ANNALES
POLONICI MATHEMATICI
LIII.1 (1991)

# On the equation $x''(t) = F(t, x(t))$ in the Sobolev space $H^1(\boldsymbol{R})$

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**Abstract.** The existence of solutions of the nonlinear equation $x''(t) = F(t, x(t))$ in the Sobolev space $H^1(\boldsymbol{R})$ is established.

**1. Introduction.** We study the existence of solutions of the nonlinear equation $x''(t) = F(t, x(t))$ in the Sobolev space $H^1(\boldsymbol{R})$. We make assumptions concerning $F$ under which the function $F(\cdot, x(\cdot))$ is locally integrable for any $x \in H^1(\boldsymbol{R})$. In this way, we may understand the above equation in the sense of distributions.

Other assumptions concerning $F$ give an a priori bound for solutions. Assumptions of this kind may be found in papers [1], [2] concerning equations on a bounded interval, and in paper [5] treating equations on the half-line.

**2. Notation.** By $H^s(\boldsymbol{R})$, for integer $s \geqslant 0$, we denote the Sobolev space

$$\{x \in L^2(\boldsymbol{R}):\ x^{(i)} \in L^2(\boldsymbol{R}),\ 0 \leqslant i \leqslant s\}$$

normed in the standard way:

$$\|x\|_s^2 = \sum_{i=0}^{s} \|x^{(i)}\|^2,$$

where $\|\cdot\|$ stands for the norm in $L^2(\boldsymbol{R})$.

We denote by $H^s_{\mathrm{loc}}$ ($H^0_{\mathrm{loc}}(\boldsymbol{R}) = L^2_{\mathrm{loc}}(\boldsymbol{R})$) the local Sobolev space (see for instance [3]) and treat it as a Fréchet space with the topology defined by the system of semi-norms

$$p_n^2(x) = \sum_{i=0}^{s} \int_{-n}^{n} |x^{(i)}(t)|^2\,dt \qquad \text{for } n = 1, 2, \ldots$$

We denote by $C_0^\infty(\boldsymbol{R})$ the space of $C^\infty$-functions on the line with compact support and by $\mathscr{D}'(\boldsymbol{R})$ the space of distributions on the line.

**3. Existence of solutions of the equation $x''(t) = F(t, x(t))$.**

THEOREM 1. *Let $F$: $\boldsymbol{R}^2 \to \boldsymbol{R}$ have the form*

$$F(t, y) = F_1(t, y) + F_2(t), \tag{1}$$

*where $F_2 \in L^2_{\mathrm{loc}}(\boldsymbol{R})$ and $F_1$ is continuous on the set $\bigcup_{i \in \boldsymbol{Z}} ]t_i, t_{i+1}[ \times \boldsymbol{R}$ and has continuous extensions to every product $[t_i, t_{i+1}] \times \boldsymbol{R}$ ($i \in \boldsymbol{Z}$). Here, $\{t_i: i \in \boldsymbol{Z}\}$ is a division of the line such that $t_i < t_{i+1}$, $t_i \to +\infty$ as $i \to +\infty$ and $t_i \to -\infty$ as $i \to -\infty$.*

*Suppose that there exist positive constants $a$, $C$ and a nonnegative function $f \in L^2(\boldsymbol{R})$ such that*

$$y(F(t, y) - a^2 y) \geq 0 \quad \text{for } |y| \geq f(t) \tag{2}$$

*almost everywhere with respect to $t$ (a.e. $t$), and*

$$|F(t, y)| \leqslant |F(t, 0)| + C|y| \quad \text{for } |y| \leqslant f(t) \text{ a.e. } t. \tag{3}$$

*Suppose finally that*

$$F(\cdot, 0) \in L^2(\boldsymbol{R}). \tag{4}$$

*Then the equation*

$$x''(t) = F(t, x(t)) \tag{5}$$

*has a solution $x$ in $H^1(\boldsymbol{R})$ for which $\|x\|_1 \leqslant M$, where*

$$M = (\min(1, a))^{-1} (\|F(\cdot, 0)\| \, \|f\| + (C + a^2) \|f\|^2)^{1/2}. \tag{6}$$

The proof is based on several lemmas.

