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Stability of iterates of Markov operators

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Abstract. Sufficient conditions are given for the weak and Cesáro convergence of iterates of Markov operators to the one-dimensional projection. These conditions are based on the idea of "lower function" due by A. Lasota and J. Yorke [4], [5].

1. Stability and lower function. Let (X, Σ, m) be a measure space with a non-negative σ-finite measure m. Denote by $D = D(X, \Sigma, m)$ the subspace of $L^1 = L^1(X, \Sigma, m)$ containing all non-negative normalized functions, i.e., the densities

$$D = \{f \in L^1 : f \geqslant 0,\ \|f\| = 1\}.$$

A linear mapping $P\colon L^1 \to L^1$ will be called a *Markov operator* if $P(D) \subset D$. From this condition it follows that P is a positive contraction on L^1. The following properties of Markov operators are well known:

$$(1.1)\qquad (Pf)^+ \leqslant Pf^+ \quad \text{and} \quad (Pf)^- \leqslant Pf^- \quad \text{for } f \in L^1,$$

where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$,

$$(1.2)\quad Pf = f \quad \text{implies} \quad Pf^+ = f^+ \quad \text{and} \quad Pf^- = f^- \quad \text{for } f \in L^1.$$

The operator adjoint to P in L^∞ will be denoted by P^*. A function $f \in D$ satisfying $f = Pf$ is called *invariant under* P. Let μ be a probability measure $\mu(X) = 1$ absolutely continuous with respect to m. By f_μ we denote the Radon–Nikodym derivative $d\mu/dm$; observe that $f_\mu \in D$. When f_μ is invariant, the measure μ is also said to be *invariant under* P. Now consider a measure space (X, Σ, μ). If μ is invariant under P, then

$$(1.3)\qquad \operatorname{supp} f \subset \operatorname{supp} f_\mu \Rightarrow \operatorname{supp} Pf \subset \operatorname{supp} f_\mu \quad \text{for } f \in L^1,$$

where $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$. For a measure μ invariant under P, we denote by P_μ the Markov operator on $L^1_\mu = L^1(X, \Sigma, \mu)$ given by the formula

$$(1.4)\qquad \int_A P_\mu f\, d\mu = \int_A P(ff_\mu)\, dm \quad \text{for } A \in \Sigma \text{ and } f \in L^1_\mu.$$

By D_μ we denote the subspace of L^1_μ containing all densities. From the definition of P_μ we obtain the following

LEMMA 1. *Let P be a Markov operator on L^1 and f_μ be an invariant density under P. Then 1_X is invariant under P_μ and for every n and $f \in D_\mu$ we have*

$$(P_\mu^n f) f_\mu = P^n(ff_\mu). \tag{1.5}$$

If P_μ is a Markov operator on L_μ^1 such that $P_\mu 1_X = 1_X$, then P_μ^* is also a positive contraction on L_μ^∞. For $f \in L^1$ and $g \in L^\infty$ we use the notation $\langle f, g\rangle = \int_X fg\,dm$. If $f \in L_\mu^1$ and $g \in L_\mu^\infty$, we write $\langle f, g\rangle_\mu = \int_X fg\,d\mu$ to exhibit the role of μ.

Let $\{Q_n\}_{n\in N}$ be a sequence of Markov operators. Assume that there is a unique $f_\mu \in D$ invariant under Q_n for every n. The sequence $\{Q_n\}$ will be called *asymptotically stable* if

$$\lim_{n\to\infty} \|Q_n f - f_\mu\| = 0 \quad \text{for every } f \in D. \tag{1.6}$$

The sequence $\{Q_n\}$ will be called *weakly asymptotically stable* if

$$\lim_{n\to\infty} \langle Q_n f, g\rangle = \langle f_\mu, g\rangle \tag{1.7}$$

for every $f \in D$ and $g \in L^\infty$, $g \geq 0$. The sequence $\{Q_n\}$ will be called *almost weakly asymptotically stable* if for every $f \in D$ there is a increasing sequence of integers $\{n_i\}$ of density 1 such that

$$\lim_{i\to\infty} \langle Q_{n_i} f, g\rangle = \langle f_\mu, g\rangle \tag{1.8}$$

for every non-negative $g \in L^\infty$.

Remark 1. An increasing sequence of integers $\{n_i\}$ is of density 1 if $\lim_{i\to\infty} n_i/i = 1$.

