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# Lipschitz spaces of holomorphic and pluriharmonic functions on bounded symmetric domains in $C^N$ $(N > 1)$

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*This paper is dedicated to the memory of my friend and teacher Stefan Bergman*

**Abstract.** Let $D$ be a bounded symmetric domain in $C^N$ $(N > 1)$ with Bergman–Šilov boundary $b, H_p$ $(p > 0)$ the Hardy space of holomorphic functions on $D$ and $B_{pq}$ $(0 < p < 1 \leqslant q < \infty)$ a Lipschitz space with norm $\left(\int_0^1 (1-r)^{Nq(1/p-1/q)-1} M_q^q(r, f)\,dr\right)^{1/q}$. $B_{pq}$ is a Banach space. The Szegö kernel shows that $H_p$ is a proper subset of $B_{pq}$ $(q \geqslant 2)$ for some domains $D$. Results on fractional derivatives and integrals for the unit disk are generalized to $B_{pq}$ spaces. The corresponding spaces $ph_p$ and $b_{pq}$ of pluriharmonic functions are introduced. Kolmogorov's theorem is generalized for $ph_1$ and $b_{pq}$ is proved to be a self-conjugate space. An example for the polydisk gives a function $F \notin H_p$ but $\operatorname{Re} F \in ph_p$.

**1. Introduction.** Let $D$ be a bounded symmetric domain in the complex vector space $C^N$ $(N > 1)$ in the canonical Harish–Chandra realization. It is known that $D$ is circular and star-shaped with respect to $0 \in D$ and has a Bergman–Šilov boundary $b$, which is circular and measurable. Let $\Gamma$ be the group of holomorphic automorphisms of $D$ and $\Gamma_0$ its isotropy subgroup with respect to 0. The group $\Gamma$ is transitive on $D$ and the holomorphic automorphisms extend continuously to the topological boundary of $D$. The group $\Gamma_0$ is transitive on $b$ and $b$ has a unique normalized $\Gamma_0$-invariant measure $d\mu_t = V^{-1} ds_t$, $V$ the euclidean volume of $b$ and $ds_t$ the euclidean volume element at $t$ [14], [18].

The Hardy space $H_p$ $(0 < p < \infty)$ is defined on $D$ by

$$H_p \equiv H_p(D) = \{f\colon f \text{ is holomorphic on } D \text{ and } \|f\|_p < \infty\},$$

where

$$M_p(r, f) = \left(\frac{1}{V}\int_b |f(rt)|^p ds_t\right)^{1/p}, \quad \|f\|_p = \sup_{0<r<1} M_p(r, f).$$

For $p \geqslant 1$, $H_p$ is a Banach space and for $0 < p < 1$ a complete linear Hausdorff space.

Let $B_{pq}$ $(0 < p < q < \infty)$ be the set of holomorphic functions on $D$ with

$$(1) \qquad \|f\|_{B_{pq}} = \Big(\int_0^1 (1-r)^{Nq(1/p-1/q)-1} M_q^q(r,f)\,dr\Big)^{1/q} < \infty .$$

Hardy and Littlewood introduced $B^p = B_{p1}$ spaces over the disk in [11], Duren and Shields studied their properties in [1]–[3] and Mitchell and Hahn over bounded symmetric domains in $C^N$ [15]. The spaces $B_{pq}$ form a proper subset of the Lipschitz spaces in [6]. We also study the analogous spaces $ph_p$ and $b_{pq}$ of pluriharmonic functions on $D$.

