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# Infinitesimal automorphisms of distributions

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**Abstract.** The main purpose of this paper is to investigate properties of sheaves of germs of infinitesimal automorphisms of distributions. Section 1 contains preliminaries. In Section 2 integrable distributions are discussed. Section 3 deals with the basic properties of sheaves of germs of infinitesimal automorphisms of distributions. The completely integrable distributions are considered in Section 4. In this section are also presented conditions for complete integrability which take into consideration the sheaves of germs of infinitesimal automorphisms. The problem of interrelations between distributions having the same sheaves of germs of infinitesimal automorphisms is discussed at the end of Section 4.

**1. Preliminaries.** Throughout the whole paper $M$ will denote a paracompact differentiable manifold of dimension $n$ and $\mathscr{V}$ – the sheaf of the germs of the vector fields of $M$. A mapping $\mathscr{M}$ which to each point $m$ of a manifold $M$ attaches a vector subspace of $T_m M$ is called a *distribution* on the manifold $M$.

A distribution $\mathscr{M}$ will be called *smooth* if there exists a set $D$ of vector fields in $M$ ($X \in D \Rightarrow \operatorname{dom} X$ open in the topology of $M$) such that $\mathscr{M}_m = \rangle\{X_m, X \in D\}\langle$ for each point $m$ of $M$. A distribution $\mathscr{M}$ is said to be *integrable* when for each point $m$ of the manifold $M$ there exists exactly one maximal integral submanifold of the distribution $\mathscr{M}$ through $m$, and is completely integrable if it is of constant dimension and involutive.

A $C^\infty$ diffeomorphism of an open subset $U$ of a manifold $M$ onto an open subset $U'$ of a manifold $M$ is called a *local diffeomorphism of* $M$. Such a local diffeomorphism $f$ which leaves a distribution invariant, i.e., $f^*(\mathscr{M}/U) \subset \mathscr{M}/U'$, is said to be an *automorphism* of the distribution $\mathscr{M}$. A set of local diffeomorphisms $P$ is said to be a *pseudogroup* if the following conditions are fulfilled:

(i) If $f\colon U \to M \in P$, then for each $V \subset U$, $V$ open in topology of $M$, $f/V \in P$.

(ii) If $f\colon U \to M$ is a diffeomorphism on its image, $U = \bigcup U_\alpha$ and $f/U_\alpha \in P$, then $f \in P$.

(iii) If $f\colon U \to M \in P$, then the inverse local diffeomorphism $f^{-1}\colon f(U) \to M \in P$.

(iv) If $f\colon U \to M \in P$, $g\colon f(U) \to M$ belong to $P$, then $gf \in P$.

(v) The identity diffeomorphism of the manifold $M$ belongs to $P$.

The set of all automorphisms of a distribution $\mathcal{H}$ is a pseudogroup. A vector field $X$ is an infinitesimal automorphism of a distribution if its flow $\varphi_t$ consists of automorphisms of the distribution (i.e., $\varphi_t$ are automorphisms of the distribution $\mathcal{H}$). The sheaf of the germs of the infinitesimal automorphisms of a distribution $\mathcal{H}$ will be denoted by $\mathcal{U}_{\mathcal{H}}$.

It is possible to associate with any given sheaf of germs of vector fields $\mathcal{L}$ a smooth distribution $\mathcal{H}_{\mathcal{L}}$

$$\mathcal{H}_{\mathcal{L}m} = \rangle\{X_m\colon (X)_m \in \mathcal{L}_m\}\langle$$

and a pseudogroup $P(\mathcal{L})$ — the smallest pseudogroup containing all $\varphi_t$, where $\varphi_t$ are flows generated by the vector fields belonging to $\mathcal{L}(U)$ ($U$ — an open set in the topology of the manifold $M$). In accordance with this definition it is possible to consider smooth distributions as sheaves of germs of vector fields.

A sheaf $\mathcal{L}$ is called *regular* if the dimension of $\mathcal{H}_{\mathcal{L}}$ is constant, and transitive if it is equal to $n$.

Let us now consider a subset $D$ of the set $V = \bigcup \mathcal{V}(U)$ ($U$ — an open set in topology of the manifold $M$). The subset $D$ is said to be *everywhere defined* if the domains of elements of $D$ form a covering of $M$. An everywhere defined pseudogroup is analogously defined.

If $P$ is an everywhere defined pseudogroup, two points $m$, $m'$ of the manifold $M$ are said to be *P-equivalent* if there exists $f \in P$ such that $f(m) = m'$; the equivalence classes of this relation are called *P-orbits.*

When a set of vector fields $D$ is given it is possible to construct a pseudogroup $P_D$ in an analogous way as in the case of sheaves. A $P_D$-orbit will be called a *D-orbit*. Orbits of the set $D$ can be given a natural topology. Let $m \in M$, $\xi \in D^k$, $T \in R^k$ and let $\varrho_{\xi,m}$ denote the following mapping

$$\varrho_{\xi,m}\colon T \mapsto \xi_T(m) = \varphi^1_{t_1} \circ \ldots \circ \varphi^k_{t_k}(m),$$

where $T = (t_1, \ldots, t_k)$, $\xi = (X_1, \ldots, X_k)$, $\varphi^i_t$ is the flow generated by $X_i$. A $D$-orbit is a union of images of these mappings. The orbit is given the finest topology in which the mappings $\varrho_{\xi,m}$ are continuous. This topology is finer than the topology induced by the topology of the manifold $M$. As it is easily seen, the $D$-orbits are connected in this topology.

The two theorems which are given below are well known but very useful in the theory of distributions. The proofs can be found in [4].

THEOREM 1. *Let $X_1, \ldots, X_p$ be vector fields on a manifold $M$, which are linearly independent at each point of the manifold and $[X_i, X_j] = 0$ for $1 \leq i$, $j \leq p$. Then for each point $m$ of the manifold $M$ there exists a chart $(U, \varphi)$, $\varphi = (x_1, \ldots, x_n)$ at this point such that $X_i = \partial/\partial x_i$ for $i = 1, \ldots, p$.*

THEOREM 2 (Frobenius). *A smooth distribution $\mathcal{M}$ of a constant dimension $p$ is completely integrable if and only if for each point $m$ of the manifold $M$ there exists a chart $(U, \varphi)$, $\varphi = (x_1, \ldots, x_n)$ at this point such that for any $\{c_j\}$, $p < j \leqslant n$, $c_j \in R$, the sets $U_c = \{x \in U,\ x_j = c_j,\ p < j \leqslant n\}$ are integral submanifolds of the distribution $\mathcal{M}$.*

If a distribution $\mathcal{M}$ is given, two very useful distributions can be constructed:

$\mathcal{M}^*$ — the smallest involutive distribution containing $\mathcal{M}$,

$\mathcal{C}_{\mathcal{M}}$ — the so-called characteristic distribution.