A non-negative function $h \in L^1$ will be called a *lower function* for a sequence of Markov operators $\{Q_n\}_{n\in N}$ if $\|h\| > 0$ and

$$\liminf_{n\to\infty} \langle Q_n f, g\rangle \geq \langle h, g\rangle \tag{1.9}$$

for every $f \in D$ and $g \in L^\infty$, $g \geq 0$.

Remark 2. Our definition of a lower function is weaker than that of Lasota–Yorke. Thus, our lower functions should be rather called *weak lower functions*. We shall omit, however, the word weak, since no other lower function will be considered in the paper.

2. Stability conditions. Let P be a Markov operator on L^1. We define a sequence of Markov operators $\{S_n\}$ on L^1 by

$$S_n f = \frac{1}{n}(f + Pf + \ldots + P^{n-1} f), \qquad n = 1, 2, \ldots$$

THEOREM 1. *The sequence* $\{S_n\}_{n\in N}$ *is asymptotically stable iff it has a lower function.*

In order to prove this theorem we need the following lemma.

LEMMA 2. *Let* P *be a Markov operator on* $L^1(m)$. *Assume that* $f_\mu \in D$ *is invariant under* P *and* $\operatorname{supp} f_\mu = X_0$. *Then for every non-negative* $f \in L^1$ *the sequences* $a_n = \langle P^n f, \mathbf{1}_{X_0}\rangle$ *and* $b_n = \langle S_n f, \mathbf{1}_{X_0}\rangle$ *are increasing. Moreover, if there is* $r > 0$ *such that*

$$\liminf_{n\to\infty} b_n \geqslant r\|f\| \tag{2.1}$$

for every non-negative $f \in L^1$, *then*

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \|f\| \quad \textit{for every } f \in L^1,\ f \geqslant 0. \tag{2.2}$$

Proof of Lemma 2. Set $f_1 = \mathbf{1}_{X_0} P^n f$. Using (1.3) we obtain

$$\langle P^{n+1} f, \mathbf{1}_{X_0}\rangle \geqslant \langle Pf_1, \mathbf{1}_{X_0}\rangle = \langle f_1, \mathbf{1}_{X_0}\rangle = \langle P^n f, \mathbf{1}_{X_0}\rangle$$

and

$$\langle S_{n+1} f, \mathbf{1}_{X_0}\rangle = \frac{n}{n+1}\langle S_n f, \mathbf{1}_{X_0}\rangle + \frac{1}{n+1}\langle P^n f, \mathbf{1}_{X_0}\rangle \geqslant \langle S_n f, \mathbf{1}_{X_0}\rangle.$$

Thus the sequence $\{a_n\}$ is increasing and $a_n \leqslant \|f\|$. It hence follows that $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_0 + \ldots + a_{n-1}}{n} = \lim_{n\to\infty} a_n.$$

Set $\alpha = \lim_{n\to\infty} b_n$. For each $\varepsilon > 0$ there is an integer n_0 for which $\langle S_{n_0} f, \mathbf{1}_{X_0}\rangle \geqslant \alpha - \varepsilon$. Define $f_2 = \mathbf{1}_{X_0} S_{n_0} f$ and $f_3 = \mathbf{1}_{X - X_0} S_{n_0} f$. We have $\|f_2\| \geqslant \alpha - \varepsilon$ and $\|f_3\| \geqslant \|f\| - \alpha$. By (2.1) we obtain

$$\lim_{n\to\infty} \langle S_n f, \mathbf{1}_{X_0}\rangle = \lim_{n\to\infty} \langle S_n S_{n_0} f, \mathbf{1}_{X_0}\rangle = \lim_{n\to\infty} \langle S_n f_2, \mathbf{1}_{X_0}\rangle + \lim_{n\to\infty} \langle S_n f_3, \mathbf{1}_{X_0}\rangle$$
$$\geqslant (\alpha - \varepsilon) + r(\|f\| - \alpha)$$

for each $\varepsilon > 0$. On the other hand, $\lim_{n\to\infty} \langle S_n f, \mathbf{1}_{X_0}\rangle = \alpha$ and consequently $\alpha = \|f\|$.

Proof of Theorem 1. If $\{S_n\}_{n\in N}$ is asymptotically stable, then f_μ is a lower function for $\{S_n\}_{n\in N}$. The proof of the converse implication will be given in three steps.

