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On a modification of the method of Euler polygons for the ordinary differential equation

by A. PELCZAR (Kraków) *

The subject of the present paper is a problem which has been formulated by Professor T. Ważewski.

§ 1. We shall consider the ordinary differential equation

$$y' = f(x, y) \tag{1.1}$$

with the initial condition

$$y(a) = c\,. \tag{1.2}$$

By $z_n(x, \xi)$, $n = 1, 2, \ldots$, we denote the Euler polygon constructed for the interval $\langle a, \xi\rangle$ and the division $d(a_1, \ldots, a_n)$ of this interval by points $a_j = a + \frac{j}{n}(\xi - a)$ $(j = 0, 1, \ldots, n)$. For $n = 0$ we define $z_0(x, \xi) = c$. Now we put

$$\varphi_n(\xi) \overset{\text{df}}{=} z_n(\xi, \xi)\,. \tag{1.3}$$

For example, for the equation

$$y' = y \tag{1.1'}$$

with the initial condition

$$y(0) = 1 \tag{1.2'}$$

we have $a = 0$ and

$$\begin{aligned} &z_0(x, \xi) = 1\,, && x \in \langle 0, \xi\rangle\,, \\ &z_1(x, \xi) = 1 + x\,, && x \in \langle 0, \xi\rangle\,, \\ &z_2(x, \xi) = \begin{cases} 1 + x\,, & x \in \left\langle 0, \frac{\xi}{2}\right\rangle, \\ \left(1 + \frac{\xi}{2}\right)\left(1 + x - \frac{\xi}{2}\right), & x \in \left(\frac{\xi}{2}, \xi\right\rangle, \end{cases} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{1.3'}$$

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$$
(1.3') \qquad z_n(x, \xi) = \begin{cases} 1+x\,, & x \in \left\langle 0, \frac{\xi}{n}\right\rangle, \\ \left(1+\frac{\xi}{n}\right)\left(1+x-\frac{\xi}{n}\right), & x \in \left(\frac{\xi}{n}, \frac{2}{n}\xi\right\rangle, \\ \left(1+\frac{\xi}{n}\right)^2\left(1+x-\frac{2}{n}\xi\right), & x \in \left(\frac{2}{n}\xi, \frac{3}{n}\xi\right\rangle, \\ \dots\dots\dots\dots & \\ \left(1+\frac{\xi}{n}\right)^n\left(1+x-\frac{n-1}{n}\xi\right), & x \in \left(\frac{n-1}{n}\xi, \xi\right\rangle. \end{cases}
$$

In this case

$$
(1.4') \qquad \varphi_n(\xi) = \left(1+\frac{\xi}{n}\right)^n
$$

and the sequence $\{\varphi_n(\xi)\}$ is uniformly convergent to e^ξ in each interval $\langle 0, b\rangle$ $(b < +\infty)$. Hence for the equation (1.1′) with the initial condition (1.2′), the sequence $\{z_n(\xi, \xi)\}$ is uniformly convergent to the solution of the problem (1.1′), (1.2′). Moreover, in this case, the sequence of the derivatives $\{\varphi_n'(\xi)\}$ is uniformly convergent to the derivative of the solution of this problem.

Now the following problem arises: Is the sequence $\{\varphi_n(x)\}$, defined by (1.3), uniformly convergent to the solution of the equation (1.1) with the initial condition (1.2) in the general case?

Remark 1. From the definition of the sequence $\{\varphi_n(x)\}$ it follows directly that we can write it in the following form:

$$
(1.4) \qquad \begin{aligned} &\varphi_0(x) = c\,, \\ &\varphi_n(x) = \varphi_{n-1}\left(a+\frac{n-1}{n}(x-a)\right)+ \\ &\qquad +\frac{x-a}{n} f\left[a+\frac{n-1}{n}(x-a),\, \varphi_{n-1}\left(a+\frac{n-1}{n}(x-a)\right)\right] \end{aligned}
$$

or equivalently

$$
(1.5) \qquad \begin{aligned} &\varphi_0(x) = c\,, \\ &\varphi_n(x) = c+\frac{x-a}{n}\sum_{i=0}^{n-1} f\left[a+\frac{i}{n}(x-a),\, \varphi_i\left(a+\frac{i}{n}(x-a)\right)\right]. \end{aligned}
$$

The answer to the above problem is given by the following

THEOREM 1. *Let us assume that*

1. $f(x, y)$ *is defined and continuous in the set*

$$
T = \{x, y\colon\ a \leqslant x \leqslant b\,,\ |y-c| \leqslant M\}\,,
$$

2. $|f(x,y)| \leqslant M$, $b-a<1$,

3. *in the interval* $\langle a, b\rangle$ *there exists exactly one solution* $y(x)$ *of the problem* (1.1), (1.2).

