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On mappings isomorphic to r-adic transformations

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Abstract. A class of piecewise continuous transformations on the real line is shown to have continuous invariant measures. In particular it is proved that for any continuous, transformation the existence of a periodic point of period three implies the existence of a continuous, ergodic invariant measure.

1. Introduction. The present paper is stimulated by two independent results. The first one is due to Misiurewicz [6] who proved the existence of continuous invariant measures for local homeomorphisms of a circle. The second is due to Sharkovsky [11], Li and Yorke [4]. They discovered interesting properties of continuous transformations having a periodic point of period three. We shall show that the source of both results is the same. They both occur due to the existence of continuous measures for so-called pseudo-r-adic transformations.

Section 2 contains basic notations and definitions. In Section 3 we prove our main result Theorem 1 which asserts the existence of invariant measures for pseudo-r-adic transformations. The proof is based on ideas due to Rényi [9] and Parry [8]. Section 4 contains some results concerning the existence of ergodic invariant measures for transformations with periodic points of period three. Finally Section 5 is devoted to functional equations related with the problem of the existence of invariant measures.

2. Pseudo-r-adic transformations. Throughout the sequel J will denote a given interval (boundet or not) of the real line. By measure we mean a probabilistic measure defined on the σ-algebra of Borel subsets of J. A measure μ is called *continuous* if it vanishes on points (in that case the function $F(x) = \mu((a, x))$ $(a \in J)$ is continuous).

A Borel measurable transformation $\tau\colon J \to J$ is called *pseudo-r-adic* if there exists a sequence $J_1, \ldots, J_r$ of open, disjoint, bounded and non-empty intervals such that the following two conditions hold:

$$\text{(i)} \qquad \bigcap_{k=1}^{r} \tau(J_k) \supset \bigcup_{k=1}^{r} J_k.$$

(ii) For each k ($k = 1, \ldots, r$) τ is continuous on J_k and may be extended as continuous function on the closure $\bar{J}_k$ of J_k.

We say pseudo-dyadic instead of pseudo-2-adic. It is easy to see that any pseudo-r-adic transformation is pseudo-s-adic, if $s \leqslant r$. The simplest example of a pseudo-r-adic transformation is the usual r-adic transformation

$$\varrho(x) = \mathrm{rx}(\mathrm{mod}\,1), \qquad x \in [0, 1].$$

A more general example may be given by the formula

$$\tau(x) = \varphi(x)(\mathrm{mod}\,1), \qquad x \in [0, 1],$$

where $\varphi\colon [0, 1] \to [0, \infty)$ is an arbitrary continuous function such that $\varphi(0) = 0$ and $\varphi(1) \geqslant r$.

3. Existence of invariant measures. We shall prove the following

THEOREM 1. *For any pseudo-r-adic transformation* ($r \geqslant 2$) *there exists an ergodic continuous invariant measure.*

Proof. Denote by τ_k the continuous extension of τ from J_k onto $\bar{J}_k$. We shall define a family of non-empty, open intervals

$$J_{k_1 \ldots k_n} = (a_{k_1 \ldots k_n}, b_{k_1 \ldots k_n}), \tag{1}$$

where $k_i = 1, \ldots, r$ for $i = 1, \ldots, n$ and $n = 1, 2, \ldots$ When $n = 1$ family (1) reduces to a finite sequence $J_1, \ldots, J_r$ and we choose the intervals J_k as in the definition of pseudo-r-adicity. Now suppose that $J_{k_1 \ldots k_m}$ are given for all $m < n$. We write

$$\begin{aligned} b_{k_1 \ldots k_n} &= \min\{x \in Z_{k_1 \ldots k_n} : \tau_{k_1}(x) \neq \tau_{k_1}(c_{k_1 \ldots k_n})\}, \\ a_{k_1 \ldots k_n} &= \max\{x < b_{k_1 \ldots k_n} : \tau_{k_1}(x) = \tau_{k_1}(c_{k_1 \ldots k_n})\}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} Z_{k_1 \ldots k_n} &= \{x \in \bar{J}_{k_1 \ldots k_{n-1}} : \tau_{k_1}(x) \in \{a_{k_2 \ldots k_n}, b_{k_2 \ldots k_n}\}\}, \\ c_{k_1 \ldots k_n} &= \min\, Z_{k_1 \ldots k_n}. \end{aligned}$$