Step I. We construct a function invariant under P. This construction is partially based on ideas taken from [4].

Observe that for every two lower functions h_1 and h_2 the function $h = \max(h_1, h_2)$ is also a lower function. To see this, write

$$A = \{x \in X: h_1(x) \geqslant h_2(x)\}.$$

From the definition of a lower function we have

$$\liminf_{n \to \infty} \langle S_n f, g\mathbf{1}_A \rangle \geqslant \langle h_1, g\mathbf{1}_A \rangle, \tag{2.3}$$

$$\liminf_{n \to \infty} \langle S_n f, g\mathbf{1}_{X-A} \rangle \geqslant \langle h_2, g\mathbf{1}_{X-A} \rangle \tag{2.4}$$

for every $f \in D$ and $g \in L^\infty$, $g \geqslant 0$. By (2.3) and (2.4) we obtain

$$\liminf_{n \to \infty} \langle S_n f, g \rangle \geqslant \langle h_1, g\mathbf{1}_A \rangle + \langle h_2, g\mathbf{1}_{X-A} \rangle = \langle h, g \rangle.$$

Thus h is a lower function. Now write

$$r = \sup\{\|h\|: h \text{ is a lower function}\}.$$

It is evident that $0 < r \leqslant 1$. Choose a sequence $\{h_n\}$ of lower functions such that $\|h_n\| \to r$. Replacing, if necessary, h_n by $\max(h_1, \ldots, h_n)$, we may assume that $\{h_n\}$ is an increasing sequence of lower functions. The limit function $h_* = \lim h_n$ is also a lower function, because

$$\langle h_*, g \rangle \leqslant \langle h_n, g \rangle + \|h_* - h_n\|_{L^1} \|g\|_{L^\infty}$$

and h_n converge to h_* in L^1. The function h_* is evidently the greatest lower function. For any lower function h, the function Ph is also a lower function because

$$\begin{aligned} \liminf_{n \to \infty} \langle S_{n+1} f, g \rangle &= \liminf_{n \to \infty} \frac{n}{n+1} \langle PS_n f, g \rangle + \frac{1}{n+1} \langle f, g \rangle \\ &= \liminf_{n \to \infty} \frac{n}{n+1} \langle S_n f, gP^* \rangle \\ &\geqslant \langle h, gP^* \rangle = \langle Ph, g \rangle. \end{aligned}$$

Thus $Ph_* \leqslant h_*$ and, since P preserves the integral, we have $Ph_* = h_*$. Let $f_\mu = h_*/r$. The function f_μ is invariant under P and

$$\liminf_{n \to \infty} \langle S_n f, g \rangle \geqslant r \langle f_\mu, g \rangle \tag{2.5}$$

for each $f \in D$ and $g \in L^\infty$, $g \geqslant 0$.

Step II. We consider the operator P_μ on L^1_μ corresponding to f_μ. Define a sequence of Markov operators on L^1_μ by

$$\bar{S}_n f = \frac{1}{n}(f + P_\mu f + \ldots + P_\mu^{n-1} f), \quad n = 1, 2, \ldots$$

For $f \in D_\mu$ we have $ff_\mu \in D$. By Lemma 1 and by (2.5), for every $g \in L^\infty_\mu$, $g \geqslant 0$ we have

$$\liminf_{n \to \infty} \langle \bar{S}_n f, g \rangle_\mu \geqslant \langle r, g \rangle_\mu. \tag{2.6}$$

Indeed,

$$\liminf_{n \to \infty} \langle \bar{S}_n f, g \rangle_\mu = \liminf_{n \to \infty} \langle (\bar{S}_n f) f_\mu, g \rangle = \liminf_{n \to \infty} \langle S_n(ff_\mu), g \rangle$$
$$\geqslant \langle rf_\mu, g \rangle = \langle r, g \rangle_\mu.$$

