ANNALES
POLONICI MATHEMATICI
XXVI (1972)

Estimation of functions by means of the first order partial differential inequalities

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The paper deals with the estimation of functions of several variables satisfying some first order partial differential inequalities.

It consist of two parts. In the first part the estimation is performed by means of the solutions of ordinary differential equations, whereas in the second one the solutions of the first order partial differential equations are applied.

I would like to express my gratitude to Professor J. Szarski for his valuable remarks.

Some symbols and definitions (¹). We will denote by

$$X = (x_1, \ldots, x_p), \qquad Y = (y_1, \ldots, y_n)$$

the points of the p- and n-dimensional space and similarly the functions of several variables

$$U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big).$$

For partial derivatives of these functions we use the brief notation

$$u^i_y = (u^i_{y_1}, \ldots, u^i_{y_n}), \qquad i = 1, \ldots, m.$$

We apply the notations

$$|X| = (|x_1|, \ldots, |x_p|), \qquad |U(X, Y)| = \big(|u^1(X, Y)|, \ldots, |u^m(X, Y)|\big),$$
$$|u^i_y| = (|u^i_{y_1}|, \ldots, |u^i_{y_n}|).$$

If ϑ is a number and $H = (\eta_1, \ldots, \eta_m)$ a point of the m-dimensional space, then we will write

$$\vartheta H = (\vartheta\eta_1, \ldots, \vartheta\eta_m).$$

(¹) These definitions and some symbols are taken from [1] and [2].

Finally, if Y and $\tilde{Y}$ are two points of the n-dimensional space, then we will write

$$Y \leqslant \tilde{Y} \quad \text{if} \quad y_j \leqslant \tilde{y}_j, \quad j = 1, \ldots, n.$$

Definition of class $\mathscr{D}$. A function $U(X, Y) = U(x_1, \ldots, x_p, y_1, \ldots, y_n)$ will be called a *function of class $\mathscr{D}$* in a pyramid

$$(0) \qquad \sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma; \quad |y_k - \mathring{y}_k| \leqslant a_k - L \sum_{r=1}^{p} |x_r - \mathring{x}_r|, \quad k = 1, \ldots, n,$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty; \quad \gamma \leqslant \min_k \left(\frac{a_k}{L}\right)$$

if $U(X, Y)$ is continuous in the pyramid, possesses Stolz's differential with regard to (X, Y) on its side surface and has first derivatives with respect to Y and Stolz's differential with regard to X in its interior.

Comparison systems of ordinary differential equations. A system of differential equations

$$(1) \qquad \frac{dv^i}{dt} = \sigma_i(t, v^1, \ldots, v^m), \quad i = 1, \ldots, m,$$

will be called a *comparison system of ordinary differential equations* if its right-hand sides are continuous and non-negative in the closed region

$$t \geqslant 0, \quad v^i \geqslant 0, \quad i = 1, \ldots, m,$$

and σ_i is increasing with respect to $v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^m$.

Comparison systems of first order partial differential equations. A system of equations

$$(2) \qquad v^i_{\xi_l} = h^i_l(\xi_1, \ldots, \xi_p, Y, v^1, \ldots, v^m, v^i_{y_1}, \ldots, v^i_{y_n}),$$
$$i = 1, \ldots, m; \quad l = 1, \ldots, p,$$

will be called a *comparison system of partial differential equations* if the following conditions are satisfied:

1° functions $h^i_l(\Xi, Y, V, Q)$ $(i = 1, \ldots, m;\ l = 1, \ldots, p)$ are defined and non-negative for $V \geqslant 0$, $Q \geqslant 0$ and for (Ξ, Y) in the pyramid

$$(3) \qquad \xi_l \geqslant 0, \quad \sum_{j=1}^{p} \xi_j < \gamma, \quad l = 1, \ldots, p,$$

$$|y_k - \mathring{y}_k| \leqslant a_k - L \sum_{j=1}^{p} \xi_j, \quad k = 1, \ldots, n,$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty; \quad \gamma \leqslant \min_k \left(\frac{a_k}{L}\right);$$