If $\mathcal{E}(M, \mathcal{M})$ denotes the set of all involutive distributions on $M$ containing $\mathcal{M}$, then $\mathcal{M}^* = \bigcap \mathcal{E}(M, \mathcal{M})$ and for any open set $V$ in the topology of $M$, $\mathcal{M}^*/V = (\mathcal{M}/V)^*$. $\mathcal{C}_{\mathcal{M}}$ can be defined in the following way:

$\mathcal{C}_{\mathcal{M}} = \mathcal{M}_{\mathcal{L}}$ where $\mathcal{L} = \mathcal{W}_{\mathcal{M}} \cap \mathcal{S}_{\mathcal{M}}$, $\mathcal{S}_{\mathcal{M}}$ — the sheaf of the germs of the vector fields belonging to $\mathcal{M}$ (notation: $X \in \mathcal{M}$).

The following relations are straightforward:

$$\mathcal{C}_{\mathcal{M}} \subset \mathcal{M} \subset \mathcal{M}^*.$$

When an everywhere defined set of vector fields $D$ is given, it is possible to construct a distribution $\mathcal{M}^D$, the smallest distribution on $M$ for which elements of $D$ are infinitesimal automorphisms. This distribution plays a very important role in the theory of integrable distributions.

The following relations are apparent:

$$\mathcal{M}_D \subset \mathcal{M}_D^* \subset \mathcal{M}^D,$$

where

$$\mathcal{M}_{D^m} = \rangle\{X_m\colon\ X \in D\}\langle.$$

**2. Integrable distributions.** Since the orbits of sets of vector fields are of great importance in the theory of integrable distributions, it seems necessary to give the basic theorem on orbits which, at the same time, links them with integrable distributions.

THEOREM 3 [5]. *Let $D$ be an everywhere defined set of vector fields in $M$. Then*:

(a) *If $S$ is a $D$-orbit, then $S$* (with the topology defined in Section 1) *admits a unique differentiable structure making $S$ into a submanifold of $M$,*

(b) *with the topology and differentiable structure resulting from* (a), *each $D$-orbit is a maximal integral submanifold of the distribution $\mathcal{M}^D$,*

(c) *$\mathcal{M}^D$ is an integrable distribution,*

(d) *$\mathcal{M}^D$ is an involutive distribution.*

Also in that paper an important theorem on integrable manifolds is proven. Many classical results are simple corollaries of this theorem.

THEOREM 4 [5]. *If $\mathcal{M}$ is a smooth distribution generated by a set of vector fields $D$, then the following conditions are equivalent:*

(i) *$\mathcal{M}$ is an integrable distribution,*

(ii) *for any point $m$ of the manifold $M$, there exists an integral submanifold of the distribution $\mathcal{M}$ containing the point $m$,*

(iii) *the distribution $\mathcal{M}$ is $D$-invariant, i.e., the elements of $D$ are infinitesimal automorphism of the distribution $\mathcal{M}$,*

(iv) *for any point $m$ of the manifold $M$, there exist vector fields $X^1, \ldots, X^k$ belonging to $D$ such that*

(a) $\mathcal{M}_m = \rangle\{X^1(m), \ldots, X^k(m)\}\langle$,

(b) *for any vector field $X$ belonging to $D$ there exist $\varepsilon > 0$ and smooth functions $f_j^i$ $(1 \leqslant i, j \leqslant k)$ defined on the interval $(-\varepsilon, \varepsilon)$ for which the following equations are true:*

$$[X, X^i](\varphi_t(m)) = \sum_{j=1}^{k} f_j^i(t) X^j(\varphi_t(m)), \qquad -\varepsilon < t < \varepsilon, \ i = 1, \ldots, k,$$

*where $\varphi_t$ is the flow generated by the vector field $X$,*

(v) $\mathcal{M} = \mathcal{M}^D$.

A set of vector fields $D$ is said to *fulfil the reachability condition* if $D$-orbits are exactly the connected components of the manifold $M$.

THEOREM 5 (Chow) [5]. *Let $D$ be an everywhere defined set of vector fields. Then $D$ fulfils the reachability condition if and only if $\mathcal{M}^D$ is the distribution of constant dimension $n$.*

As an orbit of the set of all vector fields belonging to the distribution $\mathcal{M}$ will be called an *$\mathcal{M}$-orbit*, the following proposition is true:

PROPOSITION 1. *If $\mathcal{M}$ is a smooth distribution, the following conditions are equivalent:*

(i) *$\mathcal{M}$ is an integrable distribution,*

(ii) *$\mathcal{M}$-orbits are integral submanifolds of the distribution $\mathcal{M}$,*

(iii) $\mathcal{M} = \mathcal{C}_{\mathcal{M}}$,

(iv) *$\mathcal{M}$-orbits are equal to $\mathcal{C}_{\mathcal{M}}$-orbits,*

(v) $\mathcal{D}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}}$.

Proof. Let $D = U\mathcal{D}_{\mathcal{M}}(U)$, where $U$ is an open subset of the manifold $M$. The distribution $\mathcal{M}$ can be considered as the distribution generated by the set $D$.

(i) $\Rightarrow$ (ii). (i) is equivalent to the equality $\mathcal{M} = \mathcal{M}^D$ (Theorem 4), then the $D$-orbits which are the $\mathcal{M}$-orbits, are the integral submanifolds of the distribution $\mathcal{M}$.

(ii) $\Rightarrow$ (i). Since the $\mathcal{M}$-orbits are the $D$-orbits then they are the integral submanifolds of the distribution $\mathcal{M}$. They are also the integral submanifolds of the distribution $\mathcal{M}^D$ (Theorem 3), so $\mathcal{M} = \mathcal{M}^D$ and $\mathcal{M}$ is an integrable distribution.

(i)$\Rightarrow$(v). Since Theorem 4 states that the elements of $D$ are infinitesimal automorphism of the distribution $\mathcal{M}$ so $\mathcal{D}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}}$.

(v)$\Rightarrow$(iii). It is a consequence of the definition of the characteristic distribution.

(iii)$\Rightarrow$(iv). It is a consequence of the definition of an orbit.

(iv)$\Rightarrow$(i). If we denote by $D'$ the set $\bigcup \mathcal{U}_{\mathcal{M}}(U) \cap \mathcal{D}_{\mathcal{M}}(U)$ (where $U$ is an open subset of $M$), then $\mathcal{C}_{\mathcal{M}} = \mathcal{M}_{D'}$ is $D'$ invariant, so $\mathcal{C}_{\mathcal{M}} = \mathcal{M}^{D'}$ according to Theorem 4. A $\mathcal{C}_{\mathcal{M}}$-orbit is an integral submanifold of the distribution $\mathcal{C}_{\mathcal{M}}$, and an $\mathcal{M}$-orbit is an integral submanifold of the distribution $\mathcal{M}^D$ hence $\mathcal{M}^D = \mathcal{C}_{\mathcal{M}}$, and $\mathcal{M} = \mathcal{M}^D = \mathcal{C}_{\mathcal{M}}$.

**3. Basic properties of the sheaf $\mathcal{U}_{\mathcal{M}}$.** In this section we will gather the properties of sheaves of germs of vector fields and particularly of the sheaf $\mathcal{U}_{\mathcal{M}}$ which will be used later.

