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On the μ-observability of some non-linear difference equations

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The problem of observability and the problem of μ-observability of ordinary differential systems was considered by E. J. Roytenberg in [3] and [4]. We shall show that an analogical treatment may be applied to solutions of discrete difference systems of equations.

We introduce the notions of μ-observability and of asymptotical μ-observability of the solutions of some non-linear discrete systems of difference equations. Then we formulate some sufficient conditions for a solution of such a system to be asymptotically μ-observable and we construct a system of difference equations which is satisfied by a certain function realizing observability.

We take into consideration the system of discrete difference equations

$$x_i(n+1) = \sum_{k=1}^{m} a_{ik}(n)\cdot x_k(n) + \varphi_i\big(x_1(n), \ldots, x_m(n)\big) + p_i(n) \tag{1}$$

for $n = 0, 1, 2, \ldots$, with the following initial conditions

$$x_i(\tau) = \mathring{x}_i, \quad i = 1, 2, \ldots, m,$$

where τ is natural number.

System (1) written in a matrix form is the following:

$$x(n+1) = A(n)\cdot x(n) + \varphi\big(x(n)\big) + p(n), \quad x(\tau) = \mathring{x}, \tag{2}$$

where $A(n) = [a_{ik}(n)]_{m,m}$, $\varphi(t) = [\varphi_i(t)]_{m,1}$ and $p(n) = [p_i(n)]_{m,1}$ are given functions and

$$x(n) = [x_1(n), \ldots, x_m(n)] = [x_i(n)]'_{m,1} \ (^1)$$

is an unknown function. We assume that the functions which appear in (2) satisfy the following conditions: the operator $A(n)$ is defined for $n \geqslant \tau$ and it is bounded, i.e. we have $\|A(n)\| < K = \text{const}$ for $n \geqslant \tau$; the vector function $p(n)$ is defined for $n \geqslant \tau$, and the function $\varphi(t)$ defined for $\tau \leqslant t < +\infty$ is of C^2 regularity class.

(1) $[x_i(n)]'_{m,1}$ denotes a matrix which is transposed with respect to $[x_i(n)]_{1,m}$.

We introduce the vector function

$$y(n) = C(n)\cdot x(n), \tag{3}$$

which is defined for $n \geqslant \tau$ and is called a *trace* of the solution $x(n)$ of (2). Here $C(n) = [c_{ik}(n)]_{m,m}$ is a given bounded operator, i.e. $\|C(n)\| < c = \text{const}$ for $n \geqslant \tau$ and $x(n)$ is an unknown function which has to satisfy equation (2).

DEFINITION 1. If the value $x(\tau_1)$ of a solution $x(n)$ can be determined from a trace $y(n)$ which is known in the interval $\langle \tau, \tau_1 \rangle$, where τ_1 is a natural number, then we say that the solution is observable.

DEFINITION 2. If there exists a function $\zeta(n)$ defined for $n \geqslant \tau$ and such that:

1° $\zeta(n)$ may be determined with the aid of a trace $y(n)$, which is defined on $\langle \tau, \tau_1 \rangle$,

2° we have

$$\|\zeta(\tau_1) - x(\tau_1)\| \leqslant \mu, \tag{4}$$

where μ is some positive number, then we say that $x(n)$ is μ-observable at the moment $n = \tau_1$.

DEFINITION 3. If we replace condition 2° in Definition 2 by the following one:

2′ if for an arbitrary number $\mu > 0$ there exists a number $T(\mu)$ such that the inequality

$$\|\zeta(n) - x(n)\| \leqslant \mu \quad \text{holds for } n \geqslant T(\mu),\ n \in (\tau, +\infty), \tag{5}$$

then we say that $x(n)$ is *asymptotically μ-observable.*

It is evident that if $\tau_1 > T(\mu)$, then inequality (5) implies (4). Thus μ-observability at the moment $n = \tau_1$ is a consequence of asymptotic μ-observability.

The function $\zeta(n) = [\zeta_i(n)]_{1,m}$ will be called a *function realizing the observability of $x(n)$*. The construction of the function $\zeta(n)$ will be given later (cf. Theorem 2).

