ANNALES
POLONICI MATHEMATICI
LIII.3 (1991)

Extensions of two theorems on the neutrix convolution product of distributions

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Abstract. The neutrix convolution product $f \circledast g$ of two distributions f and g is defined as the neutrix limit of the sequence $\{f_n * g\}$, where $\{f_n\}$ is a certain sequence converging to f. Extending earlier results, the neutrix convolution product $x_-^\lambda \circledast x_+^{s-\lambda}$ is evaluated for $\lambda < -1, \lambda \neq -2, -3, \ldots$ and $s = 0, \pm 1, \pm 2, \ldots$

In the following we let $\mathscr{D}$ be the space of infinitely differentiable functions with compact support and let $\mathscr{D}'$ be the space of distributions defined on $\mathscr{D}$. The convolution product $f*g$ of two distributions f and g in $\mathscr{D}'$ is then usually defined as follows:

Definition 1. Let f and g be distributions in $\mathscr{D}'$ satisfying either of the following conditions:

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

Then the *convolution product* $f*g$ is defined by

$$\langle (f*g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathscr{D}$.

It follows that if the convolution product $f*g$ exists by this definition then

$$f*g = g*f, \tag{1}$$

$$(f*g)' = f*g' = f'*g. \tag{2}$$

This definition of the convolution product is rather restrictive and in order to extend the convolution product to a larger class of distributions the neutrix convolution product was introduced in [1]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathscr{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leqslant \tau(x) \leqslant 1$,

(iii) $\tau(x) = 1$ for $|x| \leqslant \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geqslant 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$

DEFINITION 2. Let f and g be distributions in $\mathscr{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$ Then the *neutrix convolution product* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense that

$$\underset{n\to\infty}{\text{N-lim}} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathscr{D}$, where N is the neutrix, see van der Corput [4], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n \qquad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g$ is in the sense of Definition 1, the distribution f_n having bounded support since the support of τ_n is contained in the interval $[-n-n^{-n}, n+n^{-n}]$.

The following theorem was proved in [1] and shows that Definition 2 is an extension of Definition 1.

THEOREM 1. *Let f and g be distributions in $\mathscr{D}'$ satisfying either condition* (a) *or condition* (b) *of Definition* 1. *Then the neutrix convolution product $f \circledast g$ exists and $f \circledast g = f * g$.*

The next theorem was also proved in [1].

THEOREM 2. *Let f and g be distributions in $\mathscr{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and $(f \circledast g)' = f \circledast g'$.*

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

The next two theorems were proved in [2].

THEOREM 3. *The neutrix convolution product $x_-^\lambda \circledast x_+^{s-\lambda}$ exists and*

$$x_-^\lambda \circledast x_+^{s-\lambda} = (-1)^{s+1} B(-s-1, s+1-\lambda) x^{s+1} + \frac{(-1)^{s+1}(\lambda)_{s+1}}{(s+1)!}[\pi \cot(\pi\lambda) x_+^{s+1} - x^{s+1} \ln|x|], \tag{3}$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ *and* $s = -1, 0, 1, 2, \ldots$, *where*

$$(\lambda)_s = \begin{cases} 1, & s = 0, \\ \prod_{i=0}^{s-1} (\lambda - i), & s \geq 1. \end{cases}$$

In this theorem, B denotes the Beta function but is defined as in [3] by

$$B(\lambda, \mu) = \text{N-}\lim_{n\to\infty} \int_{1/n}^{1-1/n} t^{\lambda-1}(1-t)^{\mu-1}\,dt.$$

This definition is in agreement with the usual definition of $B(\lambda, \mu)$ when $\lambda, \mu \neq 0, -1, -2, \ldots$ but defines $B(\lambda, \mu)$ when λ or μ take the values $0, -1, -2, \ldots$

THEOREM 4. *The neutrix convolution product* $x_-^\lambda \circledast x_+^{-s-\lambda}$ *exists and*

$$x_-^\lambda \circledast x_+^{-s-\lambda} = \frac{\pi \cot(\pi\lambda)}{(-1-\lambda)_{s-1}} \delta^{(s-2)}(x) - \frac{(-1)^s(s-2)!}{(-1-\lambda)_{s-1}} x^{-s+1}, \tag{4}$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ *and* $s = 2, 3, \ldots$

We now prove the following generalizations of Theorems 3 and 4.