PROPOSITION 2 [2]. *The sheaf $\mathcal{U}_{\mathcal{M}}$ is a sheaf of Lie algebras of germs of vector fields.*

PROPOSITION 3. *If $M$ is a connected manifold and $\mathcal{M}$ a distribution on $M$, then $M$ is a $\mathcal{U}_{\mathcal{M}}$ orbit if and only if $\mathcal{U}_{\mathcal{M}}$ is a transitive sheaf.*

Proof. *Sufficiency.* That is Proposition 11 of [2].

*Necessity.* Chow's theorem states that $\dim \mathcal{M}_m^{U_{\mathcal{M}}} = n$ (for any $m$ of $M$), where

$$U_{\mathcal{M}} = \{X\colon \forall m \in \operatorname{dom} X\, (X)_m \in \mathcal{U}_{\mathcal{M}m}\}.$$

Now it is sufficient to show that $\mathcal{M}^{U_{\mathcal{M}}} = \mathcal{M}_{\mathcal{U}_{\mathcal{M}}}$, i.e., $\mathcal{M}_{\mathcal{U}_{\mathcal{M}}}$ is $\mathcal{U}_{\mathcal{M}}$-invariant. It is a consequence of Proposition 2.

PROPOSITION 4 [2]. *If $\mathcal{M}$ is a smooth distribution of constant dimension, $X$ is an infinitesimal automorphism of $\mathcal{M}$ if and only if $[X, Y] \in \mathcal{M}$ for any $Y \in \mathcal{M}$.*

PROPOSITION 5 [2]. *If $\mathcal{U}_{\mathcal{M}}$ is a transitive sheaf, then $\mathcal{M}$ is a regular distribution of constant dimension on connected components of the manifold $M$.*

LEMMA 1 [2]. *Let $\mathcal{M}$ be a distribution, $\mathcal{M}^*$ the smallest involutive distribution containing $\mathcal{M}$, and $\mathcal{C}_{\mathcal{M}}$ the characteristic distribution; then*

(a) $\mathcal{U}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}^*}$,

(b) $\mathcal{U}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{C}_{\mathcal{M}}}$.

Remark. If $\mathcal{U}_{\mathcal{M}}$ is a transitive sheaf and $M$ a connected manifold, then the stalks of $\mathcal{U}_{\mathcal{M}}$ have the same real dimension. It is sufficient to notice that $\varphi \in P(\mathcal{U}_{\mathcal{M}})$ induce isomorphisms of the stalks.

LEMMA 2. *When the dimension of the stalks of the sheaf $\mathcal{S}_{\mathcal{M}}$ for a smooth distribution $\mathcal{M}$ is at most denumerable, it is equal to 0 (i.e., $\mathcal{M} \equiv 0$).*

**Proof.** Let $\mathcal{M} \not\equiv 0$. There exists $m_0 \in M$ such that $\mathcal{M}_{m_0} \neq 0$. Since $\mathcal{M}$ is a smooth distribution, it is possible to find an open neighbourhood $U$ of the point $m_0$ such that for any point $m$ of $U$, $\mathcal{M}_m \neq 0$. Then there exists a map $(U_0, \varphi)$ at the point $m_0$ $(U_0 \subset U, \varphi = (x_1, \ldots, x_n))$ for which $\partial/\partial x_1 \in \mathcal{M}$.

Further on the vector fields $\partial/\partial x_i$ will be denoted by $\partial_i$.

So, for each smooth function $f$ on $U_0$, $f\partial_1 \in \mathcal{M}$; hence the dimension of the stalks of the sheaf $\mathcal{S}_{\mathcal{M}}$ at the points of $U_0$ is neither denumerable nor finite. A contradiction.

PROPOSITION 6. *Let $\mathcal{L}$ be a sheaf of vector spaces of germs of vector fields. If for any point $m$ of the manifold $M$ the codimension of the stalk of $\mathcal{L}$ is at most denumerable, $\mathcal{L}$ is a transitive sheaf.*

**Proof.** Since the sheaf $\mathcal{L}$ is regular if and only if the dimension of the distribution $\mathcal{M}_{\mathcal{L}}$ is constant, it is sufficient to prove that the dimension is equal to $n$. If $\dim \mathcal{M}_{\mathcal{L}m} = k < n$, it is possible to construct locally a supplementary distribution $S$, i.e., $\mathcal{M}|_V \oplus S = \mathcal{M}_0|V$, where $\mathcal{M}_0 = (m \mapsto TmM)$ and $V$ is an open set. The germ of a vector field $X \in S$ at a point $m \in V$ does not belong to $\mathcal{L}_m$ so the dimension of the stalks of $\mathcal{S}_S$ is at most denumerable. Hence, according to Lemma 2, $S$ is a trivial distribution ($S \equiv 0$). A contradiction.

**Remark.** If $\mathcal{L}$ is a normal sheaf (i.e. $P(\mathcal{L})^*(\mathcal{L}) \subset \mathcal{L}$), then $\mathcal{M}_{\mathcal{L}}$ is a completely integrable distribution if and only if $\mathcal{L}$ is a regular sheaf.

**Proof.** Complete integrability of the distribution $\mathcal{M}_{\mathcal{L}}$ implies a constant dimension, so $\mathcal{L}$ is a regular sheaf. To prove the necessity of the condition it is sufficient to show that the distribution $\mathcal{M}_{\mathcal{L}}$ is involutive. The hypothesis entails that $\varphi^* X \in \mathcal{L}(\varphi(V))$ when $X \in \mathcal{L}(V)$, $\varphi \in P(\mathcal{L})$. Taking instead of $\varphi$, $\varphi_t$ – the flow of a vector field $Y$, $Y \in \mathcal{L}$, gives $\varphi_t^* X \in \mathcal{M}_{\mathcal{L}}$, so $[X, Y] \in \mathcal{M}_{\mathcal{L}}$.

The next proposition was proved in the case of a finite dimensional stalk in [6]. The proof is also valid for any sheaf of germs of vector fields.

PROPOSITION 7. *Let $\mathcal{L}$ be a transitive sheaf of Lie algebras of germs of vector fields and $\mathfrak{N}$ the normalizer sheaf of the sheaf $\mathcal{L}$ in $\mathcal{V}$. The codimension of the stalks of the sheaf $\mathcal{L}$ in the stalks of the sheaf $\mathfrak{N}$ is finite.*

The proof is based on the following lemma.

LEMMA 3 [1]. *Let $U$ be a simply connected domain in $\mathbf{R}^n$ with the coordinates $x^1, \ldots, x^n$ and let $X_1, \ldots, X_n$ be vector fields on $U$ linearly independent at each point of $U$. If a point $x_0 \in U$ and a vector $\zeta = (\zeta^1, \ldots, \zeta^n) \in \mathbf{R}^n$ are given, a vector field $Y$ on $U$ such that $Y = \sum_{i=1}^{n} \zeta^i(x)\partial_i$, $\zeta^i(x_0) = \zeta^i$, $[Y, X_i] = 0$ for $i = 1, \ldots, n$ is uniquely determined.*

Let a point $m$ and a simply connected open neighbourhood $U$ of that point be given such that there exist vector fields $X_1, \ldots, X_n$ on $U$ which are linearly independent at points of $U$ and $X_i \in \mathscr{L}(U)$ for $i = 1, \ldots, n$. The kernel $\mathscr{K}(U)$ of the representation $\mathfrak{N}(U)$ on $\mathscr{L}(U)$ consists of vector fields $Y$ for which $[Y, X] = 0$ for any $X \in \mathscr{L}(U)$. In particular, $[Y, X_i] = 0$ for $i = 1, \ldots, n$. According to Lemma 3, $\mathscr{K}(U)$ has a finite dimension. Since every point has a base consisting of simply connected sets, the sheaf has stalks of finite dimension. The following inequality is always true: $\dim \mathfrak{N}_m \leqslant \dim \mathscr{L}_m + \dim \mathscr{K}_m$ so $\operatorname{codim}_{\mathfrak{N}_m} \mathscr{L}_m \leqslant \dim \mathscr{K}_m$.