On $D$ there exists a complete orthonormal system of complex homogeneous polynomials $\{\varphi_{k\nu}\}$ $(k = 0, 1, \ldots; m_k = \binom{N+k-1}{k})$ [13], normalized over $b$ and every holomorphic function $f$ on $D$ has a series expansion

$$(2) \qquad f(z) = \sum a_{k\nu}(f)\varphi_{k\nu}(z), \qquad a_{k\nu} = a_{k\nu}(f) = \lim_{r\to 1} \int_b f_r(t)\bar{\varphi}_{k\nu}(t)\,ds_t,$$

where the convergence is uniform on compact subsets of $D$ [9]. Here $\sum = \sum_{k=0}^{\infty}\sum_{\nu=1}^{m_k}$ and $f_r$ is the slice function defined by $f_r(z) = f(rz)$ $(z \in \bar{D}$, $0 < r < 1)$. If $f$ is also integrable on $D \cup b$, it satisfies a maximum principle. If $f \in H_p$ $(1 \leqslant p < \infty)$, it has Cauchy and Poisson integral representations [9].

Section 2 considers elementary properties of $B_{pq}$ spaces. Theorem 3 gives examples of functions in $B_{pq}$ but not in $H_p$ for some of the classical symmetric domains. In Section 3 properties of fractional derivatives and integrals for functions in $B_{pq}$ are obtained, which generalize results for the unit disk in [1] (Lemma 1 and Theorem 4). In Section 4 the spaces $ph_p$ and $b_{pq}$ of pluriharmonic functions are considered. Theorem 5 generalizes Kolmogorov's theorem and Theorem 6 proves that $b_{pq}$ $(0 < p < q)$ is a self-conjugate space. An example of a function not in $H_p$ for $p = 1/(k+1)$, $k$ a positive integer, but whose real part is in $ph_p$, is given for the polydisk. We could also obtain a representation formula for bounded linear functionals on $B_{pq}$ (cf. [15], Theorem 1) but this problem is not considered here.

The following are used in proofs of theorems. The formula

$$(3) \qquad \frac{1}{2\pi}\int_b ds_t \int_0^{2\pi} g(e^{i\theta}t)\,d\theta = \int_b g(t)\,ds_t \qquad (g \in L_1(b))$$

is obtained by using Fubini's theorem, the circularity of $b$ and the circular invariance of the measure $ds_t$. For $q \geqslant 1$ Minkowski's inequality in infinite form is

$$(4) \qquad \Big(\int_A \Big|\int_B g(z,\xi)\,d\mu_\xi\Big|^q d\mu_z\Big)^{1/q} \leqslant \int_B \Big(\int_A |g(z,\xi)|^q d\mu_z\Big)^{1/q} d\mu_\xi,$$

where $A$ and $B$ are measurable sets with positive measures $d\mu_z$, $d\mu_\xi$ respectively and $g$ is integrable on $A \times B$ [20].

Notation. $C$ is a constant depending on the indicated parameters but not on the function, which is not necessarily the same at each occurrence; $z$ is a point in $D$, $t$ in the Bergman–Šilov boundary $b$ and $w$ in the unit disk $\Delta = \{w\colon |w| < 1\}$.

## 2. Elementary properties of $B_{pq}$ spaces

1. The space $B_{pq}$ $(0 < p < q, q \geqslant 1)$ is a Banach space with norm (1.1). Let $f \in B_{pq}$ and $0 < r < 1$. Then

(i) $|f(z)| \leqslant C_{pqN}(1-r)^{-N(1/p+1/q+1)}\|f\|_{B_{pq}}$.

(ii) The slice function $f_r \to f$ in $B_{pq}$ norm as $r \to 1$.

(iii) $H_p$ is a dense subset of $B_{pq}$.

(iv) $\|f\|_{B_{pq}} \leqslant C_{pqN}\|f\|_p$. This follows from [15], Theorem 4, with $k = q$.