Let us consider an auxiliary difference equation of the form

$$\zeta(n+1) = A(n)\cdot\zeta(n) + \varphi\big(\zeta(n)\big) + p(n) + u(n), \tag{6}$$

with the initial condition $\zeta(\tau) = \mathring{\zeta}$ such that its solution $\zeta(n)$ is known. We assume that $u(n)$ is some vector function, defined for $n \geqslant \tau$. We assume that $\mathring{\zeta}$ is an element of the ball $S_\varrho(\mathring{x})$, i.e. $\mathring{\zeta} \in S_\varrho(\mathring{x}) = \{x;\ \|x - \mathring{x}\| < \varrho\}$.

Let

$$\eta(n) = C(n)\cdot\zeta(n) \tag{7}$$

be a trace of the solution $\zeta(n)$ of (6).

We choose the vector $u(n)$ in such a way that for $n = T(\mu)$ the inequality $\|\zeta(n)-x(n)\| \leqslant \mu$ holds, where $\mu > 0$ is a given number.

Let us consider a vector function

$$z(n) = \zeta(n)-x(n). \tag{8}$$

It follows from (2) and (6) that $z(n)$ satisfies the difference equations

$$z(n+1) = A(n)\cdot z(n)+\varphi\big(\zeta(n)\big)-\varphi\big(x(n)\big)+u(n) \tag{9}$$

with the initial condition

$$\mathring{z} = z(\tau) = \zeta(\tau)-x(\tau) = \mathring{\zeta}-\mathring{x},$$

where $\|\mathring{z}\| = \|\mathring{\zeta}-\mathring{x}\| < \varrho$.

We expand the function φ in a Taylor series. We obtain

$$\varphi(x+z) = \varphi(x)+\frac{\varphi'(x)}{1!}\cdot z+F(z, n), \quad \text{where } \|F(z, n)\| = o(\|z\|).$$

If we put $M(n) = \varphi'\big(x(n)\big)$, then (9) may be rewritten as follows:

$$z(n+1) = A(n)\cdot z(n)+M(n)\cdot z(n)+u(n)+F(z, n). \tag{10}$$

The first approximation for (10) is the following:

$$z(n+1) = [A(n)+M(n)]\cdot z(n)+u(n). \tag{11}$$

We assume that $u(n)$ is of the form

$$u(n) = B(n)\cdot v(n), \tag{12}$$

where $v(n) = \eta(n)-y(n)$ and $B(n)$ is a certain bounded operator defined for $n \geqslant \tau$.

We obtain from (3), (7) and (8)

$$v(n) = \eta(n)-y(n) = C(n)\cdot\zeta(n)-C(n)\cdot x(n) = C(n)\cdot z(n).$$

Then the exppression (12) obtains the form

$$u(n) = B(n)\cdot v(n) = B(n)\cdot C(n)\cdot z(n). \tag{13}$$

We write

$$A_1(n) = A(n)+M(n)+B(n)\cdot C(n). \tag{14}$$

If we use (13) and (14), then we may rewrite (11) in the form

$$z(n+1) = A_1(n)\cdot z(n). \tag{15}$$

Moreover, assume that the norm of the operator $M(n)$ satisfies the following conditions:

$$\beta \leqslant \|M(n)\| \leqslant \gamma \quad \text{for } n \geqslant \tau$$

on a set of all admissible solutions $x(n)$ of (2).

The sufficient conditions for the asymptotic μ-observability of solutions of (2) may be formulated in the following

THEOREM 1. *A sufficient condition for $x(n)$ to be an asymptotically μ-observable solution of* (2) *is that there exist a positive operator $V(n)$ such that*

$$0 < \alpha_1\cdot\big(z(n), z(n)\big) \leqslant \big(V(n)\cdot z(n), z(n)\big) \leqslant \alpha_2\cdot\big(z(n), z(n)\big) \tag{16}$$

and

$$\big(V(n+1)\cdot z(n+1), z(n+1)\big) \leqslant \beta_1\cdot\big(z(n), z(n)\big), \tag{17}$$

where $z(n)$ is a solution of (15) *and $\alpha_1, \alpha_2, \beta_1$ are some positive constants,* $0<\beta_1<\alpha_1$.

In order to prove this we must give some preliminaries.

