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Biholomorphic invariance of capacity and the capacity of an annulus

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Abstract. In this paper we prove the biholomorphic invariance of the capacity defined in [6] and demonstrate the existence of an extremal function for condensers biholomorphically equivalent to an annulus. We are also concerned with a conjecture of S. S. Chern, H. I. Levine and L. Nirenberg [3] and we give a partial positive answer to it.

Introduction. Let D be a bounded domain in C^n whose boundary consists of two components which are $(2n-1)$-dimensional smooth differentiable manifolds. We denote by $\operatorname{Adm} D$ the class of C^2-smooth real valued function on $\bar{D}$ which are supposed to be C^2-plurisubharmonic on D and equal zero on one component of the boundary and one on the other component. In our considerations we assume that $\operatorname{Adm} D \neq \emptyset$. We call the domain D a *condenser*.

Let us define the functional

$$\operatorname{Adm} D \ni u \mapsto J(u) = \int_D du \wedge d^c u \wedge [dd^c u]^{n-1} \in R. \tag{1}$$

With any condenser D we associate the real number

$$\operatorname{Cap} D = \inf_{u \in \operatorname{Adm} D} J(u), \tag{2}$$

which is called its *capacity*.

In the present paper a theorem on the invariance of capacities under biholomorphic mappings is proved. We demonstrate also the existence of an extremal function u for condensers biholomorphically equivalent to an annulus, whence, in particular, follows that two annuli with a different ratio of radii are not biholomorphically equivalent. Finally, we are concerned with the conjecture of Chern, Levine and Nirenberg [3] in the case where $D = D_1 - \bar{D}_0$, D_0 and D_1 are assumed to be strictly pseudoconvex. In the case where $\operatorname{Adm} D$ contains an extremal function, a positive answer was obtained by Bedford and Taylor [1].

1. Biholomorphic invariance of capacity. To begin with we introduce

DEFINITION 1. Two condensers D and D^* are called *biholomorphically equivalent* if there exists a diffeomorphism $f\colon \bar{D} \to \bar{D}^*$ such that $\tilde{f} = f|D$ is a biholomorphic mapping between D and D^*.

Our further considerations will be concerned with the existence property of an extremal function. By an *extremal function* in the class $\operatorname{Adm} D$ we mean any $u^* \in \operatorname{Adm} D$ such that

$$\inf_{u \in \operatorname{Adm} D} J(u) = J(u^*).$$

It turns out [6] that $u^* \in \operatorname{Adm} D$ is an extremal function if and only if

$$[dd^c(u^*|D)]^n = 0.$$

LEMMA 1. *If two condensers D and D^* are biholomorphically equivalent, then the existence of an extremal function in $\operatorname{Adm} D$ implies the existence in $\operatorname{Adm} D^*$ and conversely.*

Proof. The proof is trivial and follows from standard calculations.

Let $P(R; r) = \{z \in C^n\colon r < |z|^2 < R,\ 0 < r < R\}$. Then we have

LEMMA 2. *$P(R; r)$ is a condenser and the class $\operatorname{Adm} P(R; r)$ contains an extremal function.*

Proof. Trivially, $P(R; r)$ is a condenser. By simple calculations we may also verify that

$$u(z) = [\ln(R/r)]^{-1} \ln(|z|^2/r), \quad z \in \bar{P}(R; r), \tag{4}$$

is an extremal function.

We are going to prove

THEOREM 1. *Two condensers which are biholomorphically equivalent have equal capacities.*