The proofs of these properties are similar to those for the space $B_{p1}$ in [15], Theorem 11. The space $B_{pq}$ $(0 < p < q < 1)$ is a complete linear Hausdorff space which satisfies (ii)–(iv) with (i) replaced by $|f(rz)| \leqslant C_{pqN}(1-r)^{-N/P}\|f\|_{B_{pq}}$. $B_{p2}$ is a Hilbert space with inner product

$$(f, g) = \int_0^1 \int_b (1-r)^{2N(1/p-1/2)-1} f(rt)\,\bar{g}(rt)\,dr\,ds_t.$$

THEOREM 1. *The spaces $B_{pq}$ satisfy the inclusion relation $B_{pq} \subset B_{p'q'}$ $(0 < p < q, 0 < p' < q', 1 \leqslant q' \leqslant q)$ if $1/p' - 1/q' > 1/p - 1/q$.*

Proof. In the expression $\|f\|_{B_{p'q'}}^{q'}$ use Hölder's inequality on the integrals $M_{q'}^{q'}(r, f)$ and $\int_0^1 dr$ with exponent $q/q' \geqslant 1$ in each case. This gives

$$\|f\|_{B_{p'q'}}^{q'} \leqslant \|f\|_{B_{pq}}^{q'} \Big(\int_0^1 (1-r)^{\alpha}\,dr\Big)^{q'-1}$$

$$\text{with } \alpha = Nq'q''\left[\left(\frac{1}{p'} - \frac{1}{p}\right) - \left(\frac{1}{q'} - \frac{1}{q}\right)\right] - 1$$

so that $\int_0^1 (1-r)^{\alpha}\,dr$ converges if and only if $(1/p' - 1/q') > (1/p - 1/q)$.

THEOREM 2. *Let $0 < p < 1$. The space $B_{p1}$ has the Schur property, that is, if $\{f_n\}$ in $B_{p1}$ is a weak Cauchy sequence, then $\{f_n\}$ converges in norm to some element $B_{p1}$. The space $B_{p2}$ does not have this property.*

Proof. Let $T$ be a bounded linear transformation on $B_{p1}$. By hypothesis the sequence of numbers $\{T(f_n)\}$ is a Cauchy sequence. By property (i) evaluation at each point of $D$ is a bounded linear functional on $B_{p1}$ so that $\{f_n\}$ converges pointwise on $D$. Since $B_{p1}$ is a subset of Lebesgue space $L_1([0, 1] \times b)$, where the measure $V^{-1}(1-r)^{N(1/p-1)-1}\,dr\,ds_t$ is finite on $L_1$ for $0 < p < 1$, the result follows from a general result in $L_1$ spaces

[19], Theorem 5, p. 122. Since $B_{p2}$ is a Hilbert space it does not have the Schur property [2], p. 261.

**2.** The Szegö kernel, $S(z,\bar{t})$ $(z \in D, t \in b)$, of $D$ is an example of a function in $B_{pq}$ (for $q \geqslant 2$) that is not in $H_p$ for some classical symmetric spaces. This function is holomorphic on $D \times \bar{D}$ and $S(rz,\bar{t}) = S(z, r\bar{t})$. Let $S_{\bar{t}}$ be the partial function given by $S(z,\bar{t}) = S_{\bar{t}}(z)$.

THEOREM 3. *The Szegö kernel $S_{\bar{t}}$ of $D$ belongs to $B_{pq}$ for $0 < p < 1$, $q \geqslant 2$. If $D$ is the classical symmetric space $R_I(2,2)$ $(R_{II}(2))$ [13], $S_{\bar{t}} \notin H_p$ for $\frac{1}{2} \leqslant p < 1$ $(\frac{2}{3} \leqslant p < 1)$.*

Proof. Let $q \geqslant 2$. Then

$$\|S_{\bar{t}}\|_{B_{pq}}^{q'} \leqslant \int_0^1 (1-r)^{Nq(1/p-1/q)-1} \max_{v \in b} |S(rt,v)|^{q-2} M_2^2(r, S_{\bar{t}})\, dr. \tag{1}$$

By Cauchy's formula and Theorem 4.5.1 of [13]

$$M_2^2(r, S_{\bar{t}}) = S(rt, r\bar{t}) = V^{-1}(1-r^2)^{-N}, \tag{2}$$

and by the maximum principle

$$\max_{v \in b} |S(rt,\bar{v})| = V^{-1}(1-r)^{-N}. \tag{3}$$

Note that the bounds in (2) and (3) are sharp. Setting (3) and (2) into (1), gives $\|S_{\bar{t}}\|_{B_{pq}} < \infty$. From Theorem 2 of [15] on $R_I(2,2)$ $S_{\bar{t}} \notin H_p$ for $\frac{1}{2} \leqslant p < 1$ (but $\in H_p$ for $0 < p < \frac{1}{2}$). Similarly $S_{\bar{t}} \notin H_p$ on $R_{II}(2)$ if $\frac{2}{3} < p < 1$ but $\in B_{pq}$.