By Lemma 1 the function $\mathbf{1}_X$ is invariant under P_μ. We shall prove that $\mathbf{1}_X$ is a unique density invariant under P_μ. Let $f_0 \in D_\mu$ and let f_0 be invariant under P_μ. This implies that $\bar{S}_n f_0 = f_0$ and by (2.6) we have $f_0 \geqslant r$. Write $\alpha_0 = \max\{\alpha > 0: f_0 \geqslant \alpha\}$. It is evident that $r \leqslant \alpha_0 \leqslant 1$. If $\alpha_0 < 1$ we put $f_1 = (1-\alpha_0)^{-1}(f_0 - \alpha_0)$. For f_1 we have $f_1 \in D_\mu$ and $P_\mu f_1 = f_1$. This implies that $f_1 \geqslant r$. Consequently $f_0 \geqslant r(1-\alpha_0)+\alpha_0$, which contradicts the definition of α_0. Since $f_0 \geqslant 1$ and $f_0 \in D_\mu$, we obtain $f_0 = \mathbf{1}_X$. The operator P_μ is a positive contraction on L^1_μ. Since $P_\mu \mathbf{1}_X = \mathbf{1}_X$, the operator P_μ is also a positive contraction on L^∞_μ and μ is a finite measure invariant under P_μ. From the well-known ergodic properties of Markov operators ([1]) it follows that, for every $f \in D_\mu$, the sequence $\bar{S}_n f$ converges in L^1_μ norm to some $\bar{f} \in D_\mu$ such that $P_\mu \bar{f} = \bar{f}$. But in our case $\mathbf{1}_X$ is a unique density invariant under P_μ; this implies that $\bar{S}_n f$ converges to $\mathbf{1}_X$ in L^1_μ norm.

Step III. We prove that, for every $f \in D$, the sequence $S_n f$ converges to f_μ in L^1 norm.

Set $X_0 = \operatorname{supp} f_\mu$. If $f \in D$ and $\operatorname{supp} f \subset X_0$, then there is $\bar{f} \in D_\mu$ such that $f = \bar{f} f_\mu$. By Lemma 1, $P^n f = (P^n_\mu \bar{f}) f_\mu$. According to Step II, $\bar{S}_n \bar{f}$ converges to $\mathbf{1}_X$ in L^1_μ norm. This implies that $S_n f$ converges to f_μ in L^1 norm. In fact,

$$\lim_{n \to \infty} \|S_n f - f_\mu\| = \lim_{n \to \infty} \|(\bar{S}_n \bar{f}) f_\mu - f_\mu\| = \lim_{n \to \infty} \|\bar{S}_n \bar{f} - \mathbf{1}_X\|_\mu = 0.$$

Now let $f \in L^1$, $f \geqslant 0$. Using (2.5) and substituting $g = \mathbf{1}_{X_0}$ we obtain

$$\liminf_{n \to \infty} \langle S_n f, \mathbf{1}_{X_0} \rangle \geqslant r\|f\|.$$

By Lemma 2, $\lim_{n \to \infty} \langle S_n f, \mathbf{1}_{X_0} \rangle = \|f\|$. Fix $f \in D$. For any given $\varepsilon > 0$ there is an integer n_0 such that $\langle S_{n_0} f, \mathbf{1}_{X_0} \rangle \geqslant 1-\varepsilon$. Set $f_1 = \mathbf{1}_{X_0} S_{n_0} f$. Since $\operatorname{supp} f_1 \subset X_0$, the sequence $S_n f_1$ converges to $\|f_1\| f_\mu$ in L^1 norm. For sufficiently large n we have

$$\|S_n f - f_\mu\| \leqslant \|S_n f - S_n S_{n_0} f\| + \|S_n S_{n_0} f - S_n f_1\| + \big\|S_n f_1 - \|f_1\| f_\mu\big\| + \\ + \big\|(1 - \|f_1\|) f_\mu\big\|$$
$$\leqslant \|S_n f - S_n S_{n_0} f\| + 3\varepsilon.$$

It is easy to see that $\lim_{n\to\infty} \|S_n f - S_n S_{n_0} f\| = 0$. Consequently $S_n f$ converges to f_μ in L^1 norm. Condition (1.6) implies that f_μ is a unique density invariant under P. This completes the proof.

THEOREM 2. *Let P be a Markov operator. Assume that there exists a lower function for the sequence $\{P^n\}_{n\in N}$. Then $\{P^n\}_{n\in N}$ is almost weakly asymptotically stable.*

In order to prove Theorem 2 we need the following lemma which may be of an independent interset. The equivalence between conditions (b) and (c) below is proved in [6]. $\tilde{L}^p_\mu$ denotes the complex Lebesgue space.