Then the sequence $\{\varphi_n(x)\}$ *defined by* (1.3) *is uniformly convergent to* $y(x)$.

Proof. Let us perform the division $d = d(\beta_0, \beta_1, \ldots, \beta_m)$ of the interval $\langle a, b\rangle$ by the points β_j $(\beta_j < \beta_{j+1})$. We put

$$(1.6)\qquad \delta(d) \overset{\text{df}}{=} \max_{j,k=0,\ldots,m} (|\beta_j - \beta_k|) .$$

By $E(s)$ we denote the set of all Euler's polygons ψ constructed for the interval $\langle a, b\rangle$ and a division d of this interval such that $\delta(d) < s$. We have (cf. [1], III, § 8)

$$(1.7)\qquad \forall_{\varepsilon>0}\exists_{\eta}\forall_{\psi\in E(s)}\forall_{x\in\langle a,b\rangle}[s \leqslant \eta \Rightarrow |\psi(x)-y(x)| \leqslant \varepsilon]$$

where $y(x)$ is the unique solution of the problem (1.1), (1.2). Let $\xi \in \langle a, b\rangle$. By $\hat{z}_n(x, \xi)$ we denote the Euler polygon for the division $\hat{d}_{n,\xi}$, which is given by the points

$$(1.8)\qquad a,\ a+\frac{1}{n}(\xi-a),\ \ldots,\ a+\frac{n-1}{n}(\xi-a),\ \xi,$$
$$a+\frac{n+1}{n}(\xi-a),\ \ldots,\ a+\frac{k}{n}(\xi-a),\ b$$

where k is such an integer that

$$a+\frac{k}{n}(\xi-a) \leqslant b < a+\frac{k+1}{n}(\xi-a) .$$

Of course, the definition of the sequence $\{\hat{z}_n(x, \xi)\}$ implies directly that

$$(1.9)\qquad \hat{z}_n(\xi,\xi) = z_n(\xi,\xi) = \varphi_n(\xi) .$$

By $s(n, \zeta)$ we shall denote $\delta(\hat{d}_{n,\zeta})$. It is easy to see that for each $\zeta \leqslant \xi$ we have

$$s(n,\zeta) \leqslant s(n,\xi) .$$

In particular

$$\forall_{\xi\in\langle a,b\rangle}\ [s(n,\xi) \leqslant s(n,b)] .$$

Moreover

$$\forall_{\eta}\exists_{N}\ [n \geqslant N \Rightarrow s(n,b) \leqslant \eta] .$$

Hence

$$(1.10)\qquad \forall_{\eta}\exists_{N}\forall_{\xi\in\langle a,b\rangle}\ [n \geqslant N \Rightarrow s(n,\xi) \leqslant \eta] .$$

From (1.7) and (1.10) it follows that

$$\forall_{\varepsilon}\exists_{N}\forall_{\xi\in\langle a,b\rangle}\forall_{x\in\langle a,b\rangle}\ [n \geqslant N \Rightarrow |\hat{z}_n(x,\xi)-y(x)| \leqslant \varepsilon]$$

and from (1.9) we have

$$\forall_\varepsilon \exists_N \forall_{\xi \in \langle a,b\rangle} \; [n \geqslant N \Rightarrow |\varphi_n(\xi) - y(\xi)| \leqslant \varepsilon]\,,$$

which completes the proof of Theorem 1.

§ 2. Remark 2. If we assume that $f(x, y)$ has both partial first derivatives, then each $\varphi_n(x)$ has a first derivative. We prove this easily by induction with respect to n (making use of (1.4)).

Now the following problem appears: Is the sequence of derivatives $\{\varphi_n'(x)\}$ uniformly convergent to the derivative $y'(x)$ of the solution of the problem (1.1), (1.2)?