2° for every fixed l the functions $h_l^i(\Xi, Y, V, Q)$ $(i = 1, \ldots, m)$ are increasing with respect to $v^1, \ldots, v^m$;

3° inequalities

$$(4) \quad h_l^i(\Xi, Y, V, Q) - h_l^i(\Xi, Y, \tilde{V}, \tilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} \xi_r, V - \tilde{V}\Big) + L \sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, \ldots, m; \ l = 1, \ldots, p)$$

are satisfied whenever $V \geqslant \tilde{V}$, where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of ordinary differential equations with $\sigma_i(t, 0) \equiv 0$ $(i = 1, \ldots, m)$ and with the right-hand maximum solution $\Omega(t, 0)$ through the origin being identically zero.

By the solution of the comparison system (2) will be meant a sequence of non-negative functions

$$V(\Xi, Y) = \big(v^1(\Xi, Y), \ldots, v^m(\Xi, Y)\big)$$

of class $\mathscr{D}$ in the pyramid (3) satisfying equations (2) and such that

$$(5) \qquad v_y^i(\Xi, Y) \geqslant 0, \quad i = 1, \ldots, m.$$

I. Estimation by means of the solutions of ordinary differential equations.

THEOREM 1.1 (1.1a) [2]. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathscr{D}$ *in the pyramid*

$$(1.1) \qquad |x| < \gamma; \quad |y_k - \mathring{y}_k| \leqslant a_k - L|x|, \quad k = 1, \ldots, n,$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty; \quad \gamma \leqslant \min_k \left(\frac{a_k}{L}\right)$$

and satisfy the differential inequalities

$$(1.2) \qquad |u_x^i(x, Y)| \leqslant L \sum_{k=1}^{n} |u_{y_k}^i| + \sigma_i(|x|, |U|), \quad i = 1, \ldots, m,$$

with

$$(1.3) \qquad \vartheta\eta_i \leqslant u^i(0, Y) \leqslant \eta_i, \quad i = 1, \ldots, m$$

$$[(1.3a) \qquad -\eta_i \leqslant u^i(0, Y) \leqslant -\vartheta\eta_i, \quad i = 1, \ldots, m],$$

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(2) This notation means that we deal with two different theorems, namely 1.1 and 1.1a.

where $\eta_i > 0$, $0 < \vartheta \leqslant 1$ *and* $\sigma_i(t, V)$ *are increasing with respect to all* $v^1, \ldots, v^m$ *and are the right-hand members of a comparison system* (1); *its right-hand maximum solution* $\Omega(t; H) = \big(\omega^1(t; H), \ldots, \omega^m(t; H)\big)$ *through* $(0, H)$, *is supposed to be defined in the interval* $[0, a)$.

If, moreover, ϱ_i *is the positive minimum solution of the equation*

$$(1.4)\qquad \omega^i(\varrho; H) = (1+\vartheta)\eta_i \ (^3), \qquad i = 1, \ldots, m,$$

then the inequalities

$$(1.5)\qquad u^i(x, Y) \geqslant -\omega^i(|x|; H) + (1+\vartheta)\eta_i > 0, \qquad i = 1, \ldots, m$$

$$[(1.5a)\qquad u^i(x, Y) \leqslant \omega^i(|x|; H) - (1+\vartheta)\eta_i < 0, \qquad i = 1, \ldots, m]$$

are satisfied in the pyramid

$$(1.6)\qquad |x| < \min(a, \gamma, \varrho_1, \ldots, \varrho_m); \qquad |y_k - \mathring{y}_k| \leqslant a_k - L|x|, \qquad k = 1, \ldots, n.$$

Remark 1. If the assumptions of Theorem 1.1 are satisfied, with σ_i not necessarily increasing with respect to v^i, then the following inequalities are valid:

$$(*)\qquad -\omega^i(|x|; H) \leqslant u^i(x; H) \leqslant \omega^i(|x|; H), \qquad i = 1, \ldots, m,$$

in the corresponding set (cf. [2], Theorem 37.1). In the above inequalities the $u^i(x; H)$ are estimated by non-negative functions from above and non-positive ones from below. On the contrary, in our case, the functions $u^i(x; H)$ (1.5) [(1.5a)] are estimated by positive functions from below (negative ones from above).