LEMMA. *If a non-negative function $\chi(n)$ defined for $n\geqslant\tau$ satisfies the inequality*

$$\chi(n) < \delta+\sum_{i=\tau}^{n-1}\big(\eta+L\cdot\chi(i)\big), \qquad \chi(\tau)<\delta, \tag{18}$$

where δ, γ and L are positive constants, then we have

$$\chi(n) < (L+1)^{n-\tau}\cdot\delta+\frac{\eta}{L}\,[(L+1)^{n-\tau}-1]. \tag{19}$$

Proof. If we write $h(n) = \delta+\sum_{i=\tau}^{n-1}\big(\eta+L\cdot\chi(i)\big)$, then we obtain

$$h(n+1) = \delta+\sum_{i=\tau}^{n}\big(\eta+L\cdot\chi(i)\big) = \delta+\sum_{i=\tau}^{n-1}\big(\eta+L\cdot\chi(i)\big)+\eta+L\cdot\chi(n)$$
$$= h(n)+L\cdot\chi(n)+\eta \leqslant (L+1)\cdot h(n)+\eta.$$

The theorems on difference inequalities (cf. [5], Theorem 1) imply

$$h(n+1) < (L+1)^{n-\tau-1}\cdot h(\tau+1)+\sum_{r=1}^{n}(L+1)^{n-\tau-1-r}\cdot\eta.$$

We have from our assumptions

$$h(\tau+1) < \delta+\eta+L\cdot\chi(\tau) < (L+1)\cdot\delta+\eta.$$

Thus we obtain

$$h(n) < (L+1)^{n-\tau}\cdot\delta+(L+1)^{n-\tau-1}\cdot\eta+\eta\cdot[(L+1)^{n-\tau-2}+\ldots+1]$$
$$= (L+1)^{n-\tau}\cdot\delta+\eta\cdot\frac{(L+1)^{n-\tau-1}-1}{L+1-1}$$
$$= (L+1)^{n-\tau}\cdot\delta+\frac{\eta}{L}\cdot[(L+1)^{n-\tau}\cdot(L+1)^{-1}-1]$$
$$< (L+1)^{n-\tau}\cdot\delta+\frac{\eta}{L}\cdot[(L+1)^{n-\tau}-1], \qquad \text{because } L+1>1.$$

Hence we obtain

$$\chi(n) < h(n) < (L+1)^{n-\tau}\cdot\delta+\frac{\eta}{L}[(L+1)^{n-\tau}-1].$$

It should be noted that this lemma is a discrete analogy of a certain form of Gronwall's lemma (cf. [1], p. 12 or [6], p. 54).

Let us consider a difference equation of the form

$$(20)\qquad s(n+1) = D(n)\cdot s(n)+g(n), \quad s(\tau) = \mathring{s},$$

where the operator $D(n) = [d_{ik}(n)]_{m,m}$ and the vector function $g(n) = [g_i(n)]_{m,1}$ are given for $n \geqslant \tau$ and $s(n) = [s_i(n)]_{m,1}$ is an unknown function.

A solution of the homogeneous difference equation

$$(21)\qquad s(n+1) = D(n)\cdot s(n), \quad s(\tau) = \mathring{s},$$

is of the form

$$(22)\qquad s(n) = \prod_{i=\tau}^{n-1} D(n-1+\tau-i)\cdot s(\tau) = \prod_{i=0}^{n-1} D(n-1-i)\cdot\Big[\sum_{i=0}^{\tau-1} D(\tau-1-i)\Big]^{-1}\cdot s(\tau).$$

If we write

$$(23)\qquad S(n) = \prod_{i=0}^{n-1} D(n-1-i),$$

then we may write (22) in the form

$$(24)\qquad s(n) = S(n)\cdot S^{-1}(\tau)\cdot s(\tau) = S(n,\tau)\cdot s(\tau),$$

where $S(n,\tau) = S(n)\cdot S^{-1}(\tau)$.

We look for a solution of a non-homogeneous difference equation of the form

$$(25)\qquad s(n) = S(n)\cdot S^{-1}(\tau)\cdot w(n).$$

Then we obtain from (20) and (25)

$$S(n+1)\cdot S^{-1}(\tau)\cdot w(n+1) = D(n)\cdot S(n)\cdot S^{-1}(\tau)\cdot w(n)+g(n) = S(n+1)\cdot S^{-1}(\tau)\cdot w(n)+g(n).$$

Hence

$$\Delta w(n) = w(n+1)-w(n) = S(\tau)\cdot S^{-1}(n+1)\cdot g(n).$$

Then (cf. [2], p. 19–22)

$$(26)\qquad w(n) = \Delta^{-1}S(\tau)\cdot S^{-1}(n+1)\cdot g(n) = \sum_{j=\tau}^{n-1} S(\tau)\cdot S^{-1}(j+1)\cdot g(j)+s(\tau).$$