Proof. Let D and D^* be two condensers which are biholomorphically equivalent, and let f be a mapping which establishes the equivalence between D and D^*. Denote by $w = (w^1, \ldots, w^n)$ and $z = (z^1, \ldots, z^n)$ coordinate systems in D^* and D, respectively. If $u \in \operatorname{Adm} D$, then $u^* = u \circ f^{-1} \in \operatorname{Adm} D^*$. By simple calculations we see that

$$\begin{aligned} &d(u^*|D^*) \wedge d^c(u^*|D^*) \\ &\qquad = 2i \sum_{i,p,k,s=1}^{n} \left(u_{|i} \circ (f|D)^{-1}\, \bar{u}_{|\bar{k}} \circ (f|D)^{-1}\right) z^i_{|s} z^k_{|\bar{p}}\, dw^s \wedge d\bar{w}^p, \end{aligned} \tag{5}$$

$$dd^c(u^*|D^*) = 2i \sum_{i,p,k,s=1}^{n} \left(u_{|p\bar{k}} \circ (f|D)^{-1}\right) z^p_{|s} \bar{z}^k_{|i}\, dw^s \wedge d\bar{w}^i. \tag{6}$$

Taking into account (5) and (6), we obtain

(7) $$J^*(u^*) = \int_{D^*} du^* \wedge d^c u^* \wedge [dd^c u^*]^{n-1}$$
$$= c \int_{D^*} \Big(\sum_{i,k=1}^{n} u_{|i} u_{|\bar{k}} M_{ik}\Big) \circ (f|D)^{-1} |dz/dw|^2 dV^*,$$

where $c = 2^{2n}(n-1)!$, dz/dw is the Jacobian matrix of $(f|D)^{-1}$, $M_{ik} = (-1)^i(-1)^k m_{ik}$, and m_{ik} is (i, k)-th minor matrix of $[u_{|i\bar{k}}]_{1 \leqslant i,k \leqslant n}$. It is easy to see that (7) is equivalent to

(8) $$J^*(u^*) = J(u).$$

By the definition of capacity there are sequences $\{u_n\} \subset \operatorname{Adm} D$ and $\{u_n^*\} \subset \operatorname{Adm} D$ such that $J(u_n) \to \operatorname{Cap} D$ and $J^*(u_n^*) \to \operatorname{Cap} D^*$ for $n \to \infty$. Consider the functions $\tilde{u}_n^* = u_n \circ f^{-1} \in \operatorname{Adm} D^*$ and $\tilde{u}_n = u_n^* \circ f \in \operatorname{Adm} D$. By (8) we have $J(\tilde{u}_n) = J^*(u_n^*)$ and $J^*(u_n^*) = J(u_n)$ for $n = 1, 2, \ldots$ Therefore

$$\operatorname{Cap} D = \lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} J^*(\tilde{u}_n^*) \geqslant \operatorname{Cap} D^*,$$

$$\operatorname{Cap} D \leqslant \lim_{n\to\infty} J(\tilde{u}_n) = \lim_{n\to\infty} J^*(u_n^*) = \operatorname{Cap} D^*,$$

which shows that $\operatorname{Cap} D = \operatorname{Cap} D^*$ and proves the lemma.

Let $R^* > r^* > 0$ and $R > r > 0$ be such that $R^*/r^* \neq R/r$. We have

THEOREM 2. *The condensers $P(R^*; r^*)$ and $P(R; r)$ are not biholomorphically equivalent.*

Proof. By Lemma 2 it follows that

$$u(z) = [\ln(R/r)]^{-1} \ln(|z|^2/r) \quad \text{for } z \in \bar{P}(R; r)$$

and

$$u^*(z) = [\ln(R^*/r^*)]^{-1} \ln(|z|^2/r^*) \quad \text{for } z \in \bar{P}(R^*; r^*)$$

are extremal in $\operatorname{Adm} P(R; r)$ and $\operatorname{Adm} P(R^*; r^*)$, respectively. In the case of an annulus we may easily compute the capacities involved:

(9) $$\operatorname{Cap} P(R; r) = \int_{P(R;r)} du \wedge d^c u \wedge [dd^c u]^{n-1}$$
$$= \int_{P(R;r)} d\big(u d^c u \wedge [dd^c u]^{n-1}\big) - \int_{P(R;r)} u [dd^c u]^n$$
$$= \int_{\partial P(R;r)} u d^c u \wedge [dd^c u]^{n-1} = \int_{C_1} d^c u \wedge [dd^c u]^{n-1}$$
$$= [\ln(R/r)]^{-n} \int_{C_1} d^c \ln|z|^2 \wedge [dd^c \ln|z|^2]^{n-1}$$
$$\equiv [\ln(R/r)]^{-n} \Gamma,$$

and likewise

$$\operatorname{Cap} P(R^*; r^*) = [\ln(R^*/r^*)]^{-n} \Gamma^*. \tag{10}$$

We may assume that $R^* > R$; by the Stokes theorem we have

$$\Gamma^* - \Gamma = \int_{C_1^* - C_1} d^c \ln|z|^2 \wedge [dd^c \ln|z|^2]^{n-1} = \int_{P(R^*;R)} [dd^c \ln|z|^2]^n = 0. \tag{11}$$

From (9), (10) and (11) we obtain

$$\operatorname{Cap} P(R^*/r^*)/\operatorname{Cap} P(R; r) = [\ln(R/r)]^n/[\ln(R^*/r^*)]^n \neq 1.$$

In view of Theorem 1 this proves the desired result.

We end the section with two remarks:

Remark 1. For every condenser $D = D_1 \setminus \bar{D}_0$ the domain D_1 is a domain of holomorphy. If there exists some $u \in \operatorname{Adm} D$ such that $u(x) \neq 0$ for $x \in D$, then D_0 is also a domain of holomorphy.

Remark 2. Let $D = D_1 \setminus \bar{D}_0$ and $\tilde{D} = \tilde{D}_1 \setminus \bar{\tilde{D}}_0$ be two condensers which are biholomorphically equivalent. Then D_1 and $\tilde{D}_1$ are also biholomorphically equivalent.

Example. We shall give an example of a condenser which is not biholomorphically equivalent to an annulus.

Let D_1 be a strictly pseudoconvex domain which is supposed to be not biholomorphically equivalent to a ball. We also assume that D_1 has the compact closure in C^n. By assumption, one can find (see [4]) a neighbourhood $U \supset \partial D_1$ and a strictly C^2-plurisubharmonic function g such that

$$U \cap D_1 = \{x \in U: g(x) < 0\}, \quad \partial D_1 = \{x \in U: g(x) = 0\}$$

and $dg(x) \neq 0$ for every $x \in U$. For sufficiently small $\varepsilon > 0$ the set

$$\partial D_0 \overset{\mathrm{df}}{=} \{x \in U: g(x) = -\varepsilon\}$$

is a connected $(2n-1)$-dimensional C^2-smooth manifold and

$$D = \{x \in U: 0 > g(x) > -\varepsilon\}$$

is a condenser (as an admissible function we may take $U(x) = 1/\varepsilon(g(x)+\varepsilon)$, $x \in \bar{D}$).

By Lemma 3 the condenser D is not biholomorphically equivalent to an annulus.

2. The Chern–Levine–Nirenberg conjecture. Now, we will be concerned with the conjecture posed by Chern, Levine and Nirenberg [3].

Let $D = D_1 \setminus \bar{D}_0$ be a condenser. By $\Gamma(D)$ we shall denote a class of $(2n-1)$-dimensional closed currents (in the sense of de Rhan) in $\bar{D}$, which contains ∂D_1 and $-\partial D_0$ (here we assume

$$\partial D_k(\varphi) \overset{\mathrm{df}}{=} \int_{\partial D_k} i_k^* \varphi \quad \text{for } k = 0, 1$$

and for every $(2n-1)$-form φ on $\bar{D}$, where i_k denotes the C^2-embedding of ∂D_k in $\bar{D}$ and ∂D_1 and ∂D_0 have orientations induced by D) such that for every $\gamma \in \Gamma(D)$ there are positive n-currents G and $\tilde{G}$ on $\bar{D}$ (i.e. taking non-negative values on every positive n-form on $\bar{D}$) with the properties:

$$bG = \partial D_1 - \gamma, \quad b\tilde{G} = \gamma + \partial D_0,$$

b denotes the boundary operator.