Remark 2. Theorem 37.1 in [2] holds under the assumption that $\sigma_i(t, v^1, \ldots, v^m)$ is increasing with respect to the variables $v^1, \ldots, v^{i-1}$, $v^{i+1}, \ldots, v^m$, while in Theorem 1.1 we assume, moreover, that σ_i is increasing also with respect to v^i. This is an essential assumption, as can be seen from the following example (the case $m = 1$, $L = 0$, $\vartheta = \frac{1}{2}$).

Example. Let the function $\sigma(v)$ be non-negative and continuous for $v \geqslant 0$ and let $\sigma(\eta) < \sigma(\frac{1}{2}\eta)$, where η is a fixed positive number. Let the function $u(x)$ satisfy the differential equation

$$u_x(x) = -\sigma(|u|)$$

with

$$u(0) = \tfrac{1}{2}\eta.$$

Further, let $\omega(x)$ be the right-hand maximum solution of the equation

$$\frac{dv}{dx} = \sigma(v)$$

satisfying the initial condition $\omega(0) = \eta$.

(3) If (1.4) has no solutions, then we put $\varrho_i = +\infty$.

It follows from the above assumptions that

1° $u(0) = \frac{1}{2}\eta = \frac{3}{2}\eta - \omega(0)$,

2° $u_x(0) = -\sigma(\frac{1}{2}\eta) < -\sigma(\eta) = \left(\frac{3}{2}\eta - \omega(x)\right)'_{x=0}$.

It follows from 1° and 2° that in some right-hand neighbourhood of 0 the inequality

$$u(x) < -\omega(x) + \tfrac{3}{2}\eta$$

holds. In other words, inequality (1.5) of Theorem 1.1 is not valid in this case.

Remarks 1 and 2 apply to all the following theorems.

Proof.

(A) First we will consider the case $x \geqslant 0$. We get from Theorem 37.1 of [1]

$$|u^i(x, Y)| \leqslant \omega^i(x; H), \quad i = 1, \ldots, m, \tag{1.7}$$

for

$$0 \leqslant x < \min(a, \gamma).$$

It follows from (1.2), (1.7) and from the assumption that σ_i are increasing with respect to all v^j that

$$u_x^i \geqslant -\sigma_i(x; |U|) - L\sum_{k=1}^{n}|u_{y_k}^i| \geqslant -\sigma_i\big(x; \Omega(x; H)\big) - L\sum_{k=1}^{n}|u_{y_k}^i| \geqslant -\omega_x^i - L\sum_{k=1}^{n}|u_{y_k}^i|, \quad i = 1, \ldots, m. \tag{1.8}$$

Consider the function

$$v^i(x, Y; H) = u^i(x, Y) + \omega^i(x; H) - (1+\vartheta)\eta_i \tag{1.9}$$

for $0 \leqslant x < a$ and $i = 1, \ldots, m$. Then from (1.3) we get

$$v^i(0, Y; H) \geqslant 0 \quad \text{with } v_{y_k}^i = u_{y_k}^i \tag{1.10}$$

and from (1.8)

$$v_x^i = u_x^i + \omega_x^i \geqslant -L\sum_{k=1}^{n}|v_{y_k}^i|$$

for $0 \leqslant x < \min(a, \gamma)$. Hence, from (1.10) and by example 59.1 [1] we get

$$v^i(x, Y; H) \geqslant 0, \quad i = 1, \ldots, m.$$

Therefore, it follows from (1.9) that the inequalities

$$u^i(x, Y) \geqslant -\omega^i(x; H) + (1+\vartheta)\eta_i, \quad i = 1, \ldots, m, \tag{1.11}$$

are satisfied in the interval $0 \leqslant x < \min(a, \gamma)$.

Since ϱ_i is the minimum positive solution of equation (1.4), we infer that in the interval

$$0 \leqslant x < \min(a, \gamma, \varrho_1, \ldots, \varrho_m)$$

the inequalities

$$-\omega^i(x; H) + (1+\vartheta)\eta_i > 0, \quad i = 1, \ldots, m, \tag{1.12}$$

are satisfied.