LEMMA 1. *If $x \in H^1(\boldsymbol{R})$ then $x$ is a continuous function tending to 0 at $\pm\infty$ and*

$$\sup_{t \in \boldsymbol{R}} |x(t)| \leqslant 2^{-1/2} \|x\|_1. \tag{7}$$

Proof. See [3], Corollary 7.9.4. We prove only (7):

$$x^2(t) = \int_{-\infty}^{t} x(t)x'(t)dt - \int_{t}^{+\infty} x(t)x'(t)dt \leqslant \int_{-\infty}^{+\infty} |x(t)x'(t)|dt$$
$$\leqslant \|x\| \, \|x'\| \leqslant 2^{-1}(\|x\|^2 + \|x'\|^2) = 2^{-1} \|x\|_1^2,$$

and (7) follows.

Write equation (5) in the form

$$x''(t) - a^2 x(t) = G(t, x(t)), \tag{8}$$

where $G(t, y) = F(t, y) - a^2 y$. Observe that a.e. $t$

$$|G(t, y)| \leqslant |F(t, 0)| + (C + a^2)|y| \quad \text{for } |y| \leqslant f(t). \tag{9}$$

Let

$$G_n(t, y) = \begin{cases} G(t, y) & \text{for } |t| \leqslant n, \\ 0 & \text{for } |t| > n, \end{cases} \qquad n = 1, 2, \ldots \tag{10}$$

LEMMA 2. *$G_n$ has the following properties*:

(i) *If $x_k \to x$ in $H^1(\mathbf{R})$, then*

$$G_n(t, x_k(t)) \to G_n(t, x(t)) \quad \text{as } k \to \infty$$

*uniformly outside a set of measure zero.*

(ii) *If $\|x\|_1 \leqslant N$, then*

$$|G_n(t, x(t))| \leqslant K + |F_2(t)|$$

*a.e. t for some constant $K = K(N, n)$.*

(iii) *$G_n(\cdot, x(\cdot)) \in L^2(\mathbf{R})$ for $x \in H^1(\mathbf{R})$.*

Proof. (i) Let $x_k \to x$ in $H^1(\mathbf{R})$ and $\|x_k\|_1$, $\|x\|_1 \leqslant N$. Then (7) implies $|x_k(t)|$, $|x(t)| \leqslant 2^{-1/2}N$ for $t \in \mathbf{R}$. From (1), $G - F_2$ is uniformly continuous on any set of the form $]t_i, t_{i+1}[ \times [-2^{-1/2}N, 2^{-1/2}N]$, because it has a continuous extension to the compact set $[t_i, t_{i+1}] \times [-2^{-1/2}N, 2^{-1/2}N]$. Then $G_n(t, x_k(t)) \to G_n(t, x(t))$ as $k \to \infty$, uniformly for $t \in ]t_i, t_{i+1}[$, since $x_k(t) \to x(t)$ uniformly due to (7). From the finiteness of $\{i \in \mathbf{Z}\colon ]t_i, t_{i+1}[ \cap [-n, n] \neq \emptyset\}$ we get the assertion.

We prove (ii) likewise using the boundedness of a continuous function on a compact set.

(iii) $G_n(\cdot, x(\cdot))$ is measurable and vanishes outside a compact set, thus it belongs to $L^2(\mathbf{R})$ by (ii).

Now, consider the equation

$$x''(t) - a^2 x(t) = \lambda G_n(t, x(t)) \tag{11}$$

with the parameter $\lambda \in [0, 1]$, and compute an a priori estimate of the norm of its solutions:

LEMMA 3. *If $x = x_{\lambda,n} \in H^1(\mathbf{R})$ is a solution of* (11), *then $\|x\|_1 \leqslant M$, where $M$ is defined by* (6).