THEOREM 5. *The neutrix convolution product* $x_-^\lambda \circledast x_+^{s-\lambda}$ *exists and satisfies equation* (3) *for* $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $s = -1, 0, 1, 2, \ldots$

Proof. We first of all assume that equation (3) holds for $-k < \lambda < -k+1$ and $s = -1, 0, 1, 2, \ldots$, where k is some positive integer. This is certainly true when $k = 1$ by Theorem 3. Put

$$(x_-^\lambda)_n = x_-^\lambda \tau_n(x).$$

The convolution product $(x_-^\lambda)_n * x_+^{s-\lambda}$ exists by Definition 1 and so equations (2) hold. Then

$$[(x_-^\lambda)_n * x_+^{s-\lambda}]' = -\lambda(x_-^{s-\lambda})_n * x_+^{s-\lambda} + [x_-^\lambda \tau_n'(x)] * x_+^{s-\lambda}. \tag{5}$$

If $-k < \lambda < -k+1$, we have by our assumption

$$\begin{aligned} \text{N-}\lim_{n\to\infty} \langle [(x_-^\lambda)_n * x_+^{s-\lambda}]', \varphi(x)\rangle &= -\text{N-}\lim_{n\to\infty} \langle (x_-^\lambda)_n * x_+^{s-\lambda}, \phi'(x)\rangle \\ &= -\langle x_-^\lambda \circledast x_+^{s-\lambda}, \phi'(x)\rangle = \langle (x_-^\lambda \circledast x_+^{s-\lambda})', \phi(x)\rangle \end{aligned} \tag{6}$$

for arbitrary ϕ in $\mathscr{D}$.

Further, if ϕ has its support contained in the interval $[a, b]$ and $n > -a$, it follows that

$$\langle [x_-^\lambda \tau_n'(x)] * x_+^{s-\lambda}, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{s-\lambda}\,dy\,dx, \tag{7}$$

since the support of $(-y)^\lambda \tau_n'(y)$ is contained in the interval $[-n-n^{-n}, -n]$. On the domain of integration $(-y)^\lambda$ and $(x-y)^{s-\lambda}$ are locally summable functions. Integrating by parts, it follows that

$$\int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^{s-\lambda}\,dy = n^\lambda (x+n)^{s-\lambda}$$
$$+\int_{-n-n^{-n}}^{-n} [\lambda(-y)^{\lambda-1}(x-y)^{s-\lambda}+(s-\lambda)(-y)^\lambda (x-y)^{s-\lambda-1}]\tau_n(y)\,dy.$$

Now with $s \geqslant 0$,

$$n^\lambda (x+n)^{s-\lambda} = n^s \sum_{i=0}^{s} \frac{(s-\lambda)_i}{i!\,n^i} x^i + O(1/n)$$

and so

(8)
$$\operatorname*{N-lim}_{n\to\infty} n^\lambda (x+n)^{s-\lambda} = \frac{(s-\lambda)_s}{s!} x^s.$$

Further

$$|[\lambda(-y)^{\lambda-1}(x-y)^{s-\lambda}+(s-\lambda)(-y)^\lambda(x-y)^{s-\lambda-1}]\tau_n(y)| = O(n^{s-1})$$

and so

$$\left|\int_{-n-n^{-n}}^{-n} [\lambda(-y)^{\lambda-1}(x-y)^{s-\lambda}+(s-\lambda)(-y)^\lambda(x-y)^{s-\lambda-1}]\tau_n(y)\,dy\right|$$
$$= O(n^{-n+s-1}) \to 0$$

as n tends to infinity.