(B) The case $x < 0$ can be reduced to case (A) by the substitution $s = -x$.

The proof of Theorem 1.1a is quite similar.

THEOREM 1.2 (1.2a). *Let the functions*

$$U(X, Y) = \big(u^1(x_1, \ldots, x_p, y_1, \ldots, y_n), \ldots, u^m(x_1, \ldots, x_p, y_1, \ldots, y_n)\big)$$

be of class $\mathscr{D}$ in the pyramid

$$\sum_{j=1}^{p} |x_j - \mathring{x}_j| < \gamma; \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{j=1}^{p} |x_j - \mathring{x}_j|, \quad k = 1, \ldots, n, \tag{1.13}$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty; \quad \gamma \leqslant \min_k \left(\frac{a_k}{L}\right).$$

Suppose that the differential inequalities

$$|u^i_{x_j}| \leqslant L\sum_{k=1}^{n} |u^i_{y_k}| + \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U|\Big) \tag{1.14}$$

are satisfied for $i = 1, \ldots, m$ and $j = 1, \ldots, p$ with initial inequalities

$$\vartheta\eta_i \leqslant u^i(\mathring{X}, Y) \leqslant \eta_i, \quad i = 1, \ldots, m, \tag{1.15}$$

$$[(1.15a) \qquad -\eta_i \leqslant u^i(\mathring{X}, Y) \leqslant -\vartheta\eta_i, \quad i = 1, \ldots, m],$$

where $\eta_i > 0$ and $0 < \vartheta \leqslant 1$ and $\sigma_i(t, Y)$ satisfies the assumptions of Theorem 1.1.

If, moreover, ϱ_i is the minimum positive solution of the equation

$$\omega^i(\varrho; H) = (1+\vartheta)\eta_i, \quad i = 1, \ldots, m, \tag{1.16}$$

then the inequalities

$$u^i(X, Y) \geqslant -\omega^i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big) + (1+\vartheta)\eta_i > 0, \quad i = 1, \ldots, m, \tag{1.17}$$

$$[(1.17a) \quad u^i(X, Y) \leqslant \omega^i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big) - (1+\vartheta)\eta_i < 0, \quad i = 1, \ldots, m]$$

are satisfied in the pyramid

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \min(a, \gamma, \varrho_1, \ldots, \varrho_m), \tag{1.18}$$

$$|y_k - y_k| \leqslant a_k - L \sum_{j=1}^{p} |x_j - \mathring{x}_j|, \quad k = 1, \ldots, n.$$

Proof. We introduce Mayer's transformation

$$X = \mathring{X} + \Lambda_x, \quad \text{where } \Lambda = (\lambda_1, \ldots, \lambda_p) \tag{1.19}$$

under the assumption

$$0 < \lambda = \sum_{j=1}^{p} |\lambda_j| < \min(a, \gamma) \tag{1.20}$$

and we consider the function

$$v^i(x, Y, \Lambda) = u^i(\mathring{X} + \Lambda x, Y), \quad i = 1, \ldots, m. \tag{1.21}$$

Then it follows from (1.14) and (1.21) that the inequalities

$$|v^i_x| \leqslant \lambda \sum_{k=1}^{n} |v^i_{y_k}| + \lambda\sigma_i(\lambda|x|, |V|), \quad i = 1, \ldots, m, \tag{1.22}$$

are satisfied in the pyramid

$$|x| < \gamma/\lambda; \quad |y_k - \mathring{y}_k| \leqslant a_k - \lambda L|x|, \quad k = 1, \ldots, n. \tag{1.23}$$

It follows from the assumptions of σ_i that the right-hand maximum solution of the comparison system

$$\frac{d\tilde{v}^i}{dx} = \lambda\sigma_i(\lambda x, \tilde{v}^1, \ldots, \tilde{v}^m), \quad i = 1, \ldots, m, \tag{1.24}$$

satisfying initial conditions $\tilde{\omega}^i(0; H) = \eta_i$, is given by the formula

$$\tilde{\omega}^i(x; H) = \omega^i(\lambda x; H)$$

and is defined in the interval $[0, a/\lambda)$.