Here $\{a, b\}$ denotes the set with elements a and b. Using assumptions (i) and (ii) it is easy to verify (by induction) that the intervals defined by formulas (1) and (2) satisfy the following conditions

$$J_{k_1 \ldots k_n} \subset J_{k_1 \ldots k_{n-1}}, \tag{3}$$

$$J_{k_1 \ldots k_n} \cap J_{k'_1 \ldots k'_n} = \emptyset \quad \text{for } (k_1, \ldots, k_n) \neq (k'_1, \ldots, k'_n), \tag{4}$$

$$\tau(J_{k_1 \ldots k_n}) = J_{k_2 \ldots k_n}. \tag{5}$$

Now let $\varkappa = (k_1, k_2, \ldots)$ be an infinite sequence of integers such that $1 \leqslant k_i \leqslant r$. We write

$$J_\varkappa = \bigcap_{n=1}^{\infty} J_{k_1\ldots k_n}.$$

We shall show that except of a countable number of sequences $\varkappa = (k_1, k_2, \ldots)$ the set $J_\varkappa$ contains exactly one point. In fact, if $J_\varkappa$ is not a one point set, then either the interior of $J_\varkappa$ is non-empty or the set $J_\varkappa$ is empty. From (4) it follows that for different sequences $\varkappa, \varkappa'$ the corresponding sets are disjoint. Thus the set of sequences $\varkappa$ for which $\operatorname{int} J_\varkappa \neq \emptyset$ is at most countable. On the other hand if $J_\varkappa = \emptyset$, then there exists an integer n_0 such that either

$$\inf J_{k_1\ldots k_{n_0+m}} = a_{k_1\ldots k_{n_0}} \text{ for } m \geqslant 0 \tag{6}$$

or

$$\sup J_{k_1\ldots k_{n_0+m}} = b_{k_1\ldots k_{n_0}} \text{ for } m \geqslant 0. \tag{7}$$

For given n_0 the number of sequences satisfying (6) or (7) is less than r^{n_0}. Thus the family of sequences $\varkappa$ for which $J_\varkappa = \emptyset$ is also at most countable.

For any $x \in (0,1)$ consider the sequence $\varkappa(x) = \big(k_1(x), k_2(x), \ldots\big)$, where $k_n(x)$ is the largest integer less than $r\varrho^{n-1}(x)+1$ (ϱ denotes the r-adic transformation). Let $I' \subset (0,1)$ be the set of all irrational numbers such that $J_{\varkappa(x)}$ contains exactly one point. Since for irrational numbers $x \neq y$ implies $\varkappa(x) \neq \varkappa(y)$, the set $(0,1)\setminus I'$ is at most countable. Write

$$I = \bigcap_{n=0}^{\infty} \varrho^n(I').$$

The set I is invariant under ϱ ($\varrho(I) \subset I$) and $(0,1)\setminus I$ is also at most countable. For each $x \in I$ the unique point of $J_{\varkappa(x)}$ will be denoted by $\psi(x)$. The function $\psi\colon I \to J$ is invertible and Borel measurable. In fact $x \neq y$ $(x, y \in I)$ implies $\varkappa(x) \neq \varkappa(y)$ and according to (4) $J_{\varkappa(x)} \cap J_{\varkappa(x)} = \emptyset$. This proves that ψ is invertible. In order to prove the measurability observe that

$$\psi(x) = \lim_{n\to\infty} \psi_n(x), \quad \text{where } \psi_n(x) = \inf J_{k_1(x)\ldots k_n(x)} \text{ for } x \in I.$$

Any function ψ_n is piecewise constant. Namely

$$\psi_n(x) = \text{const} \quad \text{for } x \in I \cap \left(\frac{m-1}{r^n}, \frac{m}{r^n}\right), \; m = 1, \ldots, r^n.$$

Thus ψ is Borel measurable as a limit of Borel measurable functions.