## 3. Properties of fractional derivatives and integrals for functions of space $B_{pq}$

Let $f$ be holomorphic on $D$ and $\gamma \geqslant 0$. The *$\gamma$-th fractional derivative* of $f$ is

$$f^{[\gamma]}(z) = \sum \frac{\Gamma(k+1+\gamma)}{\Gamma(k+1)} a_{k\nu} \varphi_{k\nu}(z), \tag{1}$$

and the *$\gamma$-th fractional integral* is

$$f_{[\gamma]}(z) = \sum \frac{\Gamma(k+1)}{\Gamma(k+1+\gamma)} a_{k\nu} \varphi_{k\nu}(z). \tag{2}$$

(See (1.2) for the series expansion of $f$.) Since series (1) and (2) converge absolutely and uniformly on compact subsets of $D$ [15] and $\varphi_{k\nu}$ are holomorphic on $D$, $f^{[\gamma]}$ and $f_{[\gamma]}$ are holomorphic on $D$.

Lemma 1 gives a connection between $A_k(t)$ defined by

$$A_k(t) = \sum_{\nu=1}^{m_k} a_{k\nu} \varphi_{k\nu}(t)$$

and $B_{pq}$. Let $M_{1,q}(r, f_t)$ be the $q$-th mean of the partial function $f_t(w) = f(wt)$ ($t \in b$, $w \in$ the unit disk $\Delta$) and $\|f_t\|_{1,B_{pq}}$ be given by (1.1) with $M_q^q(r, f)$ replaced by $M_{1,q}^q(r, f_t)$.

LEMMA 1. *Let* $0 < p < q$, $q \geqslant 1$.

(i) *If* $f \in B_{pq}$, *then*

$$|A_k(t)| \leqslant C_{pqN} k^{N(1/p-1/q)} \|f_t\|_{1,B_{pq}}$$

*for almost all* $t \in b$ *and* $\|f_t\|_{1,B_{pq}} \in L_q(b)$.

(ii) *Let* $|A_k(t)| \leqslant C_{pqN} k^{\alpha} |h(t)|$, *where* $h \in L_q(b)$. *If* $1 \leqslant q \leqslant 2$ *and* $\alpha < N(1/p-1/q)-\frac{1}{2}$, *then* $f \in B_{pq}$. *If* $q > 2$ *and* $\alpha < N(1/p-1/q)-1+1/q$, *then* $f \in B_{pq}$.

Proof. Let $f(z) = \sum a_{k\nu} \varphi_{k\nu}(z) \in B_{pq}$ and $w \in \Delta$. Then for $t \in b$, $f(wt) = f_t(w) = \sum_{k=0}^{\infty} A_k(t) w^k$ is analytic on $\Delta$ with Fourier coefficients

$$A_k(t) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f_t(w)}{w^{k+1}} dw \quad (r < 1). \tag{3}$$

Using Hölder's inequality on the right of (3) gives

$$|A_k(t)| \leqslant r^{-k} M_{1,q}(r, f_t). \tag{4}$$

Form $\|f_t\|_{1,B_{pq}}^q$, integrate over $b$, use Fubini's theorem and (1.3) to obtain

$$\frac{1}{V} \int_b \|f_t\|_{1,B_{pq}}^q ds_t = \|f\|_{B_{pq}}^q < \infty. \tag{5}$$