LEMMA 3. *Let P_μ be a Markov operator on L^1_μ, where μ is a probability measure invariant under P_μ ($P_\mu \mathbf{1}_X = \mathbf{1}_X$). Then the conditions*

(a) *there is $r > 0$ such that*

$$\liminf_{n\to\infty} \langle P^n_\mu f, g\rangle_\mu \geqslant \|f\| \langle r, g\rangle_\mu \tag{2.7}$$

for every $f \in L^1_\mu$, $f \geqslant 0$ and every $g \in L^\infty_\mu$, $g \geqslant 0$,

(b) *if $P_\mu f = \lambda f$ for some $f \in \tilde{L}^\infty_\mu$ and some $|\lambda| = 1$, then f is constant,*

(c) *for every $g \in L^\infty_\mu$ there is a sequence $\{n_k\}$ of density* **1**, *such that all weak*-cluster points of $\{P^{n_k}_\mu g\}$ are constant,*

(d) *for every $f \in D_\mu$ there is a sequence $\{n_k\}$ of density* **1**, *such that $\{P^{n_k}_\mu f\}$ is weak-convergent to $\mathbf{1}_X$,*

satisfy the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Proof (a) $\Rightarrow$ (b). We prove (b) for every $f \in \tilde{L}^1_\mu$. Let $P_\mu f = \lambda f$ for $f \in \tilde{L}^1_\mu$ and $|\lambda| = 1$. Write $f = f_1 + i f_2$, where $f_1 \in L^1_\mu$ and $f_2 \in L^1_\mu$. We can select an increasing sequence of integers $\{n_k\}$ such that $\lambda^{n_k} \to 1$. For this sequence, $P^{n_k}_\mu f$ converges to f in $\tilde{L}^1_\mu$. Also $P^{n_k}_\mu f_1$ converges to f_1 and $P^{n_k}_\mu f_2$ converges to f_2. Every Markov operator is contractive and consequently the sequence $k \to \|P^{n_k}_\mu f_1\|$ is decreasing. On the other hand, $\|P^{n_k}_\mu f_1\|$ converges to f_1. Thus $\|P^{n_k}_\mu f_1\| = \|f_1\|$ for every k. Now, according to (1.1) we have $(P^{n_k}_\mu f_1)^+ = P^{n_k}_\mu f_1^+$ and $(P^{n_k}_\mu f_1)^- = P^{n_k}_\mu f_1^-$. From the fact that $P^{n_k}_\mu f_1$ converges to f_1 it follows that for any $\varepsilon > 0$ there is an integer $k_0(\varepsilon)$ such that

$$\|P^{n_k}_\mu f_1^+ - f_1^+\| = \|(P^{n_k}_\mu f_1)^+ - f_1^+\| < \varepsilon$$

and

$$\|P^{n_k}_\mu f_1^- - f_1^-\| = \|(P^{n_k}_\mu f_1)^- - f_1^-\| < \varepsilon$$

for $k \geqslant k_0(\varepsilon)$. Hence $P^{n_k}_\mu f_1^+$ converges to f_1^+ and $P^{n_k}_\mu f_1^-$ converges to f_1^-. We are going to prove that $f_1^+ = \|f_1^+\|$. Let $\alpha_0 = \max\{\alpha > 0\colon f_1^+ \geqslant \alpha\}$.

Applying (2.7) to $f = f_1^+ - \alpha_0$ we obtain

$$f_1^+ \geqslant \alpha_0 + r(\|f_1^+\| - \alpha_0).$$

The last inequality implies that $\alpha_0 = \|f_1^+\|$ and consequently f_1^+ is constant. Analogously, the functions f_1^-, f_2^+ and f_2^- are constant. This completes the proof of (b).