Now we will discuss the relation between sheaves of germs of infinitesimal automorphisms of regular distributions for which $\mathscr{U}_{\mathscr{M}'} \supset \mathscr{U}_{\mathscr{M}}$ and $\mathscr{M}' \subset \mathscr{M}$.

LEMMA 4. *Let $U$ be an open neighbourhood of 0 in $\boldsymbol{R}^n$ with the coordinates given by the inclusion into $\boldsymbol{R}^n$, $\mathscr{M}$, $\mathscr{M}'$ two distributions on $U$ for which there exist vector fields $X_1, \ldots, X_p$; $X_i = \sum_{j=1}^{n} g_i^j \partial_j$ and $\mathscr{M}' = \rangle X_1, \ldots, X_k \langle$, $\mathscr{M} = \rangle X_1, \ldots, X_p \langle$, $1 \leqslant k \leqslant p < n$, and additionally let the $k_0$-th row of the matrix $(g_i^j)$ be linearly dependent. If there exists a vector field $X = \sum_{i=1}^{n} f_i \partial_i$ belonging to $\mathscr{M}$ such that for a certain denumerable set of points $\{m_r\}$ of $U$, $\partial_j(f^i)(m_r) = 0$ for $i, j = 1, \ldots, n$ and there exists $i_0$ for which $f_{i_0}(m_r) \neq 0$ for any $r$ and, moreover, $\dim \mathscr{U}_{\mathscr{M}'}(U)/\mathscr{U}_{\mathscr{M}}(U) < \infty$, then $f\partial_{k_0}$ are infinitesimal automorphisms of the distribution $\mathscr{M}'$ for functions which are solutions of the equations*

$$(1) \qquad \Big(\sum_{j=1}^{n} f_j \partial_j(f)\Big)(m_r) = c_r$$

*for an at most finite-dimensional space of sequences $\{c_r\}$.*

Proof. Let us assume for simplicity sake that $k_0 = n$. We are going to find what conditions functions $f$ have to fulfil so that $f\partial_n$ will not be an infinitesimal automorphism of the distribution $\mathscr{M}$. It is enough to check that it is not true that $[X, f\partial_n] \in \mathscr{U}_{\mathscr{M}}(U)$.

Computing we get:

$$\Big[\sum_{i=1}^{n} f_i \partial_i, f\partial_n\Big] = \sum_{i=1}^{n} f_i \partial_i(f)\partial_n - \sum_{i=1}^{n} f\partial_n(f_i)\partial_i = \sum_{s=1}^{n} \gamma^s \Big(\sum_{i=1}^{n} g_s^i \partial_i\Big).$$

Comparing the functions standing next to vector fields $\partial_i$, we get the following equations:

$$(2) \qquad \begin{aligned} \sum_{s=1}^{p} \gamma^s g_s^i &= -f\partial_n(f_i), \quad i = 1, \ldots, n-1, \\ \sum_{s=1}^{p} \gamma^s g_s^n &= -f\partial_n(f_n) + \sum_{i=1}^{n} f_i \partial_i(f). \end{aligned}$$

As the $n$th row of the matrix is linearly dependent, solving system (2) without the last equation at the points $m_r$ we obtain a solution to (2). It means that

$$(3)\qquad (-f\partial_n(f_n)+\sum_{i=1}^{n} f_i\partial_i(f))(m_r)$$

has a given value. For a given function $f$ for which $f$ is an infinitesimal automorphism of the distribution $\mathcal{M}$, the function $f$ at point $m_r$ must fulfil the equations

$$(4)\qquad (\sum_{i=1}^{n} f_i\partial_i(f))(m_r)=0.$$

So for $f\partial_n$ not to be an infinitesimal automorphism of the distribution $\mathcal{M}$ it is sufficient to be a solution to the following equations:

$$(5)\qquad (\sum_{i=1}^{n} f_i\partial_i(f))(m_r)=c_r,$$

where $(c_r)$ is a non-zero sequence.

The codimension of the space $\mathcal{U}_{\mathcal{M}}(U)$ in $\rangle\{f\partial_n\colon f\in F\}\langle\cap\mathcal{U}_{\mathcal{M}}(U)+$ $+\mathcal{U}_{\mathcal{M}}(U)$ is finite (where $F=\{f\in C^{\infty}(U)\colon f$ is a solution of (5) for a non-zero sequence $(c_r)\}$), i.e., there exist vector fields $X_1,\ldots,X_m$ on $U$, $X_i=\sum_{k=1}^{n} h_i^k\partial_k$ such that for a given infinitesimal automorphism of the distribution $\mathcal{M}'$ in the shape of $f\partial_n$ it is possible to find $\beta\in\boldsymbol{R}^m$ that the vector field $f\partial_n-\sum_{j=1}^{m}\beta_j X_j$ is an infinitesimal automorphism of the distribution $\mathcal{M}$. This fact means that $f\partial_n-\sum_{j=1}^{m}\beta_j X_j$ is a solution to equations (2), i.e.,

$$(6)\qquad \begin{aligned}\sum_{s=1}^{p}\gamma^s g_s^i &= -\sum_{j,k}\beta_j f_k\partial_k(h_j^i)-f\partial_n(f_i)+\sum_{j,k}\beta_j h_j^k\partial_k(f_i),\\ \sum_{s=1}^{n}\gamma^s g_s^n &= \sum_{j,k}\beta_j h_j^k\partial_k(f_n)+\sum_{k} f_k\partial_k(f)-f\partial_n(f_n)-\sum_{j,k}\beta_j f_k\partial_k(h_j^n),\end{aligned}$$

i.e.,

$$(7)\qquad \begin{aligned}\sum_{s}\gamma^s g_s^i &= -\sum_{k,j}\beta_j f_k\partial_k(h_j^i),\\ \sum_{s}\gamma^s g_s^n &= \sum_{k} f_k\partial_k(f)-\sum_{j,k}\beta_j f_k\partial_k(h_j^k).\end{aligned}$$

Since the last equation ($n$th) is linearly dependent at each point $m_r$, the following equation is true for certain numbers $\alpha_1,\ldots,\alpha_{n-1}$

$$(8)\qquad \sum_{j=1}^{n}(f_j\partial_j(f))(m_r)=(\sum_{j,k,s}\alpha_k\beta_j f_s\partial_s(h_j^k)+\sum_{k,j}\beta_k f_j\partial_j(h_k^n))(m_r).$$

Equation (8) shows that the space $F$ can be at most finite dimensional one as it is contained in the space generated by the sequences $f_k \partial_k (h_i^j)(m_r)$.