The functions $v^i(x, Y, \Lambda)$ are of class $\mathscr{D}$ in the pyramid (1.23). Moreover, we get from (1.21)

$$v^i(0, Y, \Lambda) = u^i(\mathring{X}, Y), \quad i = 1, \ldots, m. \tag{1.25}$$

Therefore, in view of (1.15) [(1.15a)] we have

$$\vartheta\eta_i \leqslant v^i(0, Y, \Lambda) \leqslant \eta_i, \quad i = 1, \ldots, m$$

$$[-\eta_i \leqslant v^i(0, Y, \Lambda) \leqslant -\vartheta\eta_i, \quad i = 1, \ldots, m].$$

Thus we see that, with a fixed Λ satisfying inequality (1.20), the functions $v^i(x, Y, \Lambda)$ satisfy the assumptions of Theorem 1.1 (1.1a), where a is replaced by a/λ, and respective γ by γ/λ, $\tilde{\sigma}_i(x, V)$ by $\sigma_i(x, V) = \lambda\sigma_i(\lambda x, V)$, L by λL, ϱ_k by ϱ_k/λ.

By Theorem 1.1 (1.1a) we have in the pyramid

$$(1.26)\quad |x| < \min\left(\frac{a}{\lambda}\frac{\gamma}{\lambda}, \frac{\varrho_1}{\lambda}, \ldots, \frac{\varrho_m}{\lambda}\right); \quad |y_k - \mathring{y}_k| \leqslant a_k - \lambda L|x|, \quad k = 1, \ldots, n,$$

the inequalities

$$(1.27)\quad v^i(x, Y, \Lambda) \geqslant -\omega^i(\lambda|x|; H) + (1+\vartheta)\eta_i > 0, \qquad i = 1, \ldots, m$$

$$[(1.27\mathrm{a})\quad v^i(x, Y, \Lambda) \leqslant \omega^i(\lambda|x|; H) - (1+\vartheta)\eta_i < 0, \qquad i = 1, \ldots, m].$$

Let a point (X, Y) belong to the pyramid (1.18); then, taking $\lambda_j = x_j - \mathring{x}_j$, we get the inequality

$$\lambda = \sum_{j=1}^{p} |\lambda_j| < \min(a, \gamma, \varrho_1, \ldots, \varrho_m),$$

whence it follows that the point $(x = 1, y_1, \ldots, y_n)$ belongs to the pyramid (1.26), where inequalities (1.27) [(1.27a)] are satisfied.

Since we have

$$v^i(1, Y, \Lambda) = u^i(\mathring{X} + \Lambda, Y) = u^i(X, Y), \qquad i = 1, \ldots, m,$$

we conclude by (1.27) [(1.27a)] that inequalities (1.27) [(1.17a)] hold in the pyramid (1.18), which completes the proof.

II. Estimation by means of the solutions of first order partial differential equations.

THEOREM 2.1 (2.1a). *Suppose we are given the comparison system of partial differential equations* (cf. Introduction)

$$(2)\qquad v^i_\xi = h^i(\xi, Y, v^1, \ldots, v^m, v^i_{y_1}, \ldots, v^i_{y_n}), \qquad i = 1, \ldots, m,$$

which has the solution $V(\xi, Y)$ of class $\mathcal{D}$ in the pyramid

$$(2.1)\qquad 0 \leqslant \xi < \gamma; \quad |y_k - \mathring{y}_k| \leqslant a_k - L\xi, \qquad k = 1, \ldots, n,$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty, \quad \gamma \leqslant \min_k\left(\frac{a_k}{L}\right).$$

Moreover, we assume that $V(\xi, Y)$ satisfies the condition

$$(2.2)\qquad \min_{Y\in\Delta_0} V(0, Y) = H = (\eta_1, \ldots, \eta_m) > 0,$$

where

$$\Delta_\xi\colon\ |y_k - \mathring{y}_k| \leqslant a_k - L\xi, \qquad k = 1, \ldots, n.$$

Next we assume that the functions

$$U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big),$$

of class $\mathscr{D}$ in the pyramid (1.1) *satisfy the inequalities*

$$(2.3) \qquad |u^i_x(x, Y)| \leqslant h^i(|x|, Y, |U|, |u^i_y|), \quad i = 1, \ldots, m,$$

with

$$(2.4) \qquad \vartheta H \leqslant U(0, Y) \leqslant H \quad \textit{for } |y_k - \mathring{y}_k| \leqslant a_k, \; k = 1, \ldots, n$$

$$[(2.4\text{a}) \qquad -H \leqslant U(0, Y) \leqslant -\vartheta H \quad \textit{for } |y_k - \mathring{y}_k| \leqslant a_k, \; k = 1, \ldots, n],$$

where $0 < \vartheta \leqslant 1$.