(c) $\Rightarrow$ (d). Let $g \in L_\mu^\infty \cap D_\mu$ and $\{n_k\}$ be a sequence from (c). P_μ preserves the integral and consequently a unique w^*-cluster point of $\{P_\mu^{n_k} g\}$ is $\mathbf{1}_X$. This implies that $\{P_\mu^{n_k} g\}$ is w^*-convergent to $\mathbf{1}_X$. Consider a sequence $\{g_m\}_{m \in N}$, where $g_m \in D_\mu \cap L_\mu^\infty$ and g_m converges to $f \in D_\mu$ in L_μ^1 norm. According to (c), there are sets of integers A_m, of density 1, such that

$$\lim_{\substack{n \to \infty \\ n \in A_m}} \langle P_\mu^n g_m, h \rangle_\mu = \langle \mathbf{1}, h \rangle_\mu$$

for every $h \in L_\mu^1$. By Lemma 1 of [3], there is a set $A = \{q_k\}$ of density 1, such that $A \setminus A_m$ is finite for each m. Since $L_\mu^\infty \subset L_\mu^1$, we have

$$\lim_{k \to \infty} \langle P_\mu^{q_k} g_m, h \rangle_\mu = \langle \mathbf{1}, h \rangle_\mu$$

for every m and $h \in L_\mu^\infty$. From the inequality

$$\begin{aligned} |\langle P_\mu^{q_k} f, h \rangle_\mu - \langle \mathbf{1}, h \rangle_\mu| &\leqslant |\langle P_\mu^{q_k} f, h \rangle_\mu - \langle P_\mu^{q_k} g_m, h \rangle_\mu| + \\ &\qquad + |\langle P_\mu^{q_k} g_m, h \rangle_\mu - \langle \mathbf{1}, h \rangle_\mu| \\ &\leqslant \|f - g_m\|_{L_\mu^1} \|h\|_{L_\mu^\infty} + |\langle P_\mu^{q_k} g_m, h \rangle_\mu - \langle \mathbf{1}, h \rangle_\mu| \end{aligned}$$

it follows that $P_\mu^{q_k} f$ is w-convergent to $\mathbf{1}_X$.

Proof of Theorem 2. By an argument similar to that in Theorem 1, there exists a density f_μ invariant under P such that

$$\liminf_{n \to \infty} \langle P^n f, g \rangle \geqslant r \|f\| \langle f_\mu, g \rangle \tag{2.8}$$

for every $f \in L^1$, $g \in L^\infty$, $f \geqslant 0$ and $g \geqslant 0$. The operator P_μ corresponding to f_μ satisfies condition (a). Thus, according to Lemma 3, for every $f \in D_\mu$ there is a sequence $\{n_k\}$ of density 1, such that $P_\mu^{n_k} f$ is w-convergent to $\mathbf{1}_X$. Set $X_0 = \operatorname{supp} f_\mu$. Let $f \in D$ and let $\operatorname{supp} f \subset X_0$. Then, according to Lemma 1, there is a sequence $\{n_k\}$ of density 1 such that $P^{n_k} f$ is w-convergent to f_μ. Using (2.8) and substituting $g = \mathbf{1}_{X_0}$ we obtain

$$\liminf_{n \to \infty} \langle S_n f, \mathbf{1}_{X_0} \rangle \geqslant \liminf_{n \to \infty} \langle P^n f, \mathbf{1}_{X_0} \rangle \geqslant r \|f\|$$

for every $f \in L^1$ and $f \geqslant 0$. By Lemma 2, $\lim_{n \to \infty} \langle P^n f, \mathbf{1}_{X_0} \rangle = \|f\|$. Fix $f \in D$ and define $a_k = \langle P^k f, \mathbf{1}_{X_0} \rangle$. Set $f_k = a_k^{-1} \mathbf{1}_{X_0} P^k f$. Since $f_k \in D$ and $\operatorname{supp} f_k$

$\subset X_0$, there are sets of integers A_k of density 1 such that

$$w\text{-}\lim_{\substack{n\in A_k\\ n\to\infty}} P^n f_k = f_\mu.$$

Let $\bar{A}_k = \{k+n\colon n\in A_k\}$. By Lemma 1 of [3], there is a set $A = \{m_p\}$ of density 1, such that $A\setminus\bar{A}_k$ are finite for each k. Now, the sequence $P^{m_p}f$ is w-convergent to f_μ. In fact, we have

$$|\langle P^{m_p}f, g\rangle - \langle f_\mu, g\rangle| \leqslant \|g\|_{L^\infty}(a_k^{-1}-a_k) + |\langle P^{m_p-k}f_k, g\rangle - \langle f_\mu, g\rangle|.$$

The first term on the right-hand side is small for large k (since a_k converges to 1). The second is small for large p and fixed k (since $P^{m_p-k}f_k$ w-converges to f_μ). Condition (1.8) implies that f_μ is a unique density invariant under P. This completes the proof.