In view of (25) and (26) we obtain the following solution of difference equation (20):

$$s(n) = S(n)\cdot S^{-1}(\tau)\cdot s(\tau) + \sum_{j=\tau}^{n-1} S(n)\cdot S^{-1}(j+1)\cdot g(j). \tag{27}$$

Proof of Theorem 1. We consider equation (15) and write

$$\psi(n) = \big(V(n)\cdot z(n), z(n)\big) = \big(V(n)\cdot S(n,\tau)\cdot z(\tau), S(n,\tau)\cdot z(\tau)\big), \tag{28}$$

where $S(n,\tau)$ (cf. (24)) is a solution of equation (15). Thus we have

$$\psi(n+1) = \big(V(n+1)\cdot z(n+1), z(n+1)\big) \leqslant \beta_1\cdot\big(z(n), z(n)\big)$$
$$\leqslant \frac{\beta_1}{\alpha_1}\big(V(n)\cdot z(n), z(n)\big) = \frac{\beta_1}{\alpha_1}\psi(n),$$

and, moreover,

$$\psi(\tau) = \big(V(\tau)\cdot z(\tau), z(\tau)\big) \leqslant \alpha_2\cdot\big(z(\tau), z(\tau)\big) = \alpha_2\cdot\|z(\tau)\|^2.$$

From the theorems on difference inequalities (cf. [5]) we obtain

$$\psi(n) \leqslant \left(\frac{\beta_1}{\alpha_1}\right)^{n-\tau}\cdot\alpha_2\|z(\tau)\|^2. \tag{29}$$

Thus we have

$$\|z(n)\|^2 = \big(z(n), z(n)\big) \leqslant \frac{1}{\alpha_1}\big(V(n)\cdot z(n), z(n)\big) = \frac{1}{\alpha_1}\psi(n)$$
$$\leqslant \frac{\alpha_2}{\alpha_1}\cdot\left(\frac{\beta_1}{\alpha_1}\right)^{n-\tau}\cdot\|z(\tau)\|^2.$$

If we write

$$N = \sqrt{\frac{\alpha_2}{\alpha_1}}, \quad \gamma = \sqrt{\frac{\beta_1}{\alpha_1}},$$

then

$$\|z(n)\| \leqslant N\cdot\gamma^{n-\tau}\|z(\tau)\|, \tag{30}$$

where $z(n) = S(n)\cdot S^{-1}(\tau)\cdot z(\tau)$.

Because of

$$\frac{1}{\|z(\tau)\|}\|z(n)\| = \left\|\frac{z(n)}{\|z(\tau)\|}\right\| \leqslant N\cdot\gamma^{n-\tau}$$

we see that

$$\|S(n)\cdot S^{-1}(\tau)\| \leqslant N\cdot\gamma^{n-\tau}. \tag{31}$$

Then we obtain from (10) and (15)

$$z(n+1) = A_1(n)\cdot z(n) + F\big(z(n), n\big), \tag{32}$$

where

$$\|F(z(n), n)\| \leqslant q \cdot \|z(n)\| \quad \text{for } \|z(n)\| < \varrho \text{ and } q > 0.$$

(27) and (32) imply

$$z(n) = S(n)\cdot S^{-1}(\tau)\cdot z(\tau) + \sum_{j=\tau}^{n-1} S(n)\cdot S^{-1}(j+1)\cdot F(z(j), j).$$

Thus we have

$$\|z(n)\| \leqslant N\cdot \nu^{n-\tau}\|z(\tau)\| + \sum_{j=\tau}^{n-1} N\cdot \nu^{n-j-1}\cdot q\cdot \|z(j)\|$$

or

$$(33) \qquad \frac{\|z(n)\|}{\nu^n} \leqslant N\cdot \nu^{-\tau}\|z(\tau)\| + \frac{N\cdot q}{\gamma}\sum_{j=\tau}^{n-1} \nu^{-j}\|z(j)\|.$$