Thus $\|f_t\|_{1,B_{pq}}$ is an integrable function of $t$ and finite for almost all $t$. Since $|f|^q$ is plurisubharmonic, $|f_t(w)|^q$ is subharmonic in $w$ in every component of the open set $O_t = \{w\colon wt \in D\}$ [17]. Thus its mean is non-decreasing so that

$$\|f_t\|_{1,B_{pq}}^q \geqslant M_{1,q}^q(r, f_t) \int_r^1 (1-\varrho)^{Nq(1/p-1/q)-1} d\varrho$$

or

$$M_{1,q}^q(r, f_t) \leqslant C_{pqN} (1-r)^{-Nq(1/p-1/q)} \|f\|_{1,B_{pq}}^q. \tag{6}$$

Substituting (6) in the right of (4) and setting $r = 1-1/k$ gives (i).

(ii) Let $1 \leqslant q \leqslant 2$. By Hölder's inequality and the hypotheses on $A_k(t)$

$$M_{1,q}^q(r, f_t) \leqslant M_{1,2}^q(r, f_t) = \Big(\sum_{k=0}^{\infty} |A_k(t)|^2 r^{2k}\Big)^{q/2}$$
$$\leqslant C_{pqN} |h(t)|^q \Big(\sum_{k=0}^{\infty} k^{2\alpha} r^{2k}\Big)^{q/2} = O\big(|h(t)|^q (1-r)^{-(1+2\alpha)q/2}\big).$$

Hence $\|f_t\|_{1,B_{pq}}$ is finite for almost all $t$ if $\alpha < N(1/p-1/q)-\frac{1}{2}$. The finiteness of $\|f\|_{B_{pq}}$ follows from (5) and the hypothesis on $h$.

For $q > 2$ we note that the Fourier coefficients of $f_r(z)$ are $a_{k\nu}r^k$. Hence the Fourier coefficients of $f_{t,r}(w)$ are $A_k(t)r^k$ for $k \geq 0$ and 0 for $k < 0$. Also $f_{t,r} \in L_q(0, 2\pi)$ for all $t \in b$. Thus the Hausdorff–Young inequality [20] and the hypothesis on $A_k(t)$ in (ii) give

$$M_{1,q}(r, f_t) \leq \Big(\sum_{k=0}^{\infty} |A_k(t)r^k|^{q'}\Big)^{1/q'} \quad (1/q+1/q' = 1)$$

$$\leq C\Big(\sum_{k=0}^{\infty} k^{\alpha q'} r^{kq'}\Big)^{1/q'} |h(t)| = O\big((1-r)^{-1/q'-\alpha}|h(t)|\big).$$

The rest of the proof follows as in the case $1 \leq q \leq 2$.

The exponent $N(1/p-1/q)-\frac{1}{2}$ is best possible for $1 \leq q \leq 2$. An example may be constructed similarly as in [1], Theorem 4, which shows that there exists a function $f_0(z) = \sum a_{k\nu}\varphi_{k\nu}(z)$ with $A_k(t) = O(k^{N(1/p-1/q)-\frac{1}{2}})$ which $\notin B_{pq}$.

We now prove

THEOREM 4. *Let* $0 < p < p' < q$, $q \geq 1$ *and* $\beta = N(1/p-1/p')$.

(i) *If* $f \in B_{pq}$, *then* $f_{[\beta]} \in B_{p'q}$.

(ii) *If* $f \in B_{p'q}$, *then* $f^{[\beta]} \in B_{pq}$.

This generalizes [1], Theorem 5, to bounded symmetric domains and spaces $B_{pq}$.