If we denote the minimum positive solution of the equation (4)

$$(2.5) \qquad \max_{Y \in \Delta_\xi} v^i(\varrho, Y) = (1+\vartheta)\eta_i, \quad i = 1, \ldots, m,$$

by ϱ_i, *then in the pyramid*

$$(2.6) \quad |x| < \min(a, \gamma, \varrho_1, \ldots, \varrho_m), \quad |y_k - \mathring{y}_k| \leqslant a_k - L|x|, \quad k = 1, \ldots, n,$$

the following inequalities hold:

$$(2.7) \qquad u^i(x, Y) \geqslant -v^i(|x|, Y) + (1+\vartheta)\eta_i > 0, \quad i = 1, \ldots, m$$

$$[(2.7\text{a}) \qquad u^i(x, Y) \leqslant v^i(|x|, Y) - (1+\vartheta)\eta_i < 0, \quad i = 1, \ldots, m].$$

Proof. First we will prove the theorem in case (A): $x \geqslant 0$. Theorem 61.1 of [1] implies the inequalities

$$(2.8) \qquad |u^i(x, Y)| \leqslant v^i(|x|, Y), \quad i = 1, \ldots, m.$$

If we consider the functions

$$(2.9) \qquad \bar{u}^i(x, Y) = u^i(x, Y) + v^i(x, Y) - (1+\vartheta)\eta_i, \quad i = 1, \ldots, m,$$

then we obtain from the comparison system (2) and inequality (2.3)

$$(2.10) \quad \bar{u}^i_x(x, Y) \geqslant -h^i(x, Y, |U|, |u^i_y|) + h^i(x, Y, V, v^i_y), \quad i = 1, \ldots, m.$$

It is clear that (2.10), condition 2° for comparison systems and (2.8) yield

$$\bar{u}^i_x(x, Y) \geqslant -[h^i(x, Y, V, |u^i_y|) - h^i(x, Y, V, v^i_y)], \quad i = 1, \ldots, m.$$

Then, in view of (4) and (5), it follows that in the pyramid

$$0 \leqslant x < \min(a, \gamma), \quad |y_k - \mathring{y}_k| \leqslant a_k - Lx, \quad k = 1, \ldots, n,$$

(4) If (2.5) has no solutions, we put $\varrho_i = +\infty$.

we have

$$(2.11)\quad \bar{u}_x^i(x, Y) \geqslant -L\sum_{k=1}^{n}\big||u_{y_k}^i|-v_{y_k}^i\big| = -L\sum_{k=1}^{n}\big||u_{y_k}^i|-|v_{y_k}^i|\big|$$
$$\geqslant -L\sum_{k=1}^{n}|u_{y_k}^i+v_{y_k}^i| = -L\sum_{k=1}^{n}|\bar{u}_{y_k}^i|, \quad i = 1, \ldots, m.$$

From (2.1), (2.2) and (2.4) we get

$$(2.12)\quad \bar{u}^i(0, Y) = u^i(0, Y)+v^i(0, Y)-(1+\vartheta)\eta_i \geqslant 0, \quad i = 1, \ldots, m.$$

By example 59.1 in [1], (2.11) and (2.12) imply

$$(2.13)\qquad \bar{u}^i(x, Y) \geqslant 0, \quad i = 1, \ldots, m,$$

in the pyramid

$$0 \leqslant x < \min(a, \gamma); \quad |y_k-\mathring{y}_k| \leqslant a_k-Lx, \quad k = 1, \ldots, n.$$

Since ϱ_i is the minimum positive solution of equation (2.5), we infer that in the pyramid

$$0 \leqslant x < \min(a, \gamma, \varrho_1, \ldots, \varrho_m); \quad |y_k-\mathring{y}_k| \leqslant a_k-L|x|, \quad k = 1, \ldots, n,$$

the inequalities

$$(2.14)\quad u^i(x, Y) \geqslant -v^i(x, Y)+(1+\vartheta)\eta_i > 0, \quad i = 1, \ldots, m,$$

are satisfied.