To answer this problem we shall prove the following

THEOREM 2. *Let us assume that*

1. *$f(x, y)$ is defined and continuous in T,*

2. *$f(x, y)$ has both first derivatives, fulfilling the Lipschitz condition with respect to both variables,*

3. $|f(x,y)| \leqslant M\,, \quad \left|\dfrac{\partial f}{\partial x}\right| \leqslant M_1\,, \quad \left|\dfrac{\partial f}{\partial y}\right| \leqslant M_2\,,$

4. $(b-a)M_2 < 1\,, \quad b-a < 1.$

Then the sequence od derivatives $\{\varphi_n'(x)\}$ is uniformly convergent in $\langle a, b\rangle$ to the derivative of the solution of the problem (1.1), (1.2).

Remark 3. From the assumptions of Theorem 2 it follows that in the interval $\langle a, b\rangle$ there exists exactly one solution $y(x)$ of the problem (1.1), (1.2).

Proof. I_2. At first we shall show that there exists a number Q_1 such that

$$|\varphi_n'(x)| \leqslant Q_1 \quad \text{for each} \quad x \in \langle a, b\rangle\,, \; n = 0, 1, \ldots \tag{2.1}$$

We put

$$\lambda_n(x) = a + \frac{n-1}{n}(x-a)\,, \qquad \mu_n(x) = \varphi_{n-1}\big(\lambda_n(x)\big)\,. \tag{2.2}$$

Hence

$$\lambda_n'(x) = \frac{n-1}{n}\,, \qquad \mu_n'(x) = \varphi_{n-1}'\big(\lambda_n(x)\big)\cdot\frac{n-1}{n}\,. \tag{2.3}$$

From (1.4) we have

$$\varphi_n'(x) = \mu_n'(x) + \frac{1}{n} f[\lambda_n(x), \mu_n(x)] + \frac{x-a}{n}\{f_x[\lambda_n(x), \mu_n(x)]\lambda_n'(x) + f_y[\lambda_n(x), \mu_n(x)]\mu_n'(x)\}\,. \tag{2.4}$$

In view of (2.3) and the assumptions of the theorem, we have

$$|\varphi_n'(x)| \leqslant \left|\varphi_{n-1}'\big(\lambda_n(x)\big)\right| \left(\frac{n-1}{n} + \frac{b-a}{n}\cdot\frac{n-1}{n} M_2\right) + \frac{1}{n} M + \frac{b-a}{n}\cdot\frac{n-1}{n} M_1 .$$

Because $(n-1)/n < 1$, we have

$$|\varphi_n'(x)| \leqslant \left|\varphi_{n-1}'\big(\lambda_n(x)\big)\right| \left(1 - \frac{1}{n} + \frac{b-a}{n} M_2\right) + \frac{1}{n} M + \frac{b-a}{n} M_1 .$$

Now we want to find a constant Q_1 such that

(2.5) $$\{|\varphi_{n-1}'(x)| \leqslant Q_1 \text{ in } \langle a, b\rangle\} \Rightarrow \{|\varphi_n'(x)| \leqslant Q_1 \text{ in } \langle a, b\rangle\} .$$

It is easy to see that this condition is fulfilled by each positive solution Q of the following inequality:

(2.6) $$Q\left(1 - \frac{1}{n} + \frac{b-a}{n} M_2\right) + \frac{1}{n} M + \frac{b-a}{n} M_1 \leqslant Q .$$

Hence if we put in particular

(2.7) $$Q_1 = \frac{M + (b-a) M_1}{1-(b-a) M_2} ,$$

then $|\varphi_0'(x)| = 0 \leqslant Q_1$ and (2.5) holds, and in consequence (2.1) holds for each n.

II_2. We shall prove that there exists a constant R_1 such that

(2.8) $$|\varphi_n'(x) - \varphi_n'(y)| \leqslant R_1 |x-y| \quad \text{for} \quad x, y \in \langle a, b\rangle ,\ n = 0, 1, \ldots$$

Let $L_{11}, L_{12}, L_{21}, L_{22}$ be the Lipschitz constants for the partial derivatives of the function $f(x, y)$ (see assumption 2), i.e. we have

$$|f_x(x, y) - f_x(\bar{x}, \bar{y})| \leqslant L_{11}|x-\bar{x}| + L_{12}|y-\bar{y}| ,$$

$$|f_y(x, y) - f_y(\bar{x}, \bar{y})| \leqslant L_{21}|x-\bar{x}| + L_{22}|y-\bar{y}| .$$