It now follows from equation (7) that

(9)
$$\operatorname*{N-lim}_{n\to\infty} \langle [x_-^\lambda \tau_n'(x)] * x_+^{s-\lambda}, \phi(x)\rangle = \frac{(s-\lambda)_s}{s!}\langle x^s, \phi(x)\rangle$$

and it then follows from equations (5), (6) and (9) that

$$\operatorname*{N-lim}_{n\to\infty} \lambda\langle (x_-^{\lambda-1})_n * x_+^{s-\lambda}, \phi(x)\rangle = -\langle (x_-^\lambda \circledast x_+^{s-\lambda})', \phi(x)\rangle + \frac{(s-\lambda)_s}{s!}\langle x^s, \phi(x)\rangle.$$

This proves that the neutrix convolution product $x_-^{\lambda-1} \circledast x_+^{s-\lambda}$ exists and

(10)
$$\lambda x_-^{\lambda-1} \circledast x_+^{s-\lambda} = -(x_-^\lambda \circledast x_+^{s-\lambda})' + \frac{(s-\lambda)_s}{s!} x^s$$

for $-k < \lambda < -k+1$ and $s = 0, 1, 2, \ldots$

From equation (3) we have

(11)
$$-(x_-^\lambda \circledast x_+^{s-\lambda})' = (-1)^s B(-s-1, s+1-\lambda)(s+1)x^s$$
$$+\frac{(-1)^s(\lambda)_{s+1}}{s!}[\pi\cot(\pi\lambda)x_+^s - x^s \ln|x|]$$
$$-\frac{(-1)^s(\lambda)_{s+1}}{(s+1)!}x^s.$$

It was proved in [4] that

$$B(-s,\,-\lambda)=\frac{(-1)^s\Gamma(\lambda)}{s!\,\Gamma(\lambda-s)}\left[\psi(s)-\gamma-\frac{\Gamma'(\lambda-s)}{\Gamma(\lambda-s)}\right]=\frac{(s-\lambda)_s}{s!}\left[\psi(s)-\gamma-\frac{\Gamma'(\lambda-s)}{\Gamma(\lambda-s)}\right],$$

where γ denotes Euler's constant and

$$\psi(s)=\begin{cases}0, & s=0,\\ \sum_{i=1}^{s}1/i, & s\geqslant 1.\end{cases}$$

Thus

$$B(-s-1,\,s+1-\lambda)=\frac{(\lambda)_{s+1}}{(s+1)!}\left[\psi(s+1)-\gamma-\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)}\right]$$

and so

$$(12)\qquad (-1)^sB(-s-1,\,s+1-\lambda)(s+1)-\frac{(-1)^s(\lambda)_{s+1}}{(s+1)!}+\frac{(s-\lambda)_s}{s!}$$

$$=\frac{(-1)^s(\lambda)_{s+1}}{s!}\left[\psi(s+1)-\gamma-\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)}\right]+\frac{(-1)^s(\lambda)_{s+1}}{(s+1)!}+\frac{(-1)^s(\lambda-1)_s}{s!}$$

$$=\frac{(-1)^s(\lambda)_{s+1}}{s!}\left[\psi(s)-\gamma+\frac{1}{\lambda}-\frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)}\right]$$

$$=(-1)^s\lambda B(-s,\,s+1-\lambda),$$

since

$$(s-\lambda)_s=(-1)^s(\lambda-1)_s$$

and

$$1/\lambda-\Gamma'(-\lambda)/\Gamma(-\lambda)=-\Gamma'(1-\lambda)/\Gamma(1-\lambda).$$

Further

$$(13)\qquad \frac{(-1)^s(\lambda)_{s+1}}{s!}[\pi\cot(\pi\lambda)x_+^s-x^s\ln|x|]$$

$$=\frac{(-1)^s\lambda(\lambda-1)_s}{s!}\{\pi\cot[\pi(\lambda-1)]x_+^s-x^s\ln|x|\}$$

and it follows from equations (10)–(13) that

$$x_-^{\lambda-1}\circledast x_+^{(s-1)-(\lambda-1)}=(-1)^sB(-s,\,s+1-\lambda)x^s$$

$$+\frac{(-1)^s(\lambda-1)_s}{s!}\{\pi\cot[\pi(\lambda-1)]x_+^s-x^s\ln|x|\}.$$

Equation (3) now follows by induction for $\lambda<-1$, $\lambda\neq-2,-3,\ldots$ and $s=-1,0,1,2,\ldots$ This completes the proof of the theorem.

