}(p_j) \in \bigcup_{i=1}^{m} A_i(f)$ or $f^{i(j)+1}(p_j) \in \gamma(g)$. We define $f_1^l(p_j) = \zeta^{-1}\big(f^l(p_j)\big)$ for $l = i(j)$, $i(j)+1$. Now let k be odd and $k = i(j)$ or $i(j)+1$. If $p_j \in J(I_i)$ for some i, then either $f^k(p_j) \in B^M$ or $f^k(p_j) \notin B^M$. If $f^k(p_j) \notin B^M$ then $[f^k(p_j), f_1^k(p_j)] \cap B_m$ is empty. By repeated application of Lemma 3.2 we define $f_1^n(p_j)$ in such a way that $[f_1^n(p_j), f^n(p_j)] \cap B^l$ is empty, where $l = (n-1)/2$, $n = k-2, k-4, \ldots, 1$. If $f^k(p_j) \in B^M$ then $f^{k+1}(p_j) \in B^{M+1}$, since k is odd and $f^{2n}(p_j) = g^n(p_j)$ for any n. In this case we define $f_1(p_j) = f(p_j)$. If $f^k(p_j) = \gamma_i \in \gamma(g)$ we define $f_1(p_j) = f(p_j)$. Finally, we define $f_1(\gamma_i) = f(\gamma_i) = \gamma_i$ for all $\gamma_i \in \gamma(g)$. It now remains to show that f_1 satisfies conditions B1, B2 and B3 of Theorem 2.3. That f_1 satisfies B3 is an immediate consequence of the fact that B3 holds for f, ζ is a one to one order preserving map, and $g|I_i$ is s.m. for all i. That f_1 satisfies B1 is an immediate consequence of the definition of f_1 and the fact that f satisfies B1. To facilitate the proof of the fact that f_1 satisfies B2, we establish the following lemma.

LEMMA 3.3. *The correspondence $f^i(p_j) \to f_1^i(p_j)$ and $f^i(\gamma_j) \to f_1^i(\gamma_j)$ for all i, j defines a one to one order preserving map σ of $S(f)$ onto $S(f_1)$.*

Proof. The one to one nature of the map follows from the finite nature of $S(f)$, $S(f_1)$ and the fact that for any k, l, i, j, $f_1(g^k(p_i)) = f_1(g^l(p_j))$ if and only if $f(g^k(p_i)) = f(g^l(p_j))$. Since g is s.m. on each I_j and the map has the desired property on each σ_j, it follows that the map restricted to I_j is order preserving. To see that the map is order preserving on the complement of the terminal segments we first observe that it is the identity on $\gamma(g)$. Thus it will suffice to show for any $u, v \in B^m \cap S(f) \cap$ $\cap$ [Complement of $\bigcup_{i=1}^{m} I_i$], $f(u) \neq f_1(u)$, $f(v) \neq f_1(v)$, that if $f(v) \in (f(u), f_1(u))$ then $f_1(u) \in (f(v), f_1(v))$. However, from the definition of f_1 and Lemma 3.2 it follows that there exists a k such that g^k defines a homeomorphism of $[f(u), f_1(u)]$ and $[f(v), f_1(v)]$ onto subintervals of σ_j for some j. Since ζ is order preserving on σ_j it follows that

$$g^k(f_1(u)) = f_1(g^k(u)) \in \big(g^k(f(v)), g^k(f_1(v))\big).$$

Thus $f_1(u) \in (f_1(v), f(v))$. This completes the proof of Lemma 3.3.

To complete the proof of Theorem 3.1 it remains to establish that f_1 satisfies B2. Since $f_1^2(p_i) = g(p_i) \neq g(p_{i+1}) = f^2(p_{i+1})$, it follows that $f_1(p_i) \neq f_1(p_{i+1})$ for all i. Since the mapping σ is order preserving it follows that the set E is the same for f_1 and f. Since f satisfies B2 and σ is order preserving, it now follows that f_1 also satisfies B2. This completes the proof of Theorem 3.1.

We first observe that the set A as described in Theorem 3.1 may contain at most n points, where n is the number of partition points in $P(g)$. It may contain less. Also the number of points in $A \cap \sigma_i$ is equal to or less than the number of points in $J(I_i)$. Thus, the number of possible orderings for the sets $(A \cup B^k) \cap \sigma_i$, for all possible choices of A, is finite. Given a function g one may determine B^M and the σ_i. Thus, one may select a finite collection of sets $A(k)$, $k = 1, \ldots, r$, which realize every possible ordering for the sets $(A \cup B^k) \cap \sigma_i$, $i = 1, \ldots, m$. Theorem 3.1 thus has the following corollary.

COROLLARY 3.1. *The equation $f^2(x) = g(x)$ possesses a solution as described in Theorem 3.3 if and only if for some $k \in [1, \ldots, n]$, $A(k) = A(f)$, where f is a solution of $f^2(x) = g(x)$ as described in Theorem 2.3.*

There remains the problem of determining if $A(k) = A(f)$ for some solution f of $f^2(x) = g(x)$. However, if $A(f) = A(k)$ then it is clear that

$$A(f) \subset D\big(A(k)\big) = \bigcup_{i=1}^{m} g^{-1}\Big(\Big[A(k) \cup \Big(\bigcup_{i=1}^{m} (B^M \cap \sigma_i)\Big) \cup \gamma(g)\Big]\Big).$$

Since $g \in M[a, b]$, the set $D\big(A(k)\big)$ is finite. Thus, the number of possible choices of $f\big(P(g)\big)$ such that $f\big(P(g)\big) \subset D\big(A(k)\big)$ is finite.

COROLLARY 3.2. *The equation $f^2(x) = g(x)$ possesses a solution as described in Theorem 2.3 if and only if for some $k \in [1, \ldots, n]$ the finite set $D\big(A(k)\big)$ contains a subset $f\big(P(g)\big) \cup \gamma(g)$ such that the extension of f by iteration satisfies the conditions of Theorem 2.3.*

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