From (2.2) and (2.3) we have

(2.9) $$|\lambda_n(x) - \lambda_n(y)| \leqslant \frac{n-1}{n} |x-y| ,$$

(2.10) $$|\mu_n(x) - \mu_n(y)| \leqslant Q_1 \frac{n-1}{n} |x-y| ,$$

where Q_1 is the constant (2.7). From (2.4), (2.9) and (2.10) we infer

(2.11) $$|\varphi_n'(x) - \varphi_n'(y)|$$
$$\leqslant \frac{n-1}{n} A_n \left|\varphi_{n-1}'\big(\lambda_n(x)\big) - \varphi_{n-1}'\big(\lambda_n(y)\big)\right| + \frac{1}{n}\cdot\frac{n-1}{n} B_n |x-y| ,$$

where

$$A_n = 1 + M_2 \frac{b-a}{n}, \tag{2.12}$$

$$B_n = 2(M_1 + Q_1 M_2) + \frac{n-1}{n} C, \tag{2.13}$$

$$C = (b-a)(L_{11} + L_{12}Q_1 + L_{21}Q_1 + L_{22}Q_1^2). \tag{2.14}$$

Now we want to find a number R_1 such that

$$\{|\varphi'_{n-1}(x) - \varphi'_{n-1}(y)| \leqslant R_1|x-y|\} \Rightarrow \{|\varphi'_n(x) - \varphi'_n(y)| \leqslant R_1|x-y|\}. \tag{2.15}$$

This property characterizes each positive solution R of the following inequality:

$$R - R\left(\frac{1}{n} - M_2\frac{b-a}{n}\right) + \frac{1}{n}[2(M_1 + M_2Q_1) + C] \leqslant R. \tag{2.16}$$

Hence, if we put in particular

$$R_1 = \frac{2(M_1 + M_2Q_1) + C}{1 - M_2(b-a)} \tag{2.17}$$

then $|\varphi'_0(x) - \varphi'_0(y)| = 0 \leqslant R_1|x-y|$ and (2.15) holds. Hence (2.8) holds for each n.

III$_2$. Let $\{\varphi'_{a_n}\}$ be an arbitrary subsequence of the sequence $\{\varphi'_n\}$. From Arzelo's theorem, the assumptions of which are satisfied in view of parts I$_2$ and II$_2$, it follows that there exists a subsequence $\{\varphi'_{\beta_n}\}$ of the sequence $\{\varphi'_{a_n}\}$ uniformly convergent. From Theorem 1 it follows that the sequence $\{\varphi_{\beta_n}\}$ is uniformly convergent to $y(x)$. Hence $\{\varphi'_{\beta_n}\}$ is uniformly convergent to $y'(x)$. But the limit is independent of the choice of the sequence $\{\varphi'_{a_n}\}$. Hence $\{\varphi'_n\}$ is uniformly convergent in the interval $\langle a, b\rangle$ to the derivative $y'(x)$.

§ 3. Remark 4. It is easy to prove by induction with respect to n that if $f(x,y)$ has all derivatives $\frac{\partial^p f}{\partial x^q \partial y^r}$ ($p = 1, \ldots, k$, $q + r = p$, $q = 0, 1, \ldots, k$, $r = 0, 1, \ldots, k$), then each $\varphi_n(x)$ has all derivatives $\varphi_n^{(p)}(x)$ ($p = 1, \ldots, k$).

THEOREM 3. *Let us assume that*

1. *$f(x,y)$ is defined and continuous in T,*
2. $|f(x,y)| \leqslant M$, $\left|\frac{\partial f}{\partial y}\right| \leqslant M_2$,
3. $(b-a)\cdot M_2 < 1$, $b - a < 1$,
4. *$f(x,y)$ has all bounded partial derivatives* $\frac{\partial^p f}{\partial x^q \partial y^r}$ ($p = 1, \ldots, k$, $q + r = p$, $q = 0, \ldots, k$, $r = 0, \ldots, k$),

5. *all derivatives* $\frac{\partial^k f}{\partial x^q \partial y^r}$ $(q+r=k,\ q=0,\dots,k,\ r=0,\dots,k)$ *fulfil the Lipschitz condition with respect to both variables.*

Then the sequence $\left\{\frac{d^k}{dx^k}\varphi_n(x)\right\}$ *is uniformly convergent in* $\langle a,b\rangle$ *to* $\frac{d^k}{dx^k}y(x)$, *where* $y(x)$ *is the solution of the problem* (1.1), (1.2).