Proof. Observe that Lemma 2(iii) implies that $x \in H^2(\mathbf{R})$. Multiply (11) by $x(t)$ and integrate over $\mathbf{R}$:

$$\int_{\mathbf{R}} x(t)x''(t)dt - a^2 \int_{\mathbf{R}} x^2(t)dt = \lambda \int_{\mathbf{R}} x(t)G_n(t, x(t))dt. \tag{12}$$

We integrate by parts the first integral in (12) making use of $x(\pm\infty) = x'(\pm\infty) = 0$ (Lemma 1), to obtain

$$\|x'\|^2 + a^2\|x\|^2 = -\lambda \int_{\mathbf{R}} x(t)G_n(t, x(t))dt.$$

Let $S = \{t \in \mathbf{R}\colon |x(t)| \leqslant f(t)\}$. Inequalities (2), (3) and (9) imply

$$\begin{aligned} \min(1, a^2)\|x\|_1^2 &\leqslant \|x'\|^2 + a^2\|x\|^2 \\ &= -\lambda \int_S x(t)G_n(t, x(t))dt - \lambda \int_{\mathbf{R}\setminus S} x(t)G_n(t, x(t))dt \\ &\leqslant \int_S |x(t)G_n(t, x(t))|dt \leqslant \int_{\mathbf{R}} f(t)(|F(t, 0)| + (C + a^2)f(t))dt \\ &\leqslant \|f\|(\|F(\cdot, 0)\| + (C + a^2)\|f\|). \end{aligned}$$

We have used the Schwarz inequality in the last step.

Simple calculations finish the proof.

Inverting the operator $x \mapsto x'' - a^2 x$, we see that in $H^1(\mathbf{R})$ equation (11) is equivalent to

$$x = \lambda A_n x, \tag{13}$$

where

$$(A_n x)(t) = -(2a)^{-1} \int_{-n}^{n} e^{-a|t-s|} G(s, x(s)) ds. \tag{14}$$

We have

$$(A_n x)'(t) = 2^{-1} \int_{-n}^{n} \operatorname{sgn}(t-s) e^{-a|t-s|} G(s, x(s)) ds, \tag{15}$$

$$(A_n x)''(t) = G_n(t, x(t)) - 2^{-1} a \int_{-n}^{n} e^{-a|t-s|} G(s, x(s)) ds. \tag{16}$$

An important step in the proof of Theorem 1 is:

LEMMA 4. *The embedding $H^2_{\mathrm{loc}}(\mathbf{R}) \to H^1_{\mathrm{loc}}(\mathbf{R})$ is continuous and transforms bounded sets into precompact ones.*

The proof is in [3], Theorem 10.1.27.

LEMMA 5. *The operator $A_n$: $H^1(\mathbf{R}) \to H^1(\mathbf{R})$ defined by* (14) *is continuous and transforms bounded sets into precompact ones.*

Proof. The continuity of $A_n$ can be obtained from Lemma 2(i), (14) and (15).

Take a bounded sequence $(x_k)$, $k = 1, 2, \ldots$, in $H^1(\mathbf{R})$. Lemma 2(ii) and (14)–(16) imply the boundedness of the sequence $(A_n x_k)$, $k = 1, 2, \ldots$, in $H^2(\mathbf{R})$, hence also in $H^2_{\mathrm{loc}}(\mathbf{R})$. Using Lemma 5, we take a subsequence $(A_n x_{k_l})$ which is convergent to some $y$ in $H^1_{\mathrm{loc}}(\mathbf{R})$.

Let $\psi \in C_0^\infty(\mathbf{R})$, $\psi(t) = 1$ for $t \in [-n, n]$. We have

$$\|\psi A_n x_{k_l} - \psi y\|_1 \to 0 \quad \text{as } l \to \infty. \tag{17}$$

Observe that

$$(A_n x_{k_l})(t) = e^{a(n+t)} (A_n x_{k_l})(-n) \quad \text{for } t \leqslant -n, \tag{18}$$

$$(A_n x_{k_l})(t) = e^{a(n-t)} (A_n x_{k_l})(n) \quad \text{for } t \geqslant n. \tag{19}$$

Notice that convergence in $H^1_{\mathrm{loc}}(\mathbf{R})$ implies pointwise convergence (Lemma 1), hence

$$y(t) = e^{a(n+t)} y(-n) \quad \text{for } t \leqslant -n, \tag{20}$$

$$y(t) = e^{a(n-t)} y(n) \quad \text{for } t \geqslant n. \tag{21}$$

Now, it is easy to see that (17)–(21) imply that $A_n x_{k_l} \to y$ in $H^1(\mathbf{R})$. Lemma 5 is proved.