Let $\Gamma^*(D) = \{\Gamma(D)\} \cup \{-\Gamma(D)\}$. Define the functionals

$$\operatorname{Adm} D \ni u \mapsto R_i(u) = \int_{\partial D_i} d^c u \wedge [dd^c u]^{n-1} \in R \quad \text{for } i = 0, 1.$$

We have

THEOREM 3. *If a condenser $D = D_1 \setminus \bar{D}_0$ is such that D_1 and D_0 are strictly pseudoconvex and*

$$\inf_{u \in \operatorname{Adm} D} \int_D [dd^c u]^n = 0,$$

then

1° *the functional R_1 is convex and R_0 is concave, respectively,*

2° *the following equality holds:*

$$\inf_{u \in \operatorname{Adm} D} R_1(u) = \Big| \sup_{u \in \operatorname{Adm} D} \inf_{\gamma \in \Gamma^*(D)} \int_\gamma d^c u \wedge [dd^c u]^{n-1} \Big| = \operatorname{Cap} D.$$

Proof. By the pseudoconvexity of D_1 and D_0 there exists a real-valued C^2-function f defined in C^n which fulfils the following conditions:

(a) $D = \{x \in C^n \colon f(x) < 0\}$, $\partial D = \{x \in C^n \colon f(x) = 0\}$,

(b) there are neighbourhoods $V_1 \supset \partial D_1$ and $V_0 \supset \partial D_0$ such that $f | V_1$ and $-f | V_0$ are plurisubharmonic and $df(x) \neq 0$ for $x \in V_0 \cup V_1$.

By simple calculations we have [5]

$$\delta^2 R_k(u)(h, t) = \tfrac{1}{2} n(n-1) \int_{\partial D_k} d^c h \wedge dd^c h \wedge [dd^c u(t)]^{n-2} \quad \text{for } k = 0, 1,$$

where $h = u - \tilde{u}$, $u(t) = u + th$, $t \in [0, 1]$, and $u, \tilde{u} \in \operatorname{Adm} D$. Since $df(x) \neq 0$ for $x \in V_1 \cup V_0$, we have

$$h(x) = f(x) r_h(x), \quad x \in \bar{D},$$

where $r_h \in C^1(\bar{D})$.

Let $i_k \colon \partial D_k \to \bar{D}$, $k = 0, 1$ be C^2-imbeddings. Let $i_k^* \colon \Lambda(\bar{D}) \to \Lambda(\partial D_k)$, $k = 0, 1$ be exterior algebra homomorphisms. It is easy to check that

$$i_k^*(d^c h \wedge dd^c h) = (r_h \circ i_k)^2 i_k^* d^c f \wedge i_k^* dd^c f.$$

Hence we get

$$\delta^2 R_k(u)(h, t) = \tfrac{1}{2}n(n-1) \int_{\partial D_k} r_h^2\, d^c f \wedge dd^c f \wedge [dd^c u(t)]^{n-2} \overset{\text{df}}{=} \tfrac{1}{2}n(n-1) \int_{\partial D_k} g dS,$$

where $dS = *df/\|df\|$ is the surface area on ∂D_k, since

$$r_h^2\, df \wedge d^c f \wedge dd^c f \wedge [dd^c u(t)]^{n-2} = g df \wedge (*df/\|df\|)$$

and

$$r_h^2\, df \wedge d^c f \wedge dd^c f \wedge [dd^c u(t)]^{n-2} \geqslant 0$$

in $(D_1 \setminus \bar{D}_0) \cap V_1$, [6]. Analogously, we have

$$r_h^2\, df \wedge d^c f \wedge dd^c f \wedge [dd^c u(t)]^{n-2} \leqslant 0$$

in $(D_1 \setminus \bar{D}_0) \cap V_0$. By the above we infer that

$$g(x) \geqslant 0 \quad \text{for } x \in \partial D_1 \quad \text{and} \quad g(x) \leqslant 0 \quad \text{for } x \in \partial D_0.$$

Since for every $u \in \operatorname{Adm} D$

$$[dd^c u]^n \geqslant 0,$$

for every $\gamma \in \Gamma(D)$ and every $u \in \operatorname{Adm} D$ we have

$$(12) \quad \int_{\partial D_1} d^c u \wedge [dd^c u]^{n-2} \geqslant \int_\gamma d^c u \wedge [dd^c u]^{n-2} \geqslant \int_{-\partial D_0} d^c u \wedge [dd^c u]^{n-2}$$

(here $\int_\gamma$ is identified with γ).