COROLLARY. *The neutrix convolution product* $x_+^{\lambda} \circledast x_-^{s-\lambda}$ *exists and*

$$x_+^{\lambda} \circledast x_-^{s-\lambda} = B(-s-1,\, s+1-\lambda)x^{s+1} + \frac{(-1)^{s+1}(\lambda)_{s+1}}{(s+1)!}[\pi\cot(\pi\lambda)x_-^{s+1} + (-1)^s x^{s+1}\ln|x|]$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $s = -1, 0, 1, 2, \ldots$

Proof. The result of the corollary follows immediately on replacing x by $-x$ in equation (3).

THEOREM 6. *The neutrix convolution product* $x_-^{\lambda} \circledast x_+^{-s-\lambda}$ *exists and satisfies equation* (4) *for* $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $s = 2, 3, \ldots$

Proof. We first of all assume that equation (4) holds for $-k < \lambda < -k+1$ and $s = 2, 3, \ldots$, where k is some positive integer. This is certainly true by Theorem 4 when $k = 1$. It follows as in the proof of Theorem 5 that

$$[(x_-^{\lambda})_n * x_+^{-s-\lambda}]' = -\lambda(x_-^{\lambda-1})_n * x_+^{s-\lambda} + [x_-^{\lambda}\tau_n'(x)] * x_+^{s-\lambda}, \tag{14}$$

and

$$\operatorname*{N-lim}_{n\to\infty}\langle[(x_-^{\lambda})_n * x_+^{-s-\lambda}]', \phi(x)\rangle = \langle(x_-^{\lambda} \circledast x_+^{-s-\lambda})', \phi(x)\rangle \tag{15}$$

for arbitrary ϕ in $\mathscr{D}$. Further, if ϕ has its support contained in the interval $[a, b]$ and $n > -a$,

$$\langle[x_-^{\lambda}\tau_n'(x)] * x_+^{-s-\lambda}, \phi\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n'(y)(x-y)^{-s-\lambda}\,dy\,dx. \tag{16}$$

Integrating by parts we have

$$\int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n'(y)(x-y)^{-s-\lambda}\,dy = n^{\lambda}(x+n)^{-s-\lambda} + \int_{-n-n^{-n}}^{-n} [\lambda(-y)^{\lambda-1}(x-y)^{-s-\lambda} - (s+\lambda)(-y)^{\lambda}(x-y)^{-s-\lambda-1}]\,\tau_n(y)\,dy.$$

Now with $s \geqslant 2$,

$$\lim_{n\to\infty} n^{\lambda}(x+n)^{-s-\lambda} = 0$$

and it follows as in the proof of Theorem 5 that

$$\int_{-n-n^{-n}}^{-n} [\lambda(-y)^{\lambda-1}(x-y)^{-s-\lambda} - (s+\lambda)(-y)^{\lambda}(x-y)^{-s-\lambda-1}]\,\tau_n(y)\,dy \to 0$$

as n tends to infinity.