From the definition of the functions $k_n(x)$ it follows that $k_n\big(\varrho(x)\big) = k_{n+1}(x)$. From this and equality (5) we obtain

$$\tau\big(J_{k_1(x)\ldots k_n(x)}\big) = J_{k_1(\varrho(x))\ldots k_{n-1}(\varrho(x))}.$$

Since $J_{\varkappa(x)} \subset J_{k_1(x)\ldots k_n(x)}$, this implies

$$\tau(J_{\varkappa(x)}) \subset J_{k_1(\varrho(x))\ldots k_{n-1}(\varrho(x))}$$

and consequently

$$(8) \qquad \tau(J_{\varkappa(x)}) \subset \bigcap_{n=1}^{\infty} J_{k_1(\varrho(x))\ldots k_n(\varrho(x))} = J_{\varkappa(\varrho(x))}.$$

For $x \in I$ we have $\varrho(x) \in I$ and according to the definition of ψ

$$J_{\varkappa(x)} = \{\psi(x)\}, \qquad J_{\varkappa(\varrho(x))} = \{\psi(\varrho(x))\}.$$

Thus from (8) it follows that

$$(9) \qquad \tau(\psi(x)) = \psi(\varrho(x)) \quad \text{for } x \in I.$$

Now we define the desired measure μ by the formula

$$(10) \qquad \mu(E) = m(\psi^{-1}(E)) \quad (E \text{ Borel subset of } J),$$

where m denotes the usual Borel measure on $(0, 1)$. Since ϱ preserves the measure m, we have

$$\mu(E) = m(\psi^{-1}(E)) = m(\varrho^{-1}(\psi^{-1}(E))) = m(\psi^{-1}(\tau^{-1}(E))) = \mu(\tau^{-1}(E)).$$

This proves that μ is invariant under τ. The continuity of μ follows from the fact that ψ is invertible. In order to prove that μ is ergodic assume that $\tau^{-1}(A) = A$. Then $\psi^{-1}(\tau^{-1}(A)) = \psi^{-1}(A)$ and according to (9) $\varrho^{-1}(\psi^{-1}(A)) = \psi^{-1}(A)$. Since ϱ is ergodic with respect to m, this implies that either $m(\psi^{-1}(A)) = 0$ or $m(\psi^{-1}(A)) = 1$. Consequently by the definition of μ either $\mu(A) = 0$ or $\mu(A) = 1$. This completes the proof of Theorem 1.

Remark 1. From (9) and (10) it follows that the system (J, τ, μ) is isomorphic to $([0, 1], \varrho, m)$. Thus (J, τ, μ) is not only ergodic but also mixing and exact in the sense of Rochlin [10]. The measure μ is supported on the set

$$D = \bigcap_{n=1}^{\infty} \bigcup J_{k_1\ldots k_n},$$

where the union is taken over all sequences $(k_1, \ldots, k_n)$ of the length n.

Remark 2. The measure μ in Theorem 1 is not unique. We may repeat the proof replacing ϱ by a more general transformation

$$\varrho_p(x) = \frac{1}{p_k}(x - q_{k-1}) \quad \text{for } q_{k-1} \leqslant x < q_k,$$

where

$$q_k = \sum_{i=1}^{k} p_i, \qquad q_0 = 0,$$

and $p = (p_1, \ldots, p_r)$ is an arbitrary probability vector

$$0 < p_i < 1, \quad \sum_{i=1}^{r} p_i = 1.$$

In that case the measure μ depends upon p. In particular $\mu(J_{k_1 \ldots k_n}) = p_{k_1} \cdots p_{k_n}$.

4. Transformations with periodic points. Theorem 1 enables us to prove the existence of invariant measures for continuous mappings with periodic points. We say that $x_0 \in J$ is a periodic point of period n for a transformation $\tau\colon J \to J$ if

$$\tau^n(x_0) = x_0 \quad \text{and} \quad \tau^k(x_0) \neq x_0 \quad (k = 1, \ldots, n-1).$$

PROPOSITION 1. *Let $\tau\colon J \to J$ be a continuous mapping. Then the existence of a periodic point of period 3 for τ implies that τ^2 is pseudo-dyadic.*