Proof. I_3. From (1.5) it easily follows that for $m \geqslant 2$

$$\varphi_n^{(m)}(x) = \frac{m}{n}\sum_{i=0}^{n-1} U_i + \frac{x-a}{n}\sum_{i=0}^{n-1} V_i\,, \tag{3.1}$$

where

$$U_i = U_i(x) = \frac{d^{m-1}}{dx^{m-1}} f\left[a+\frac{i}{n}(x-a)\,,\ \varphi_i\left(a+\frac{i}{n}(x-a)\right)\right], \tag{3.2}$$

$$V_i = V_i(x) = \frac{d^{m}}{dx^{m}} f\left[a+\frac{i}{n}(x-a)\,,\ \varphi_i\left(a+\frac{i}{n}(x-a)\right)\right]. \tag{3.3}$$

Moreover, it is possible to write

$$V_i = W_i(x) + \left(\frac{i}{n}\right)^m \frac{\partial f}{\partial y}\cdot \varphi_i^{(m)}\left(a+\frac{i}{n}(x-a)\right), \tag{3.4}$$

where W_i is independent of $\varphi_i^{(m)}$. Of course U_i is also independent of $\varphi_i^{(m)}$.

II_3. It is possible to prove that for each m there exists a constant Q_m such that

$$|\varphi_n^{(m)}(x)| \leqslant Q_m \quad \text{for each} \quad x \in \langle a,b\rangle\,,\ n=0,1,\dots \tag{3.5}$$

In order to prove this, we apply the induction procedure with respect to m. For $m=0$ (3.5) holds evidently ($Q_0 = |c|+M$). In view of the inequality (2.1) (see part I_2 of the proof of Theorem 2) it holds also for $m=1$. Now we assume that there exist such constants Q_p $(p \leqslant s-1)$ that (3.5) holds for all $m=0,1,\dots,s-1$. Now, the induction procedure with respect to n proceeds in the same manner as in part I_2 of the proof of Theorem 2. In consequence there exists a constant Q_s, such that (3.5) holds for $m=s$, which finishes the induction proof of (3.5) for all m.

III_3. **Remark 5.** If we make the first and the second assumptions of Theorem 1 and, moreover, assume that $f(x,y)$ fulfils the Lipschitz condition with respect to x and y, with the constants K and L respectively, then

$$|\varphi_n(x)-\varphi_n(y)| \leqslant R_0|x-y| \tag{3.6}$$

for each $x \in \langle a,b\rangle$, $n=0,1,\dots$, where

$$R_0 = \frac{M+(b-a)K}{1-(b-a)L}\,.$$

The proof of this follows the same method as the proof of (2.1), Of course, if we suppose that $f(x, y)$ has both bounded partial derivatives, then $f(x, y)$ fulfils the condition of Lipschitz and (3.6) holds.

It is possible to prove that for each m there exists a constant R_m such that

$$(3.7) \qquad |\varphi_n^{(m)}(x) - \varphi_n^{(m)}(y)| \leqslant R_m|x-y| \quad \text{for} \quad x, y \in \langle a, b\rangle, \ n = 0, 1, \ldots$$

In order to prove the existence of that R_m we apply the induction procedure with respect to m. In view of Remark 5 and (2.8), (3.7) holds for $m = 0, 1$.

Now we assume that there exist such constants R_p $(p \leqslant s-1)$ that (3.7) holds for $m = 0, 1, \ldots, s-1$. Then U_i and V_i are polynomials of the Lipschitz functions and in consequence they are also Lipschitz functions.

Now the induction procedure with respect to n proceeds in the same manner as in part II_2 of the proof of Theorem 2. Hence, we infer the existence of such a constant R_s that (3.7) holds for $m = s$, which completes the proof of (3.7) for all m.

IV_3. In order to finish the proof of the theorem, we apply a similar reasoning to that followed in part III_2 of the proof of Theorem 2.

Reference

[1] E. Kamke, *Differentialgleichungen I*, Leipzig 1962.

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