Remark. The existence of a lower function for $\{P^n\}_{n\in N}$ does not implies the weak asymptotic stability of $\{P^n\}_{n\in N}$. This problem is discussed in Section 3.

3. Point transformations. Now we give an interpretation of our theorems in terms of the ergodic theory of point transformations. Let $T\colon X\to X$ be a transformation on a σ-finite measure space (X, Σ, m). We shall assume that T is measurable and nonsingular. The last condition means that $m(T^{-1}(A)) = 0$ if $m(A) = 0$ and $A\in\Sigma$. For a given T we define the Frobenius–Perron operator corresponding to T by

$$\int_A P_T f\,dm = \int_{T^{-1}(A)} f\,dm \quad \text{for } A\in\Sigma \text{ and } f\in L^1.$$

P_T is obviously a Markov operator and $S_n = \frac{1}{n}(I+P_T+\ldots+P_T^{n-1})$ are Markov operators. Let M be a set of all probability measures absolutely continuous with respect to m. A measure $\mu\in M$ is called *invariant under* T if $\mu(T^{-1}(A)) = \mu(A)$ for each $A\in\Sigma$. From the definition of P_T it follows that μ is invariant under T iff $P_T f_\mu = f_\mu$. A measure μ invariant under T is called *ergodic* if for each $A\in\Sigma$ the condition $A = T^{-1}(A)$ implies $\mu(A)(1-\mu(A)) = 0$. If $\mu\in M$ is a unique measure invariant under T, then μ is ergodic. From Theorem 1 follows immediately

COROLLARY 1. *Assume that $\{S_n\}_{n\in N}$ has a lower function. Then there exists a unique measure $\mu\in M$ invariant under T. This measure is ergodic. Moreover, for every measure $v\in M$ and $A\in\Sigma$ we have*

$$\lim_{n\to\infty}\frac{1}{n}\big(v(A)+v(T^{-1}(A))+\ldots+v(T^{-n+1}(A))\big) = \mu(A).$$

Now we are going to apply our results to so-called *weakly mixing* and *partially mixing dynamical systems.*

Let (X, Σ, μ) be a measure space with probability measure μ and let $T\colon X \to X$ be a measure preserving transformation. The dynamical system (X, Σ, μ, T) is called *weakly mixing* if for every $A \in \Sigma$ and $B \in \Sigma$

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} |\mu(A \cap T^{-k}(B)) - \mu(A)\mu(B)| = 0.$$

The dynamical system (X, Σ, μ, T) is called *partially mixing* if there is $r > 0$ such that for every $A \in \Sigma$, $B \in \Sigma$

$$\liminf_{n \to \infty} \mu(A \cap T^{-n}(B)) \geqslant r\mu(A)\mu(B).$$

The dynamical system (X, Σ, μ, T) is called *mixing* if for every $A \in \Sigma$, $B \in \Sigma$ we have

$$\lim_{n \to \infty} \mu(A \cap T^{-1}(B)) = \mu(A)\mu(B).$$

It is well known that mixing implies partially mixing, partially mixing implies weakly mixing and weakly mixing implies ergodicity. The inverse implications do not hold. From Theorem 2 and inequality (2.8) we obtain

COROLLARY 2. *Assume that $\{P_T^n\}_{n \in N}$ has a lower function. Then there exists a unique measure $\mu \in M$ invariant under T. The dynamical system (X, Σ, μ, T) is weakly mixing and partially mixing. Moreover, for every measure $v \in M$ there is a sequence $\{n_k\}$ of density* **1** *such that*

$$\lim_{k \to \infty} v(T^{-n_k}(A)) = \mu(A) \quad \textit{for } A \in \Sigma.$$

Remark. The existence of a lower function for $\{P^n\}$ does not imply weak asymptotic stability. To see this, consider a dynamical system (X, Σ, μ, T) which is partially mixing but not mixing. An example of such a system is given in [2]. Then by the definition of partial mixing there is $r > 0$ such that

$$\liminf_{n \to \infty} \langle P_T^n f, g \rangle_\mu \geqslant \langle r, g \rangle_\mu$$

for every $f \in D_\mu$ and $g \in L_\mu^\infty$, $g \geqslant 0$. Thus r is a lower function. On the other hand, the sequence $\{P_T^n\}$ is not weakly asymptotically stable, since weak asymptotical stability implies mixing.

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