LEMMA 6. *The equation*

$$x''(t) - a^2 x(t) = G_n\big(t, x(t)\big) \tag{22}$$

*has a solution in* $H^1(\mathbf{R})$.

Proof. Equation (22), considered in $H^1(\mathbf{R})$, is equivalent to (13) for $\lambda = 1$. Write (13) in the form

$$(I - \lambda A_n)x = 0,$$

where $I$ stands for the identity mapping. We treat $I - \lambda A_n$ as a mapping from the ball $B(0, M+\varepsilon) \subset H^1(\mathbf{R})$ into $H^1(\mathbf{R})$ ($M$ is defined by (6)) and use the Leray–Schauder degree theory (see, for instance, [4]), since $A_n$ is compact due to Lemma 5. From Lemma 3, we know that $(I - \lambda A_n)x \neq 0$ for $\|x\|_1 = M + \varepsilon$, so the Leray–Schauder degree

$$\deg\big(I - A_n,\ B(0,\ M+\varepsilon),\ 0\big) = \deg\big(I,\ B(0,\ M+\varepsilon),\ 0\big) = 1 \neq 0.$$

Therefore, equation (22) has a solution in $H^1(\mathbf{R})$.

Consider the sequence $(x_n)$, $n = 1, 2, \ldots$, of solutions of equation (22). Lemma 3 implies that $(x_n)$ is bounded in $H^1(\mathbf{R})$, and, by Lemma 2(ii), (10) and (22), $(x_n)$ is bounded in $H^2_{\text{loc}}(\mathbf{R})$.

Using Lemma 4, we choose a subsequence $(x_{n_l})$ which is convergent to some $x$ in $H^1_{\text{loc}}(\mathbf{R})$. But $\|x_{n_l}\|_1 \leqslant M$, so $x \in H^1(\mathbf{R})$ and $\|x\|_1 \leqslant M$.

We shall prove that $x$ is a solution of (8). We have $\varphi x_{n_l} \to \varphi x$ in $H^1(\mathbf{R})$ for any $\varphi \in C_0^\infty(\mathbf{R})$. Therefore, Lemma 2(i) and (10) imply that

$$G\big(\cdot, x_{n_l}(\cdot)\big) \to G\big(\cdot, x(\cdot)\big) \quad \text{in } \mathscr{D}'(\mathbf{R}). \tag{23}$$

The convergence $x_{n_l} \to x$ in $H^1_{\text{loc}}(\mathbf{R})$ implies that

$$x_{n_l} \to x \quad \text{in } \mathscr{D}'(\mathbf{R}), \tag{24}$$

hence

$$x''_{n_l} \to x'' \quad \text{in } \mathscr{D}'(\mathbf{R}). \tag{25}$$

(23)–(25) imply that $x$ is a solution of (8).

The proof of Theorem 1 is complete.

THEOREM 2. *For any solution $x$ of* (5) *in* $H^1(\mathbf{R})$, *we have* $\|x\|_1 \leqslant M$, *where $M$ is defined by* (6).

Proof. Let $x$ be a solution of (5) in $H^1(\mathbf{R})$. For $n = 1, 2, \ldots$, $x|_{[-n,n]}$ is a solution of (22) on $[-n, n]$. Extending $x|_{[-n,n]}$ by (14), we get a solution $x_n$ of (22) on the line. Since $\|x_n\|_1 \leqslant M$ (Lemma 3), we have $\|x\|_1 \leqslant M$.

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*Reçu par la Rédaction le 21.11.1988*