The formulation of the lemma is rather complicated but it can be applied to some simple cases.

PROPOSITION 8. *Let $\mathcal{M}$, $\mathcal{M}'$ be two smooth distributions for which the sheaf $\mathcal{U}_{\mathcal{M}}$ is a subsheaf of the sheaf $\mathcal{U}_{\mathcal{M}'}$. If for any point m there exists a chart $(U, \varphi)$ at this point such that*

(i) $\dim \mathcal{U}_{\mathcal{M}'}(U)/\mathcal{U}_{\mathcal{M}}(U) < \infty$,

(ii) *there exist vector fields* $X_1, \ldots, X_p$: $X_i = \sum_{s=1}^{n-1} g_i^s \partial_s$ *and*

$$\mathcal{M}' = \rangle X_1, \ldots, X_k \langle, \ \mathcal{M} = \rangle X_1, \ldots, X_p \langle, \ 1 \leqslant k \leqslant p < n,$$

(iii) *for* $1 \leqslant i \leqslant k$ $g_i^s$ *are independent of* $x_n$.

(iv) *$\mathcal{M}'$ is a distribution of a constant dimension, it is not true that there exists $k_0$ $1 \leqslant k_0 \leqslant n$ such that $\partial_{k_0} \in \mathcal{M}$.*

Proof. Let us assume that there exists $k_0$: $1 \leqslant k_0 < n$, such that $\partial_{k_0} \in \mathcal{M}$. (ii) and (iii) ensure that a vector field $f\partial_n$ is an infinitesimal automorphism of the distribution $\mathcal{M}'$ if $f$ depends only on $x_n$. Let us assume for argument sake that there exists a smooth function $g$ on $U$ that for a certain sequence of points $m_r$ without a cluster point in $U$, $g(m_r) = 1$ and $\partial_i(g)(m_r) = 0$ $(i = 1, \ldots, n)$. The vector field $g\,\partial_{k_0}$ belongs to $\mathcal{M}$ and conforms to the hypothesis of Lemma 4. If we construct functions $f_r$ of the variable $x_n$ such that $f_r(m_j) = 0$ and $\partial_n(f_r)(m_j) = \delta_j^r$, we will get a contradiction.

Construction of functions $g, f_r$. If the sequence $\{m_r\}$ has no cluster points in $U$, it is possible to construct a function with given jets at these points. Let $\{U_\alpha\}$ be a locally finite open covering of $U$ with the following property: $m_r \in U_\alpha$, $m_j \in U_{\alpha'}$, $j \neq r$; then $U_\alpha \cap U_{\alpha'} = \emptyset$. We can choose locally finite covering $\{V_\alpha\}$, $\bar{V}_\alpha \subset U_\alpha$ and build smooth functions $h_\alpha U$ on such that

(i) $h_\alpha / \bar{V}_\alpha \equiv 1$,

(ii) $\operatorname{supp} h_\alpha \subset U_\alpha$.

Having functions $k_r$ with a given jet at the point $m_r$, the function $k = \sum h_{\alpha_r} k_r$ is one of the functions we have been looking for ($\alpha_r$ is the very index for which $m_r \in U_{\alpha_r}$).

COROLLARY 1. *Let $\mathcal{M}$ be a smooth distribution of constant dimension on an open set $U$ in $\mathbf{R}^n$, $\mathcal{M}' = \rangle X_1, \ldots, X_k \langle$, $X_i = \sum_{j=1}^{n-1} f_i^j \partial_j$ and functions $f_j^i$ are not dependent on $x_n$. If $\mathcal{M} = \mathcal{M}' \oplus \partial_{k_0}$, $1 \leqslant k_0 < n$ and $\mathcal{U}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}'}$, then* $\dim \mathcal{U}_{\mathcal{M}'}(U)/\mathcal{U}_{\mathcal{M}}(U) = \infty$.

**4. Completely integrable distributions.** At the beginning we want to state some simple propositions, which are straightforward consequences of the results of the previous sections.

PROPOSITION 9. *Let $\mathcal{M}$ be a smooth distribution of a constant dimension. Then $\mathcal{M}$ is a completely integrable distribution if and only if $X \in \mathcal{M}$ implies that $X$ is an infinitesimal automorphism of the distribution $\mathcal{M}$.*

LEMMA 6 [2]. If *$\mathcal{U}_{\mathcal{M}}$ is a transitive sheaf, $\mathcal{C}_{\mathcal{M}}$ is a completely integrable distribution.*

Proof. $\mathcal{C}_{\mathcal{M}}$ is an integrable distribution of constant dimension (Proposition 4); hence $\mathcal{C}_{\mathcal{M}}$ is a completely integrable distribution.

The next proposition is stronger than the similar one formulated in [3], that is,

PROPOSITION 10. *If $\mathcal{M}$ is a distribution on a manifold $M$ and the sheaf $\mathcal{U}_{\mathcal{M}}$ is a transitive one such that* $\dim \mathcal{U}_{\mathcal{M}^*}(M)/\mathcal{U}_{\mathcal{M}}(M) < \infty$, *$\mathcal{M}$ is a completely integrable distribution.*

The stronger form is the following:

PROPOSITION 11. *Let $\mathcal{M}$ be a distribution on a manifold $M$. If the codimension of the stalks of the transitive sheaf $\mathcal{U}_{\mathcal{M}}$ in the stalks of the sheaf $\mathcal{U}_{\mathcal{M}^*}$ is at most denumerable, $\mathcal{M}$ is a completely integrable distribution.*

Proof. It is sufficient to prove that $\mathcal{M} = \mathcal{C}_{\mathcal{M}}$. Let $m \in M$. Then

$$\dim \mathcal{S}_{\mathcal{M}m}/\mathcal{S}_{\mathcal{M}m} \cap \mathcal{U}_{\mathcal{M}m} = \dim \mathcal{S}_{\mathcal{M}m} \cap \mathcal{U}_{\mathcal{M}^*m}/\mathcal{S}_{\mathcal{M}m} \cap \mathcal{U}_{\mathcal{M}m}$$
$$\leqslant \dim \mathcal{U}_{\mathcal{M}^*m}/\mathcal{U}_{\mathcal{M}m} \leqslant \aleph_0.$$

Since $\mathcal{C}_{\mathcal{M}m} = \rangle X_m\colon (X)_m \in \mathcal{U}_{\mathcal{M}m} \cap \mathcal{S}_{\mathcal{M}m}\langle$, we have

$$\mathcal{S}_{\mathcal{M}m} \cap \mathcal{U}_{\mathcal{M}m} \subset \mathcal{S}_{\mathcal{C}_{\mathcal{M}}m}.$$

As the distributions $\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$ are of constant dimension it is possible to construct a distribution $S$ such that $\mathcal{M}|U = \mathcal{C}_{\mathcal{M}}|U \oplus S$ for a certain open neighbourhood $U$ of the point $m$. Then $\dim \mathcal{S}_{Sm} \leqslant \dim \mathcal{S}_{\mathcal{M}m}/\mathcal{S}_{\mathcal{C}_{\mathcal{M}}m} \leqslant \aleph_0$.