Applying again the substitution $s = -x$, we reduce case (B): $x < 0$ to case (A).

The proof of Theorem 2.1a is similar.

THEOREM 2.2 (2.2a). *Suppose we are given the comparison system of partial differential equations*

$$(2)\qquad v_{\xi_l}^i(\Xi, Y) = h_l^i(\xi_1, \ldots, \xi_p, Y, v^1, \ldots, v^m, v_{y_1}^i, \ldots, v_{y_n}^i)$$

for $i = 1, \ldots, m$ *and* $l = 1, \ldots, p$ *and a solution* $V(\Xi, Y)$ *which is of class* $\mathcal{D}$ *in the pyramid*

$$(2.15)\quad \xi_l \geqslant 0, \quad \sum_{j=1}^{p}\xi_j < \gamma; \quad |y_k-\mathring{y}_k| \leqslant a_k-L\sum_{j=1}^{p}\xi_j, \quad k = 1, \ldots, n,$$

where

$$0 \leqslant L < +\infty; \quad 0 < a_k < +\infty; \quad \gamma \leqslant \min_k\left(\frac{a_k}{L}\right)$$

with

$$(2.16)\qquad \min_{Y\in\Delta_0} V(0, Y) = H > 0,$$

where

$$\Delta_\xi\colon\ |y_k-\mathring{y}_k|\leqslant a_k-L\sum_{j=1}^{p}\xi_j,\qquad k=1,\ldots,n.$$

We assume that the functions $U(X, Y)$ *are of class* $\mathcal{D}$ *in the pyramid* (0) *and satisfy the differential inequalities*

$$|u^i_{x_l}(X, Y)|\leqslant h^i_l(|X-\mathring{X}|, Y, |U|, |u^i_y|) \tag{2.17}$$

for $i=1,\ldots,m$ *and* $l=1,\ldots,p$ *as well as the initial inequalities*

$$\vartheta H\leqslant U(\mathring{X}, Y)\leqslant H; \tag{2.18}$$

[(2.18a) $\qquad -H\leqslant U(\mathring{X}, Y)\leqslant -\vartheta H$],

with $0<\vartheta\leqslant 1$.

If we denote the minimum positive solution of equation in ϱ

$$\max_{\sum_{j=1}^{p}\xi_j=\varrho;\, Y\in\Delta_\xi} v^i(\Xi, Y)=(1+\vartheta)\eta_i,\qquad i=1,\ldots,m, \tag{2.19}$$

by ϱ_i, then in the pyramid

$$\sum_{j=1}^{p}|x_j-\mathring{x}_j|<\min(\alpha,\gamma,\varrho_1,\ldots,\varrho_m), \tag{2.20}$$

$$|y_k-\mathring{y}_k|\leqslant a_k-L\sum_{j=1}^{p}|x_j-\mathring{x}_j|,\qquad k=1,\ldots,n,$$

we have the inequalities

$$u^i(X, Y)\leqslant -v^i(|X-\mathring{X}|, Y)+(1+\vartheta)\eta_i>0,\qquad i=1,\ldots,m \tag{2.21}$$

[(2.21a) $\quad u^i(X, Y)\geqslant v^i(|X-\mathring{X}|, Y)-(1+\vartheta)\eta_i<0,\qquad i=1,\ldots,m$].

The proof is performed by means of Mayer's transformation similarly to the case of Theorems 1.2 and 1.1.

References

[1] J. Szarski, *Differential inequalities*, Monografie Matematyczne, t. 43, Warszawa 1967.

[2] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Ann. Soc. Polon. Math. 23 (1950), p. 112-166.

Reçu par la Rédaction le 3. 2. 1971