Proof. 1° We first prove that if $f = \partial g/\partial w \in B_{pq}$ for $|w| < 1$, then $g \in B_{p'q}$. (Note that the proof in [1] does not hold for $q > 1$.) By the fundamental theorem of the calculus

$$|g(tw)| \leq \left|\int_0^r \frac{\partial g_t}{\partial \sigma}(\sigma e^{i\theta})\,d\sigma + g(0)\right| \qquad (w = re^{i\theta}).$$

Set $\theta = 0$ and form the $q$-th mean of $g$:

$$M_q(r, g) = \left\{\frac{1}{V}\int_b ds_t \left|\int_0^r \frac{\partial g_t(\sigma)}{\partial \sigma}\,d\sigma + \frac{1}{r}\,g(0)\right|^q\right\}^{1/q}. \tag{7}$$

Use (1.4) and Minkowski's inequality on the right of (7) to get

$$M_q(r, g) \leq \int_0^r M_q\left(\sigma, \frac{\partial g}{\partial \sigma}(\ )\right) d\sigma + |g(0)|$$

so that by the monotonicity of the mean

$$M_q(r, g) \leq M_q\left(r, w\frac{\partial g}{\partial \sigma}\right) + |g(0)|. \tag{8}$$

The function $g^{[1]}(wt) = g(wt)+w\partial g(wt)/\partial w$. Using the inequality $(a+b)^q \leqslant C(a^q+b^q)$, $a$, $b \geqslant 0$, and (8) we get

$$M_q^q(r, g^{[1]}) \leqslant C\left[M_q^q\left(r, w\frac{\partial g}{\partial w}\right)+|g(0)|^q\right].$$

Forming the $B_{pq}$ mean

$$\|g^{[1]}\|_{B_{pq}} \leqslant C\left[\left\|w\frac{\partial g}{\partial w}\right\|_{B_{pq}}+|g(0)|\right]. \tag{9}$$

(Thus $\partial g/\partial w \in B_{pq}$ implies that $g^{[1]} \in B_{pq}$.) We now prove that

$$\|g\|^q_{B_{p'q}} \leqslant C\,\|g^{[1]}\|_{B_{pq}}. \tag{10}$$

Theorem 4(i) follows from (9) and (10) if $\beta = 1$. By induction (i) holds for any positive integer $m$.

To prove (10) we use a weak form of an inequality for Riemann–Liouville integrals. In the integral

$$\|g\|^q_{B_{p'q}} = \int_0^1 (1-R)^{Nq(1/p'-1/q)-1} M_q^q(R, g)\,dR$$

set $R = r^{q+1}$ and use the monotonicity of the mean and the inequality $(1-r^{q+1})^b \leqslant C(1-r)^b$ to obtain

$$\|g\|^q_{B_{p'q}} \leqslant C\int_0^1 (1-r)^{Nq(1/p'-1/q)-1} M_q^q(r, g)\,r^q\,dr. \tag{11}$$

From the series expansion of $g_r^{[1]}$ we have

$$g(rt) = 2\int_0^1 g^{[1]}(r\varrho^2 t)\varrho\,d\varrho$$

so that by (1.4)

$$M_q(r, g) \leqslant 2\int_0^1 M_q(r\varrho^2, g^{[1]})\varrho\,d\varrho.$$

Set $\sigma = r\varrho^2$ in the integral on the right to get

$$M_q(r, g) \leqslant \frac{1}{r}\int_0^r M_q(\sigma, g^{[1]})\,d\sigma. \tag{12}$$

Using (12) in (11) gives

$$\|g\|^q_{B_{p'q}} \leqslant \int_0^1 (1-r)^{Nq(1/p'-1/q)-1}\Big(\int_0^r M_q(\sigma, g^{[1]})\,d\sigma\Big)^q.$$

By formula (9.2), p. 758 of [5] and the fact that $1 = N(1/p-1/p')$ we get

$$\|g\|_{B_{p'q}}^{q} \leqslant C \int_0^1 (1-r)^{Nq(1/p-1/q)-1} M_q^q(r, g^{[1]})\,dr = C\,\|g^{[1]}\|_{B_{pq}},$$

which is (10).