Proof. Denote the periodic point by x_0 and write $x_i = \tau^i(x_0)$. From our assumptions it follows that $x_1 \neq x_0$, $x_0 \neq x_2$ and $x_3 = x_0$. It is also easy to see that $x_1 \neq x_2$. In fact the equality $x_1 = x_2$ implies $x_2 = \tau(x_1) = \tau(x_2) = x_0$ which is impossible. Thus we have three different points x_0, x_1, x_2 such that

$$\tau(x_0) = x_1, \quad \tau(x_1) = x_2, \quad \tau(x_2) = x_0.$$

We may assume that $x_0 < x_1 < x_2$. In the remaining cases the proof is similar. From the continuity of τ it follows the existence of a point $\bar{x} \in (x_1, x_2)$ such that $\tau(\bar{x}) = x_1$. Thus we have

$$\tau^2\big((\bar{x}, x_2)\big) \supset \big(\tau^2(x_2), \tau^2(\bar{x})\big) = (x_1, x_2)$$

and

$$\tau^2\big((x_1, \bar{x})\big) \supset \big(\tau^2(x_1), \tau^2(\bar{x})\big) = (x_0, x_2).$$

Writing $J_1 = (x_1, \bar{x})$ and $J_2 = (\bar{x}, x_2)$ we obtain

$$J_1 \cup J_2 \subset (x_1, x_2) \subset \tau^2(J_1) \cap \tau^2(J_2)$$

which completes the proof.

PROPOSITION 2. *Let $\tau\colon J \to J$ be a Borel measurable transformation and let μ be a continuous measure invariant under τ^m for some integer $m \geqslant 1$. Then the measure*

$$\bar{\mu}(E) = \frac{1}{m} \sum_{i=0}^{m-1} \mu\big(\tau^{-i}(E)\big) \tag{11}$$

is continuous and invariant under τ. If μ is ergodic with respect to τ^m, then the same property has $\bar{\mu}$ with respect to τ.

Proof. The condition $\mu = \mu\tau^{-m}$ implies $\bar{\mu} = \bar{\mu}\tau^{-1}$. Thus $\bar{\mu}$ is invariant under τ. Now let $x \in J$ be an arbitrary point. We have

$$\tau^{-k}(x) \subset \tau^{-m}\big(\tau^{m-k}(x)\big), \quad k = 0, \dots, m-1,$$

and consequently

$$\mu\big(\tau^{-k}(x)\big) \leqslant \mu\Big(\tau^{-m}\big(\tau^{m-k}(x)\big)\Big) = \mu\big(\tau^{m-k}(x)\big), \quad k = 0, \dots, m-1.$$

Since $\tau^{m-k}(x)$ is a single point and μ is continuous, this implies $\mu\big(\tau^{-k}(x)\big) = 0$ for $k = 0, \dots, m-1$. Thus $\bar{\mu}(x) = 0$ for each $x \in J$. Now suppose that μ is ergodic and $\bar{\mu}$ is not. Then there exists a set A such that

$$A = \tau^{-1}(A), \quad 0 < \bar{\mu}(A) < 1.$$

The equality $A = \tau^{-1}(A)$ implies $A = \tau^{-k}(A)$ for all positive integers k. Thus we have

$$A = \tau^{-m}(A), \quad \mu(A) = \bar{\mu}(A) \in (0, 1)$$

which contradicts to the ergodicity of μ (with respect to τ^m).

Using Propositions 1 and 2 we may prove immediately the following

THEOREM 2. *Let $\tau\colon J \to J$ be a continuous mapping. Then the existence of a periodic point of period $3n$, for some integer $n \geqslant 1$, implies the existence of an ergodic continuous invariant measure.*

Proof. From Proposition 1 it follows that τ^{2n} is pseudo-dyadic. Thus according to Theorem 1 there exists a continuous ergodic measure μ invariant under τ^{2n}. Setting $m = 2n$ in formula (11) we obtain a continuous measure $\bar{\mu}$ invariant and ergodic with respect to τ.

Theorem 2 was proved in [3] by a different method based on the notion of strictly turbulent trajectories.