By the same considerations as above and by the fact that $u|\partial D_0 = 0$, we obtain

$$(13) \quad \int_{-\partial D_0} d^c u \wedge [dd^c u]^{n-1} \geqslant 0.$$

From (12), (13) and from the definition of $\Gamma^*(D)$ we infer that for every $\gamma \in \Gamma^*(D)$

$$(14) \quad \int_\gamma d^c u \wedge [dd^c u]^{n-1} \geqslant - \int_{\partial D_1} d^c u \wedge [dd^c u]^{n-1}.$$

From (14) it follows that

$$\Big| \sup_{u \in \operatorname{Adm} D} \inf_{\gamma \in \Gamma^*(D)} \int_\gamma d^c u \wedge [dd^c u]^{n-1} \Big| = \inf_{u \in \operatorname{Adm} D} \int_{\partial D_1} d^c u \wedge [dd^c u]^{n-1}.$$

It remains to prove the relation

$$\inf_{u \in \operatorname{Adm} D} \int_{\partial D_1} d^c u \wedge [dd^c u]^{n-1} = \operatorname{Cap} D.$$

By assumption there is a sequence $\{u_k\} \subset \operatorname{Adm} D$ such that

$$\lim_{k\to\infty} \int_D [dd^c u_k]^n = 0.$$

We will prove that

$$\lim_{k\to\infty} J(u_k) = \operatorname{Cap} D.$$

We see that

$$\lim_{k\to\infty} \delta J(u_k)(h_k, 0) = -(n+1) \lim_{k\to\infty} \int_D h_k [dd^c u_k]^n = 0.$$

Therefore, for arbitrary $\varepsilon > 0$ we may find $N(\varepsilon)$ such that for every $k > N(\varepsilon)$ we have

$$|\delta J(u_k)(h_k, 0)| < \varepsilon, \quad h_k = \tilde{u} - u_k.$$

Since J is convex [5], it follows that

$$J(u_k) \leqslant J(\tilde{u}) + \varepsilon \tag{15}$$

for arbitrary $\tilde{u} \in \operatorname{Adm} D$ and $k > N(\varepsilon)$. By (15) the sequence $\{J(u_k)\}$ is bounded and we may choose a subsequence $\{J(u_{k_i})\}$ which is convergent to some $g \geqslant 0$. Hence we have

$$g = \lim_{i\to\infty} J(u_{k_i}) \leqslant J(\tilde{u}) + \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ we get

$$g \leqslant J(\tilde{u})$$

and then from the arbitrariness of $\tilde{u}$ we infer that

$$g \leqslant \operatorname{Cap} D.$$

Together with the evident inequality

$$g \geqslant \operatorname{Cap} D$$

this yields

$$g = \operatorname{Cap} D.$$

Because the last equality holds for an arbitrary convergent subsequence $\{u_{k_i}\}$ we conclude that

$$g = \lim_{k\to\infty} J(u_k) = \operatorname{Cap} D, \tag{16}$$

which proves our assertion.

Using the Stokes theorem it is easy to verify that

(17) $$R_1(u) = \int_D du \wedge d^c u \wedge [dd^c u]^{n-1} - \int_D u[dd^c u]^n$$

and

(18) $$\delta R_1(u)(h, t) = n\Big(\int_D udd^c h \wedge [dd^c u(t)]^{n-1} - \int_D h[dd^c u(t)]^n\Big).$$