2° To prove (ii) use the formula

$$f^{[\beta]}(r^2 t) = \frac{\Gamma(\beta+1)}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta} t)}{(1-re^{-i\theta})^{\beta+1}}\,d\theta, \quad r < 1, \tag{13}$$

[15], (5.3). Form the $q$-th mean on both sides of (13). Then use (1.4), (1.3) and [4], Lemma p. 65, on the right. This gives

$$M_q^q(r^2, f^{[\beta]}) \leqslant C_{pqN} \frac{M_q^q(r, f)}{(1-r)^{\beta q}}.$$

Form $\|f^{[\beta]}\|_{B_{pq}}$ on the left with $r^2$ as variable of integration. Since $Nq(1/p-1/q)-1-\beta q = Nq(1/p'-1/q)-1$, we get $\|f\|_{B_{p'q}}$, which is finite, on the right so that $f^{[\beta]} \in B_{pq}$.

3° The proof of (i) for real positive $\beta$ is similar to the proof for the unit disk [1], Theorem 5, using (ii), (i) for $m$ a positive integer and both parts of Lemma 1.

## 4. $ph_p$ and $b_{pq}$ spaces of pluriharmonic functions

**1. Definitions and elementary properties.** A continuous real function $u$ on $D$ is pluriharmonic if for every holomorphic mapping $\gamma$ of $\Delta$ into $D$, $u \circ \gamma$ is harmonic in $\Delta$ [7]. Since $D$ is simply-connected [12], p. 311, every pluriharmonic function on $D$ is the real part of a holomorphic function [17], p. 44. A pluriharmonic function is plurisubharmonic.

Let $ph_p$ and $b_{pq}$ be the spaces of pluriharmonic functions analogous to the spaces $H_p$ and $B_{pq}$ of holomorphic functions respectively. The space $b_{pq}$ for $q \geqslant 1$ is a Banach space and properties (i) and (ii) of Section 2 holds as for $B_{pq}$ spaces. However, there is no inequality similar to that in (iv) for pluriharmonic functions if $0 < p < 1$; for $N = 1$ a counterexample [2], p. 257 shows that $ph_p \not\subset b_{p1}$. The inclusion relation of Theorem 1 holds for $b_{pq}$ and the Schur property for $b_{p1}$.

**2. Comparison of $ph_p$ and $b_{pq}$ spaces.** Let $u$ be pluriharmonic on $D$. Then $u = \operatorname{Re} f$, where $f = u+iv$ is holomorphic on $D$ and $v$ is the pluriharmonic conjugate of $u$. M. Stoll proved that if $u \in ph_q$ $(1 < q < \infty)$, then $v \in ph_q$ [16]. We prove that if $u \in ph_1$, then $v \in ph_p$ for all $p < 1$, which generalizes Kolmogorov's theorem for the unit disk [4]. If $N = 1$ and $p < 1$ a counterexample shows that $u \in ph_p$ does not imply that $v \in ph_q$ for any $q > 0$ [4], p. 65.

The proof of Theorem 5 is essentially due to my student Pui-Wah Chan.

THEOREM 5. *If* $u \in ph_1$, *then* $v \in ph_p$ *for all* $p < 1$ *and* $M_p(r, v) \leqslant CM_1(r, u)$, $0 < r < 1$.

Proof. Let $M_{1,1}(r, u_t)$ be the first mean of the partial function $u_t$ $(t \in b)$. By (1.3) $\int_b M_{1,1}(r, u_t)\, ds_t = M_1(r, u)$. Since $|u|$ is plurisubharmonic on $D$, $|u_t|$ is subharmonic on $\Delta$ for every $t \in b$ [17]. Thus $M_{1,1}(r, u)$ is monotone in $r$ for all $t \in b$ so that by the monotone convergence theorem $V\|u\|_1 = \int_b \|u_t\|_{1,1}\, ds_t$. Thus $u \in ph_1(D)$ implies $u_t \in ph_1(\Delta)$ for almost all $t \in b$. Since $v_t$ is the harmonic conjugate of $u_t$ on $\Delta$ by Kolmogorov's theorem

$$M_{1,p}(r, v_t) \leqslant C_p M_{1,1}(r, u_t), \tag{1}$$

[4], p. 57, for $0 \leqslant r < 1$ and almost all $t \in b$, where $C_p$ is independent of $u_t$. Raise both sides of (1) to the pth power and integrate over $b$. This gives $M_p^p(r, v) \leqslant C_p^p V^{-1} \int_b M_{1,1}^p(r, u_t)\, ds_t$. The result follows by using Hölder's inequality on the right with exponent $1/p > 1$.