On the ground of Lemma 2, $S = 0$ so $\mathcal{M}|U = \mathcal{C}_{\mathcal{M}}|U$; hence $\mathcal{M} = \mathcal{C}_{\mathcal{M}}$.

Remarks. (a) In this proposition it is possible to consider any involutive distribution $\mathcal{M}'$ such that $\mathcal{M}' \subset \mathcal{M}$ and $\mathcal{U}_{\mathcal{M}'} \supset \mathcal{U}_{\mathcal{M}}$.

(b) If $M$ is a connected manifold, the proposition will be true for the following hypothesis: there exists a point $m_0$ of the manifold $M$ for which $\dim \mathcal{U}_{\mathcal{M}^*m_0}/\mathcal{U}_{\mathcal{M}m_0} \leqslant \aleph_0$. The condition is sufficient, because the pseudogroup $P(\mathcal{U}_{\mathcal{M}})$ induces isomorphisms of the stalks of both sheaves $\mathcal{U}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{M}^*}$ and the inequality holds for any point of the manifold $M$.

P. Lecompte gave a characterization of distributions with $\mathcal{U}_{\mathcal{M}}(M)$ of finite codimension in his paper [3].

PROPOSITION 12 [3]. *If $\mathcal{M}$ is a distribution on a manifold $M$ such that the codimension of $\mathcal{U}_{\mathcal{M}}(M)$ is finite, there exists a finite subset $E$ of $M$ having the following properties:*

(i) $\mathcal{M}$ *is a trivial distribution on* $M \setminus E$ *(i.e.,* $\mathcal{M} = \mathcal{M}_0$ *or* $\mathcal{M} \equiv 0$),

(ii) $\mathcal{U}_{\mathcal{M}}(M) = \{X \in \mathcal{V}(M)\colon X_m = 0 \text{ for } m \in E\}$.

If we assume the $\mathcal{U}_{\mathcal{M}}$ to be regular, then the following proposition is true.

PROPOSITION 13. *Let* $\mathcal{U}_{\mathcal{M}}$ *be the regular sheaf of the germs of the infinitesimal automorphisms of a distribution* $\mathcal{M}$. *If the codimension of the stalks of the sheaf* $\mathcal{U}_{\mathcal{M}}$ *is at most denumerable,* $\mathcal{M}$ *is a trivial distribution.*

Proof. Since $\mathcal{U}_{\mathcal{M}}$ is a transitive sheaf (Proposition 6), $\mathcal{M}$ is a smooth distribution of constant dimension (Proposition 5). Then:

$$\dim \mathcal{D}_{\mathcal{M}m}/\mathcal{D}_{\mathcal{C}_{\mathcal{M}m}} \leqslant \dim \mathcal{D}_{\mathcal{M}m}/\mathcal{D}_{\mathcal{C}}\, U_m \cap \mathcal{U}_{\mathcal{M}m} \leqslant \dim \mathcal{V}_m/\mathcal{U}_{\mathcal{M}m} \leqslant \aleph_0.$$

Now, using the similar method as in the proof of Proposition 11, $\mathcal{M} = \mathcal{C}_{\mathcal{M}}$ so $\mathcal{M}$ is a completely integrable distribution. This is equivalent to the condition

$$\forall m \in M\, \exists (U, \varphi) \in \mathrm{Atl}(M, m),\ \varphi(x) = (x_1, \ldots, x_n)\colon\ \rangle \partial_1, \ldots, \partial_p \langle\, = \mathcal{M}|U,$$

where $p = \dim \mathcal{M}$. If we assume that $\mathcal{M}$ is not a trivial distribution, it means that $0 < p < n$. Let us consider smooth functions $f$ on $U$ such that $\partial_1(f) \not\equiv 0$. The cardinal number of the space of the germs at a given point of these functions is greater than $\mathfrak{c}$. Then

$$[\partial_1, f\partial_n] = f[\partial_1, \partial_n] + \partial_1(f)\partial_n = \partial_1(f)\partial_n \not\equiv 0,$$

so $f\partial_n$ is not an infinitesimal automorphism of the distribution $\mathcal{M}$, which leads to a contradiction of the hypothesis on the codimension.

Complete integrability of a distribution $\mathcal{M}$ may be characterized by relations between infinitesimal automorphisms of the characteristic distribution $\mathcal{C}_{\mathcal{M}}$ and infinitesimal automorphisms of the distribution $\mathcal{M}$. If $\mathcal{M}$ is a completely integrable distribution, then, of course, $\mathcal{C}_{\mathcal{M}} = \mathcal{M}$ and $\mathcal{U}_{\mathcal{M}} = \mathcal{U}_{\mathcal{C}_{\mathcal{M}}}$. The converse is also true, but now we will prove several very useful lemmas.

LEMMA 7. *Let* $\mathcal{M}$ *be a smooth distribution of constant dimension* $k$. *Then if for any point* $m$ *there exists a chart* $(U, \varphi)$ *at this point and* $\mathcal{U}_{\mathcal{M}}(U)$ *has the following properties*:

(i) *there exists number* $p$ *such that for* $X \in \mathcal{U}_{\mathcal{M}}(U)$, $X = \sum_{i=1}^{n} f_i \partial_i$, $\partial_j(f_i) \equiv 0$ *when* $i > p$, $j \leqslant p$,

(ii) *for any point* $m'$ *of the set* $U$, *the equations*

$$g^i(m') = 1, \qquad \partial_j(g^i)(m') = b_j,$$

*where* $i = 1, \ldots, n$, $j = 1, \ldots, n$ *and for* $i > p$, $j \leqslant p$, $b_j = 0$ *have solutions in* $\mathcal{U}_{\mathcal{M}}(U)$ *for* $n$ $(n-p$ *for* $i > p)$ *linearly independent vectors* $b$, *i.e.,* *there exists an infinitesimal automorphism* $X = \sum_{j=1}^{n} f_j \partial_j$ *such that* $f_i = g^i$, *the distribution* $\mathcal{M}$ *is completely integrable.*

Proof. Let us consider the case of $k \geqslant p$. We can find $C^\infty$ independent vector fields $X_1, \ldots, X_k$ on a perhaps smaller open set $U$ such that $\mathcal{M}/U = \rangle X_1, \ldots, X_k \langle$ and $X_i = \sum_{j=1}^{n} f_i^j \partial_j$. If $X \in \mathcal{U}_{\mathcal{M}}(U)$, $X = \sum_{i=1}^{n} g_i \partial_i$ according to Proposition 4 there is the equivalence $X \in \mathcal{U}_{\mathcal{M}}(U) \Leftrightarrow [X, X_i] \in \mathcal{S}_{\mathcal{M}}(U)$. Computing the right-hand side of the equivalence (omitting the index $i$), we get

$$[\sum_{j=1}^{n} f^j \partial_j, \sum_{s=1}^{n} g_s \partial_s] = \sum_{j,s} (f^j \partial_j(g_s) - g_j \partial_j(f_s)) \partial_s = \sum_{r=1}^{k} \gamma^r (\sum_{s=1}^{n} f_r^s \partial_s)$$
$$= \sum_s (\sum_r \gamma^r f_r^s) \partial_s,$$

where $\gamma^s$ are smooth functions on $U$. Comparing the functions standing next to the vector fields $\partial_s$, it is possible to write the equations

$$\sum_{s=1}^{n} f^s \partial_s(g_j) - g_s \partial_s(f^j) = \sum_{s=1}^{k} f_s^j \gamma^s, \quad j = 1, \ldots, n. \tag{9}$$

Taking values of equations (9) at each point $m$ of the set $U$ equations (9) have the form

$$\sum_{s=1}^{n} a_{js} X_s = \sum_{r=1}^{n} -b_r \bar{a}_{jr} + \sum_{r=1}^{n} \bar{a}_r b_{jr}, \tag{10}$$

where $f_i^j(m) = a_{ji}$, $f^j(m) = \bar{a}_j$, $g_r(m) = b_r$, $\partial_r(g_j)(m) = b_{jr}$, $\partial_r(f^j)(m) = \bar{a}_{jr}$.