It now follows from equation (16) that

(17) $$\lim_{n\to\infty} \langle [x_-^{\lambda} \tau_n'(x) * x_+^{-s-\lambda}, \phi(x)\rangle = 0,$$

and it then follows from equations (14), (15) and (17) that

$$\text{N-}\lim_{n\to\infty} \lambda \langle (x_-^{\lambda-1})_n * x_+^{-s-\lambda}, \phi(x)\rangle = \langle (x_-^{\lambda} \circledast x_+^{-s-\lambda})', \phi(x)\rangle.$$

This proves that the neutrix convolution product $x_-^{\lambda-1} \circledast x_+^{-s-\lambda}$ exists and

(18) $$\lambda x_-^{\lambda-1} \circledast x_+^{-s-\lambda} = -(x_-^{\lambda} \circledast x_+^{-s-\lambda})'$$

for $-k < \lambda < -k+1$ and $s = 3, 4, \ldots$

From equation (4) we have

$$(x_-^{\lambda} \circledast x_+^{-s-\lambda})' = \frac{\pi\cot(\pi\lambda)}{(-1-\lambda)_{s-1}}\delta^{(s-1)}(x) + \frac{(-1)^s(s-1)!}{(-1-\lambda)_{s-1}}x^{-s}$$

and it follows from equation (18) that

$$x_-^{\lambda-1} \circledast x_+^{(-s-1)-(\lambda-1)} = \frac{\pi\cot[\pi(\lambda-1)]}{(-\lambda)_s}\delta^{(s-1)}(x) - \frac{(-1)^{s+1}(s-1)!}{(-\lambda)_s}x^{-s}.$$

Equation (4) now follows by induction for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s = 3, 4, \ldots$

To prove that equation (4) holds when $s = 2$, we note that equations (5)–(7) hold in the proof of Theorem 3 when $s = -1$. However when $s = -1$, equation (8) must be replaced by

$$\lim_{n\to\infty} n^{\lambda}(x+n)^{-1-\lambda} = 0$$

and then equation (10) must be replaced by

$$\lambda x_-^{\lambda-1} \circledast x_+^{-1-\lambda} = -(x_-^{\lambda} \circledast x_+^{-1-\lambda})'$$

for $-k < \lambda < -k+1$. From equation (3)

$$(x_-^{\lambda} \circledast x_+^{-1-\lambda})' = \pi\cot(\pi\lambda)\delta(x) - x^{-1},$$

and it follows that

$$x_-^{\lambda-1} \circledast x_+^{-2-(\lambda-1)} = \frac{\pi\cot[\pi(\lambda-1)]}{(-\lambda)_1}\delta(x) - \frac{1}{(-\lambda)_1}x^{-1}.$$

Equation (4) now follows by induction for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s = 2$. This completes the proof of the theorem.

The corollary follows immediately on replacing x by $-x$ in equation (4).

COROLLARY. *The neutrix convolution product* $x_+^\lambda \circledast x_-^{-s-\lambda}$ *exists and*

$$x_+^\lambda \circledast x_-^{-s-\lambda} = \frac{(-1)^s \pi \cot(\pi\lambda)}{(-1-\lambda)_{s-1}} \delta^{(s-2)}(x) + \frac{(s-2)!}{(-1-\lambda)_{s-1}} x^{-s+1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $s = 2, 3, \ldots$

The distributions $|x|^\lambda$ and $\operatorname{sgn} x \cdot |x|^\lambda$ are defined by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad \operatorname{sgn} x \cdot |x|^\lambda = x_+^\lambda - x_-^\lambda.$$

We finally note that since the convolution products $x_+^\lambda * x_+^\mu$ and $x_-^\lambda * x_-^\mu$ exist by Definition 1 and since the neutrix convolution product is clearly distributive with respect to addition, it follows that further neutrix convolution products such as

$$x_-^\lambda \circledast |x|^{s-\lambda}, \qquad x_+^\lambda \circledast |x|^{s-\lambda}, \qquad x_-^\lambda \circledast (\operatorname{sgn} x \cdot |x|^{s-\lambda}),$$

exist for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = -1, 0, 1, 2, \ldots$ and

$$(\operatorname{sgn} x \cdot |x|^\lambda) \circledast x_+^{-s-\lambda}, \qquad |x|^\lambda \circledast |x|^{-s-\lambda}, \qquad |x|^\lambda \circledast x_-^{-s-\lambda}$$

exist for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 2, 3, \ldots$

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Reçu par la Rédaction le 25.03.1990