Now we are going to apply Theorem 2 in some special cases. Consider the mapping $\tau_a(x) = ax(1-x)$ of the unit interval $[0,1]$ into itself. $(0 \leqslant a \leqslant 4)$. This transformation was studed by Ulam [14], Lorenz [5], Smale and Williams [13]. It is known that for $a = 4$ there exists an absolutely continuous measure invariant under τ_a. The problem of the existence of a non-trivial invariant measure for $a < 4$ was open (a trivial measure supported on a fixed point or a periodic orbit always exists). An elementary computation shows that for each $a \geqslant 3.83$ the equation $\tau_a^3(x) = x$ admits a solution which is not a fixed point of τ_a or τ_a^2. Thus for each $a \in [3.83, 4]$ there exists a continuous measure invariant and ergodic with respect to τ_a.

Now consider the mapping $\sigma_\lambda(x) = x^2 + \lambda$ of the real line into itself. The transformation σ_λ was studed by Myrberg [7], Gumowski and Mira [2]. An easy computation shows that for $\lambda \leqslant -1.75$ there exists a periodic point of period 3. This implies the existence of a continuous measure ergodic and invariant under σ_λ for $\lambda \leqslant -1.75$. The results concerning τ_a and σ_λ are closely related. In fact setting $\lambda = \frac{1}{2}a - \frac{1}{4}a^2$ and $\psi(x) = \frac{1}{2} - x/a$ we have $\tau_\lambda \circ \psi = \psi \circ \sigma_\lambda$. Thus, a measure μ is ergodic and invariant with respect to σ_λ if and only if $\bar{\mu} = \mu\psi^{-1}$ is invariant with respect to τ_a. Moreover, since ψ is linear the measure μ is continuous if and only if $\bar{\mu}$ is continuous.

Our last example is the transformation

$$\tau_q(x) = \pi q \sin x \pmod{\pi}$$

of the interval $[0, \pi]$ into itself. Bunimovič [1] has shown that for each integer $q \neq 0$ there exists an absolutely continuous measure invariant with respect to τ_q. Observe that for each real q such that $|q| \geqslant 1$ the transformation τ_q is pseudo-dyadic. In fact setting

$$J_1 = \left(0, \arcsin \frac{1}{|q|}\right), \quad J_2 = \left(\pi - \arcsin \frac{1}{|q|}, \pi\right)$$

we have

$$\tau_q(J_1) \cap \tau_q(J_2) = (0, \pi) \supset J_1 \cup J_2.$$

For each q satisfying the inequality $1 \geqslant |q| \geqslant 0.94$ the mapping τ_q is continuous and admits a periodic point of period 3. Thus, according to Theorem 1 and 2, for $|q| \geqslant 0.94$ there exists a continuous measure invariant and ergodic with respect to τ_q.

5. Functional equations. The problem of the existence of invariant measures for piecewise monotonic transformations may be easily formulated in terms of functional equations. Let $0 = a_0 < \ldots < a_n = 1$ be a partition of the unit interval and let $\varphi_i\colon [a_{i-1}, a_i] \to [0, 1]$ be a given sequence of continuous invertible functions. Assume that any mapping φ_i is onto. Define the transformation $\tau_\varphi\colon [0, 1] \to [0, 1]$ by the formulas

$$\tau_\varphi(x) = \varphi_i(x) \quad \text{for } a_{i-1} \leqslant x < a_i, \ \tau_\varphi(1) = \varphi_n(1).$$

PROPOSITION 3. *A continuous measure μ is invariant under τ_φ if and only if the function $F(x) = \mu\big((0, x)\big)$ satisfies the functional equation*

$$F(x) = \sum_{i=1}^{n} \left|F\big(\varphi_i^{-1}(x)\big) - F\big(\varphi_i^{-1}(0)\big)\right|. \tag{12}$$

The proof of Proposition 3 is obvious, since the right-hand side of the functional equation (12) is equal to $\mu\big(\tau_\varphi^{-1}(0, x)\big)$.

We shall consider a more general equation

$$F(x) = \sum_{i=1}^{n} \left|F\big(\psi_i(x)\big) - F\big(\psi_i(0)\big)\right|, \tag{13}$$

where ψ_i are arbitrary mappings from the unit interval $[0,1]$ into itself. We have the following

THEOREM 3. *Assume that* $\inf\psi_{i+1} \geqslant \sup\psi_i$ $(i = 1, \ldots, n-1)$ *and that at least two functions* ψ_i *are continuous and invertible. Then there exists a continuous increasing solution* F *of equation* (13) *such that* $F(0) = 0$, $F(1) = 1$.