Because of $\alpha_1 < \alpha_2$ we have $N = \sqrt{\alpha_2/\alpha_1} > 1$. But we have $0 < \beta_1 < \alpha_1$ and $\beta_1^\tau < \alpha_1^\tau$; therefore, we obtain $N\cdot\gamma^{-\tau} > 1$ for $\nu = \sqrt{\beta_1/\alpha_1}$. Hence

$$34) \qquad \|z(\tau)\| \leqslant N\cdot \nu^{-\tau}\|z(\tau)\|.$$

Formulas (33) and (34) and our lemma imply

$$\frac{\|z(n)\|}{\nu^n} \leqslant \left(\frac{N\cdot q}{\nu} + 1\right)^{n-\tau}\cdot N\nu^{-\tau}\|z(\tau)\|$$

or

$$(35) \qquad \|z(n)\| \leqslant N\cdot (N\cdot q + \nu)^{n-\tau}\cdot \|z(\tau)\|.$$

If we assume

$$q < \frac{1-\nu}{N} = \frac{\sqrt{\alpha_1} - \sqrt{\beta_1}}{\sqrt{\alpha_2}},$$

then we obtain $q > 0$ and $N\cdot q + \nu < 1$.

Then $\|z\| < \varrho$ for each $z \in S_\varrho(\mathring{x})$ and if $\mu > 0$, then we have

$$\|z(n)\| \leqslant N\cdot (N\cdot q + \nu)^{n-\tau}\cdot \varrho < \mu.$$

Hence we obtain

$$(36) \qquad n > \tau + \frac{\ln\mu - \ln\varrho - \ln N}{\ln(N\cdot q + \nu)} \stackrel{\mathrm{df}}{=} T(\mu).$$

Then we shall have the inequality $\|z(n)\| < \mu$ for $n > T(\mu)$. Thus condition (5) is satisfied, and the solution $x(n)$ of (2) is asymptotically μ-observable (cf. Definition (8)).

If the conditions of Theorem 1 are satisfied, then the function $\zeta(n)$ realizing observability may be denitermed from the following theorem:

THEOREM 2. *A vector function* $\zeta(n)$ *which realizes an asymptotic* μ*-observability of the solution* $x(n)$ *of* (2) *is a solution of a difference equation of he form*

$$(37)\quad \zeta(n+1) = A(n)\cdot\zeta(n)+\varphi\big(\zeta(n)\big)+p(n)+B_V(n)\cdot C(n)\cdot\big(\zeta(n)-x(n)\big)$$

with the initial condition $\zeta(\tau) = \overset{\circ}{\zeta}$. B_V *is here an operator which depends on the operator* V (cf. Theorem 1).

It follows from (30) that μ, q and ϱ are given scalar quantities. We choose a_1, a_2 and β_1 in such a way that the following inequality holds:

$$(38)\qquad T(\mu) < \tau_1.$$

Then the constants a_1, a_2 and β_1 which realize inequality (38) will be denoted by a_1^1, a_2^1, β_1^1.

Then there holds a theorem which yields sufficient conditions for the μ-observability of solutions $x(n)$ of (2) at the moment τ_1.

THEOREM 3. *The* μ*-observability of the solution* $x(n)$ *of* (2) *at* $n = \tau_1$ *holds if we choose* $B(n) = B_{V_1}(n)$ *in such a way that there exists an operator* $V(n) = V_1(n)$ *which satisfies* (16) *and* (17) *for some constants* $a_1 = a_1^{\mathrm{I}}$, $a_2 = a_2^1$, $\beta_1 = \beta_1^{\mathrm{I}}$.

If the conditions of Theorem 3 are satisfied, then there holds a theorem analogous to Theorem 2.

THEOREM 4. *A vector function* $\zeta(n)$ *which realizes the* μ*-observability at the moment* $n = \tau_1$ *of the solution* $x(n)$ *of* (2) *is a solution of a difference equation of the form*

$$(37')\quad \zeta(n+1) = A(n)\cdot\zeta(n)+\varphi\big(\zeta(n)\big)+p(n)+B_{V_1}(n)\cdot C(n)\cdot\big(\zeta(n)-x(n)\big)$$

with the initial condition $\zeta(\tau) = \overset{\circ}{\zeta}$.

COROLLARY 1. *Assume that difference equation* (2) *is of the form*

$$(39)\quad x(n+1) = A(n)\cdot x(n)+\varphi\big(x(n)\big)+p(n)+p_1(n), \qquad x(\tau) = \overset{\circ}{x},$$

where the unknown perturbation $p_1(n)$ *is a vector function defined for* $n \geqslant \tau$ *and bounded* (i.e. $\|p_1(n)\| \leqslant k_1 = \text{const}$).