Since the rank of the matrix $(a_{js})$ is equal to $k < n$, it is possible to omit one of the equations which is linearly dependent and does not influence the existence and value of the solution, for instance the $l$th equation

$$\sum_{s=1}^{k} a_{ls} X_s = \sum_{r=1}^{n} -b_r \bar{a}_{lr} + \sum_{r=1}^{n} \bar{a}_r b_{lr}. \tag{11}$$

Let $1 \leqslant l \leqslant p$. The remaining equations give the unique solution of equations (10), so

$$\sum_{r=1}^{n} -b_r \bar{a}_{lr} + \sum_{r=1}^{n} \bar{a}_r b_{lr}$$

has a unique value.

Since $b_{lr}$ does not appear in other equations, we have

$$\sum_{r=1}^{n} \bar{a}_r b_{lr} = c = \text{const}. \tag{12}$$

Because of hypothesis (ii) the space of solutions of equations (12) is of a dimension $n$. A contradiction since there exists $\bar{a}_r \neq 0$. If $p < l \leqslant n$, equation (11) has the form

$$(13) \qquad \sum_{s=1}^{k} a_{ls} X_s = \sum_{r=1}^{n} -b_r \bar{a}_{lr} + \sum_{r=p+1}^{n} \bar{a}_r b_{lr}.$$

The analogous proof as in the case of $1 \leqslant l \leqslant p$ shows that $\bar{a}_r = 0$ for $r = p+1, \ldots, n$. Since the proof is independent of the choice of point $m$, $f_i^j \equiv 0$ for $j > p$. The matrix $(f_i^j(m))_{j=1,\ldots,p}^{i=1,\ldots,k}$ is of the rank $p$ at each point $m$ of the set $U$. By choosing $p$ linearly independent columns (let us assume the first $p$ columns) and in this way forming a square matrix of the maximal rank, we may multiply this matrix by the inverse matrix, so we get the identity matrix. This means that the vector fields $\partial_1, \ldots, \partial_p$ belong to the distribution $\mathcal{M}$ at a point $m$. It is possible to repeat it at each point of an open neighbourhood of $m$. That is the end of the proof of the case $k \geqslant p$. In the case of $k < p$ we get the contradiction as it is possible to omit the equation for $1 \leqslant l \leqslant p$.

LEMMA 8. *Let us have two distributions $\mathcal{M}$, $\mathcal{M}'$ such that $\mathcal{M} \supset \mathcal{M}'$, $\mathcal{M}'$ is a completely integrable distribution and $\mathcal{U}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}'}$. If the sheaf $\mathcal{U}_{\mathcal{M}}$ is transitive and there exists a chart $(U, \varphi)$ for which $\mathcal{M}'|U = \rangle\partial_1, \ldots, \partial_k\langle$ and one of the two conditions is fulfilled:*

(a) $\mathcal{U}_{\mathcal{M}}(U) = \mathcal{U}_{\mathcal{M}'}(U)$(1),

(b) *for each point $m$ of $U$ there exists a base of open neighbourhoods $\{V_\alpha\}$ for which* $\dim \mathcal{U}_{\mathcal{M}'}(V_\alpha)/\mathcal{U}_{\mathcal{M}}(V_\alpha) < \infty$, *then $\mathcal{M}$ is a completely integrable distribution. If $\mathcal{M}' \not\equiv 0$, then $\mathcal{M}' = \mathcal{M}$.*

**Proof.** Let $\dim \mathcal{M} = p$, $\dim \mathcal{M}' = k$. Then $0 \leqslant k \leqslant p \leqslant n$. It is necessary to consider the following cases:

(a) $p = n$,

(b) $k = 0$,

(c) $k = p$,

(d) $0 < k < p < n$.

As (a) $\mathcal{M}$ is a trivial distribution, so $\mathcal{U}_{\mathcal{M}}(U) = \mathcal{V}(U)$. According to Proposition 13, $\mathcal{M}'$ is also a trivial distribution.

Ad (b) $\mathcal{M}'$ is a trivial distribution, so the codimension of $\mathcal{U}_{\mathcal{M}}(U)$ is finite. According to Proposition 13, $\mathcal{M}/U$ is a trivial distribution.

Ad (c) $\mathcal{M} = \mathcal{M}'$.

Ad (d) In this case we will prove that the sheaf $\mathcal{U}_{\mathcal{M}}/U$ fulfils the hypothesis of Lemma 7. Since the sheaf $\mathcal{U}_{\mathcal{M}}$ is transitive, the proof of the lemma will be completed. The relation $\mathcal{U}_{\mathcal{M}} \subset \mathcal{U}_{\mathcal{M}'}$ ensures (i). Let us choose

(1) $\mathcal{U}_{\mathcal{M}'}(U) = \{X\colon X = \sum_{i=1}^{n} f_i \partial_i,\ \partial_j(f_i) \equiv 0,\ i > k,\ j \leqslant k\}$; see the proof of Lemma 9.

a point $m$ of the set $U$. If condition (ii) is not fulfilled at the point $m$, e.g. for $i = n$, the solutions of equations (8) form a surface at most of the dimension $(n-p-1)$. Changing the map in a neighbourhood of the point $m$, we may assume that it is of the form $(b_1^0, \ldots, b_n^0) + \{b \in \mathbf{R}^n \colon b = (0, \ldots, 0, \overset{(p+1)}{x}, \ldots, x, 0)\}$. It means that if $X = \sum_{i=1}^{n} f^i \partial_i$ is an infinitesimal automorphism of the distribution $\mathcal{M}$, $\partial_n(f^n)(m) = b_n^0$, when $f^n(m) = 1$.

Let $F_n = \{f \in C^\infty(U) \colon \exists X \in \mathcal{V}_{\mathcal{M}}(U),\ X = \sum_{i=1}^{n} f_i \partial_i,\ f_n = f\}$.

Since $\mathcal{V}_{\mathcal{M}}(U)$ is a vector space, $F_n$ is a vector subspace of $C^\infty(U)$. A smooth function $f$ on $U$ belonging to $F_n$ has to fulfil the following conditions:

(i) if $f(m) = 1$, $\partial_n(f)(m) = b_n^0$,

(ii) if $f(m) = 0$, $\partial_n(f)(m) = 0$.