For $b_{pq}$ spaces we prove much more, generalizing Theorem 1 of [2] for $b_{p1}$ spaces of harmonic functions on the unit disk.

THEOREM 6. *Let* $0 < p < 1$ *and* $p < q < \infty$. *Then* $b_{pq}$ *is a self-conjugate class, that is, if* $u \in b_{pq}$, *then* $v \in b_{pq}$.

Proof. We use partial function techniques and the methods of Duren and Shields in [2]. Since $f_t$ is analytic in $\Delta$ for $t \in b$,

$$f_t(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varrho e^{i\theta} + w}{\varrho e^{i\theta} - w}\, u_t(\varrho e^{i\theta})\, d\theta + iC_t,$$

for $|w| < \varrho < 1$, where $C_t$ is independent of $w$ [2], p. 256. Differentiate with respect to $w$ and set $w = re^{i\varphi}$, $\theta - \varphi = \theta'$. This gives

$$\left|\frac{df_t(w)}{dw}\right| \leqslant \frac{1}{\pi} \int_0^{2\pi} \frac{|u_t(\varrho e^{i(\theta' + \varphi)})|\, d\theta'}{\varrho^2 + r^2 - 2\varrho r \cos\theta'}.$$

Form the $q$-th mean on both sides and use (1.4) and (1.3) on the right. This gives

$$M_q(r, \partial f/\partial w) \leqslant \frac{2M_q(\varrho, u)}{\varrho^2 - r^2}$$

or setting $\varrho = \frac{1}{2}(1+r)$

$$M_q(r, \partial f/\partial w) \leqslant 4\, \frac{M_q(\varrho, u)}{1 - \varrho}. \tag{2}$$

Form $\|\partial f/\partial w\|_{B_{p'q}}$ on the left of (2), where $p' = Np/(p+N)$. Since

$Nq(1/p'-1/q)-1-q = Nq(1/p-1/q)-1$, the right-hand side equals $C\|u\|_{b_{pq}}$, where $u \in b_{pq}$, so that $\partial f/\partial w \in B_{p'q}$. By Theorem 4(i) $f \in B_{pq}$. Hence $v \in b_{pq}$.

**3. An example.** Let $k$ be a positive integer, $p = 1/(k+1)$ and $D$ a polydisk. The function

$$F(z) = e^{i\pi k/2}(1-z_1)^{-k-1}$$

$\notin H^p$ but $\operatorname{Re} f \in ph_p$. This follows since

$$M_p^p(r,f) = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \dots \int_0^{2\pi} |F(z)|^p d\theta_1 \dots d\theta_N = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta_1}{|1-z_1|} \sim \log\frac{1}{1-r}$$

$(z = (re^{i\theta_1}, \dots, re^{i\theta_N}))$ but

$$M_p^p(r, \operatorname{Re} F) = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \dots \int_0^{2\pi} |\operatorname{Re} F(z)|^p d\theta_1 \dots d\theta_N$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{e^{ik\pi/2}}{(1-z_1)^{k+1}}\right)\right|^p d\theta_1$$

is bounded independently of $r$ [10], p. 416–417. If $N = 1$, $F \notin B_{p1}$ [2], p. 257, so that by Theorem 6 $\operatorname{Re} F \notin b_{p1}$. Also by Theorem 1 $F \notin B_{p_1 q_1}$, $\operatorname{Re} F \notin b_{p_1 q_1}$ for any $p_1$, $q_1$ with $q_1 > 1$ and $q_1 > p_1 > (k+q_1^{-1})^{-1}$.

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