From (16) and (17) it follows that

(19) $$\lim_{k\to\infty} R_1(u_k) = \operatorname{Cap} D.$$

Since $udd^c\tilde{u} \wedge [dd^c u]^{n-1}$ is non-negative for every $u, \tilde{u} \in \operatorname{Adm} D$, therefore from (18) and the above remark we arrive at the inequality

$$\lim_{k\to\infty} \inf \delta R_1(u_k)(h_k, 0) \geqslant 0.$$

Hence, for $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for every $k > N(\varepsilon)$ we have

$$\delta R_1(u_k)(h_k, 0) > -\varepsilon, \quad h_k = \tilde{u} - u_k.$$

By the convexity of R_1 we infer that

$$R_1(u_k) \leqslant R_1(\tilde{u}) + \varepsilon, \quad k > N(\varepsilon).$$

Using then (19) and the same arguments as in the proof of (16), we obtain

$$\operatorname{Cap} D = \inf_{u\in \operatorname{Adm} D} R_1(u),$$

thus concluding the proof.

Applying Theorem A of [2] we are going to prove

THEOREM 4. *If the class of admissible functions for a condenser D contains an extremal function, then*

$$\sup_{u\in \operatorname{Adm} D} \inf_{\gamma\in\Gamma^*(D)} \Big|\int_\gamma d^c u \wedge [dd^c u]^{n-1}\Big| = \inf_{u\in \operatorname{Adm} D} \sup_{\gamma\in\Gamma^*(D)} \Big|\int_\gamma d^c u \wedge [dd^c u]^{n-1}\Big| = \operatorname{Cap} D.$$

Proof. Since for every $\gamma \in \Gamma(D)$ and every $u \in \operatorname{Adm} D$ we have

$$\int_\gamma d^c u \wedge [dd^c u]^{n-1} \geqslant 0,$$

if follows that

$$\sup_{u\in \operatorname{Adm} D} \inf_{\gamma\in\Gamma^*(D)} \Big|\int_\gamma d^c u \wedge [dd^c u]^{n-1}\Big| = \sup_{u\in \operatorname{Adm} D} \inf_{\gamma\in\Gamma(D)} \int_\gamma d^c u \wedge [dd^c u]^{n-1}.$$

By the definition of $\Gamma(D)$ we have

$$\inf_{\gamma\in\Gamma(D)} \int_\gamma d^c u \wedge [dd^c u]^{n-1} = \int_{-\partial D_0} d^c u \wedge [dd^c u]^{n-1} = -R_0(u)$$

and

$$\sup_{\gamma\in\Gamma(D)} \int_{\gamma} d^c u \wedge [dd^c u]^{n-1} = \int_{\partial D_1} d^c u \wedge [dd^c u]^{n-1} = R_1(u).$$

Let u^* be an extremal function in $\operatorname{Adm} D$. By Theorem A [2] we have

$$u - u^* = h \leqslant 0 \quad \text{in } \bar{D}. \tag{20}$$

Let f be a C^2-smooth real-valued function such that

$$D = \{x \in C^n\colon f(x) < 0\}, \quad \partial D = \{x \in C^n\colon f(x) = 0\}$$

and

$$df(x) \neq 0 \quad \text{for } x \in \partial D.$$

It can be easily seen that

$$\delta[-R_0](u^*)(h, t) = -\delta R_0(u^*)(h, t) = -n \int_{\partial D_0} d^c h \wedge [dd^c u(t)]^{n-1},$$

where $h = u - u^*$. By the same arguments as in Theorem 3 we have

$$h = u - u^* = r_h f,$$

where $r_h \in C^1(\bar{D})$. From (20) it follows that $r_h(x) \geqslant 0$ for $x \in \partial D_0$. Thus

$$\delta[-R_0](u^*)(h, t) = -n \int_{\partial D_0} r_h d^c f \wedge [dd^c u(t)]^{n-1} \leqslant 0$$

for every function $u \in \operatorname{Adm} D$ and for every $t \in [0, 1]$. The last inequality shows that the functional $-R_0$ attains its maximum for u^*. Simple calculations show also that $-R_0(u^*) = \operatorname{Cap} D$. By analogous considerations we may also prove the second part of our statement, as desired.

References

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