The second condition is the consequence of the following: Let $f(m) = 0$ and $f \in F_n$. There exists $X \in \mathcal{V}_{\mathcal{M}}(U)$, $X = \sum_{i=1}^{n-1} f_i \partial_i + f \partial_n$. For any $Y \in \mathcal{V}_{\mathcal{M}}(U)$, $Y = \sum_{i=1}^{n} g_i \partial_i$ and $\alpha, \beta \in \mathbf{R}^*$, $\alpha X + \beta Y \in \mathcal{V}_{\mathcal{M}}(U)$ and $\alpha X + \beta Y = \sum_{i=1}^{n-1} (\alpha f_i + \beta g_i) \partial_i + (\alpha f + \beta g_n) \partial_n$.

If we denote $\alpha f + \beta g$ by $h$, then

$$h(m) = \alpha f(m) + \beta g(m) = \beta,$$

$$\partial_n(h)(m) = \alpha \partial_n(f)(m) + \beta \partial_n(g_n)(m) = \alpha \partial_n(f)(m) + \beta b_n^0.$$

So for $h' = h/\beta$, we have $h'(m) = 1$

$$\partial_n(h')(m) = \frac{\alpha}{\beta} \partial_n(f)(m) + b_n^0.$$

Since $h' \in F_n$, $\partial_n(h')(m) = b_n^0$, so $\partial_n(f)(m) = 0$. $F_n$ is a vector subspace of the space $F(m)$, where

$F(m) = \{f \in C^\infty(U)^p \colon$

$\quad$ (i) $f(m) = 1 \Rightarrow \partial_n(f)(m) = b_n^0$; (ii) $f(m) = 0 \Rightarrow \partial_n(f)(m) = 0\}$,

$C^\infty(U)^p$ — functions of variables $x_{p+1}, \ldots, x_n$.

The codimension of the space $F(m)$ in $C^\infty(U)^p$ is equal to 1, so the codimension of $\mathcal{V}_{\mathcal{M}}(U)$ in $\mathcal{V}_{\mathcal{M}'}(U)$ is greater than or equal to 1. A contradiction in the case of (a). If condition (b) is fulfilled we will repeat the same for an open set $V$ which does not contain $m$, and so on. At each step we will ask whether a set $V$ fulfils the hypothesis of Lemma 7. If we do not find such an open set, it is easy to see that the codimension of $\mathcal{V}_{\mathcal{M}}(U)$ in $\mathcal{V}_{\mathcal{M}'}(U)$ is infinite. A contradiction.

Remark. The lemma will be true when we assume that $\mathcal{M}$ is a smooth distribution of constant dimension, $\mathcal{U}_{\mathcal{M}}$ is a regular sheaf and $\forall m \in M\ \exists (U, \varphi) \in \mathrm{Atl}(M, m)$: $\mathcal{M}'|U = \rangle \partial_1, \ldots, \partial_k \langle$ and $\mathcal{U}_{\mathcal{M}'}(U) = \mathcal{U}_{\mathcal{M}}(U)$.

THEOREM 7. *Let $\mathcal{M}$ be a smooth distribution and $\mathcal{U}_{\mathcal{M}}$ a transitive sheaf. If $\mathcal{U}_{\mathcal{M}} = \mathcal{U}_{\mathcal{C}_{\mathcal{M}}}$, $\mathcal{M}$ is a completely integrable distribution ($\mathcal{M} = \mathcal{C}_{\mathcal{M}}$).*

Proof. The distribution $\mathcal{C}_{\mathcal{M}}$ fulfils the hypothesis of Lemma 8.

Now we will prove a proposition on the uniqueness of distributions having a given sheaf of germs of infinitesimal automorphisms.

LEMMA 9. *A nontrivial distribution $\mathcal{M}$ is a completely integrable one if and only if $\exists k > 0\ \forall m \in M\ \exists (U, \varphi) \in \mathrm{Atl}(M, m)$:*

$$\mathcal{U}_{\mathcal{M}}(U) = \{X = \sum_{i=1}^{n} f_i \partial_i \colon \partial_j(f_i) \equiv 0,\ i > k,\ j \leqslant k\}.$$

Proof. If $\mathcal{M}$ is a completely integrable distribution, then directly from the definition $\exists k > 0$: $\forall m \in M\ \exists (U, \varphi) \in \mathrm{Atl}(M, m)$: $\mathcal{M}|U = \rangle \partial_1, \ldots, \partial_k \langle$. According to Proposition 4, $X = \sum_{i=1}^{n} f_i \partial_i \in \mathcal{U}_{\mathcal{M}}(U)$ if and only if $[X, \partial_j] \in \mathcal{M}|U$ for $j = 1, \ldots, k$. Computing, we get $[X, \partial_j] = [\sum f_i \partial_i, \partial_j] = -\sum_{i=1}^{n} \partial_j(f_i)\partial_i$ so $\partial_j(f_i) \equiv 0$ for $i > k$, $j \leqslant k$. When $\mathcal{U}_{\mathcal{M}}(U)$ is of the required form, the distribution $\mathcal{M}$ fulfils the hypothesis of Lemma 7, so $\mathcal{M}|U = \rangle \partial_1, \ldots, \partial_k \langle$.

PROPOSITION 14. *Let $\mathcal{M}$, $\mathcal{M}'$ be two smooth distributions having the same sheaf of the germs of the infinitesimal automorphisms. If $\mathcal{M}$ is a completely integrable distribution, $\mathcal{M} = \mathcal{M}'$.*

Proof. It is a corollary of Lemma 9.

To end the paper we will recall a proposition which connects complete integrability of distributions with involutiveness at a point.

We say that a distribution $\mathcal{M}$ is involutive at a point $m_0 \in M$ if

$$\left.\begin{matrix} \forall X, Y \in \mathcal{M} \\ m_0 \in \mathrm{dom}[X, Y] \end{matrix}\right\} \Rightarrow [X, Y]_{m_0} \in \mathcal{M}_{m_0}.$$

PROPOSITION 15 [2]. *If $M$ is a connected manifold, $\mathcal{U}_{\mathcal{M}}$ a transitive sheaf, then $\mathcal{M}$ is a completely integrable distribution if and only if $\mathcal{M}$ is involutive at a certain point of $M$.*

## References


[1] P. Griffiths, *Deformations of G-structures B: Geometric structures*, Math. Ann. 158, 326–351.

[2] P. Lecompte, *Infinitesimal automorphisms and integrability of distributions*, Bull. Soc. Roy. Sci. Liege 46, 1–2 (1977), 44–50.

[3] P. Lecompte, *Some applications of a result of Amemiya*, Bull. Soc. Roy. Sci. Liege 46, 11–12 (1977), 401–405.

[4] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland Publ. Amsterdam 1968.

[5] H. J. Sussmann, *Orbits of families of vector fields of distributions*, Trans. Amer. Math. Soc. 180 (1973), 171–188.

[6] T. Yagyu, *Deformations of G-structures and infinitesimal automorphisms*, J. Math. Kyoto Univ. 12, 2 (1972), 325–349.

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