Proof. Let $\psi_{k_1}, \ldots, \psi_{k_r}$ be a subsequence of continuous one-to-one mappings. Write $J_i = \psi_i((0,1))$ and

$$\tau(x) = \begin{cases} \psi_{k_i}^{-1}(x) & \text{for } x \in J_i, \\ x & \text{for } x \notin \bigcup J_i. \end{cases}$$

The transformation τ is pseudo-r-adic and according to Theorem 1 there exists a continuous measure μ invariant under τ. Since μ is supported on $\bigcup J_i$ the function $F(x) = \mu((0,x))$ is constant on each subinterval of $[0,1]\setminus\bigcup J_i$. Thus

$$\begin{aligned} F(x) &= \mu((0,x)) = \mu(\tau^{-1}(0,x)) = \mu(\tau^{-1}(0,x)\cap\bigcup J_i) \\ &= \sum_{i=1}^{r} |F(\psi_{k_i}(x)) - F(\psi_{k_i}(0))| = \sum_{i=1}^{n} |F(\psi_i(x)) - F(\psi_i(0))| \end{aligned}$$

which finishes the proof.

A special case of equation (12) corresponding to the transformation

$$\tau(x) = \begin{cases} \dfrac{x}{1-x}, & 0 \leqslant x \leqslant \tfrac{1}{2}, \\ \dfrac{1-x}{x}, & \tfrac{1}{2} \leqslant x \leqslant 1; \end{cases} \tag{14}$$

was studed by Small [12]. Since we are interested in solutions satisfying condition $F(0) = 0$, the equation may by written in the form

$$F(x) = F\left(\frac{x}{1+x}\right) + F(1) - F\left(\frac{1}{1+x}\right), \quad 0 \leqslant x \leqslant 1. \tag{15}$$

From Theorem 3 it follows the existence of a continuous increasing solution F of (15) satisfying conditions $F(0) = 0$ and $F(1) = 1$. We claim that there is no solution of (15) which are absolutely continuous on $[0,1]$ except the trivial one $F = \text{const}$. In fact suppose that F is absolutely continuous solution of (15). Then $f = dF/dx$ satisfies the equation

$$f(x) = \frac{1}{(1+x)^2}\left[f\left(\frac{x}{1+x}\right) + f\left(\frac{1}{1+x}\right)\right] \quad \text{a.e. in } [0,1]. \tag{16}$$

Denote the right-hand side of (16) by Pf. An elementary computation shows that the sequence of functions $f_n(x) = P^n 1$ converges to zero uniformly on each subinterval $[\varepsilon, 1]$ $(\varepsilon > 0)$ of $[0, 1]$. Given $\varepsilon > 0$ choose a constant $C > 0$ such that

$$\int_0^1 [(f^+(x) - C)^+ + (f^-(x) - C)^+]\,dx \leqslant \varepsilon,$$

where $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$. We have

$$\int_\varepsilon^1 |P^n f|\,dx = \int_\varepsilon^1 P^n f^+\,dx + \int_\varepsilon^1 P^n f^-\,dx \leqslant 2C\int_\varepsilon^1 P^n 1\,dx + \int_\varepsilon^1 P^n (f^+ - C)^+\,dx +$$

$$+ \int_0^1 P^n (f^- - C)^+\,dx = 2C\int_\varepsilon^1 P^n 1 + \int_0^1 (f^+ - C)^+\,dx + \int_0^1 (f^- - C)^+\,dx$$

and consequently

$$\int_\varepsilon^1 |P^n f|\,dx \leqslant 2C\int_\varepsilon^1 P^n 1 + \varepsilon.$$

Since $P^n 1$ converges on $[\varepsilon, 1]$ uniformly to zero, this implies that the sequence $P^n f$ converges in measure to zero. Thus the unique solution of equation (16) is $f \equiv 0$. This finishes the proof of the claim.

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