Then equation (32) *assumes the form*

$$(40)\qquad z(n+1) = A_1(n)\cdot z(n)+F\big(z(n), n\big)-p_1(n).$$

The following estimation is valid for the solution $z(n)$ of (40):

$$\|z(n)\| \leqslant N\cdot\nu^{n-\tau}\|z(\tau)\| + \sum_{j=\tau}^{n-1} N\cdot\nu^{n-j-1}[q\cdot\|z(j)\|+k_1]$$

$$= N\cdot\nu^{n-\tau}\|z(\tau)\| + \frac{N\cdot q}{\nu}\sum_{j=\tau}^{n-1}\nu^{n-j}\|z(j)\| + \frac{N\cdot k_1}{\nu}\sum_{j=\tau}^{n-1}\nu^{n-j}.$$

We have

$$\sum_{\nu=\tau}^{n-1} \nu^{-j} = \nu^{-\tau}+\nu^{-\tau-1}+\ldots+\nu^{-n+1}$$

$$= \nu^{-\tau}\cdot\frac{1-(1/\nu)^{n-\tau}}{1-(1/\nu)} = \frac{\nu}{\nu-1}\cdot\left(\nu^{-\tau}-\frac{1}{\nu^n}\right) < \frac{\nu}{\nu-1}\cdot\nu^{-\tau}$$

$$= \left(1+\frac{1}{\nu-1}\right)\cdot\nu^{-\tau} = \left(1-\frac{1}{1-\nu}\right)\cdot\nu^{-\tau} < \nu^{-\tau}.$$

It follows that

$$\frac{\|z(n)\|}{\nu^n} \leqslant N\cdot\nu^{-\tau}\|z(\tau)\| + \frac{N\cdot k_1}{\nu}\nu^{-\tau} + \frac{N\cdot q}{\nu}\sum_{j=\tau}^{n-1}\nu^{-j}\|z(j)\|.$$

Hence we obtain in view of the lemma

$$(41)\qquad \|z(n)\| \leqslant N\cdot(N\cdot q+\nu)^{n-\tau}\|z(\tau)\| + \frac{N\cdot k_1}{\nu}\cdot(N\cdot q+\nu)^{n-\tau}.$$

The last inequality implies that the problem of μ-observability for the solution $x(n)$ of (39) can be solved for fixed perturbations.

COROLLARY 2. *If the trace* $y(n)$ (cf. (3)) *of the solution* $x(n)$ *of* (2) *is of the form* $y(n) = C(n)\cdot x(n)+p_2(n)$, *where* $p_2(n)$ *is some bounded vector unction, i.e.* $\|p_2(n)\| \leqslant k_2 = \text{const}$ *for* $n \geqslant \tau$, *then equation* (32) *assumes he form*

$$42)\qquad z(n+1) = A_1(n)\cdot z(n)+F\big(z(n), n\big)+B_V(n)\cdot p_2(n).$$

If we write $p_1(n) = B_V(n)\cdot p_2(n)$, *then we obtain*

$$\|p_1(n)\| = \|B_V(n)\cdot p_2(n)\| \leqslant \|B_V(n)\|\cdot\|p_2(n)\| \leqslant b\cdot k_2,$$

because $B_V(n)$ *is a bounded operator, i.e.* $\|B_V(n)\| \leqslant b$ *for* $n \geqslant \tau$.

Thus we may apply Corollary 1.

COROLLARY 3. *We assume that* (3) *is of the form*

$$(43)\qquad y(n) = C(n)\cdot x(n)+p_3(n),$$

where $p_3(n)$ *is a certain vector function such that* $\|p_3(n)\| \leqslant k_3 = \text{const}$ *for* $n \geqslant \tau$. *If we write* $p_2(n) = C(n)\cdot p_3(n)$ *and make use of the fact that* $C(n)$ *is bounded operator, then we obtain*

$$\|p_2(n)\| = \|C(n)\cdot p_3(n)\| \leqslant \|C(n)\|\cdot\|p_3(n)\| \leqslant c\cdot k_3 \overset{\text{df}}{=} k_2.$$

Thus (43) assumes the form $y(n) = C(n)\cdot x(n)+p_2(n)$ and Corollary 2 may be applied.

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