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On differentiable solutions of the functional equation $\varphi(f(x)) = g(x, \varphi(x))$ for vector-valued functions

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1. Introduction. The object of this paper is the problem of the existence of solutions of the functional equation

$$\varphi(f(x)) = g(x, \varphi(x)), \tag{1}$$

which are of class C^r in an open interval (a, b). f and g denote the known functions and φ is the unknown function. The functions φ and g are vector-valued functions with ranges contained in R^m.

The bold type denotes points of the space R^m, where in the sequel m is fixed, $m \geqslant 2$.

We define the norm of an element $u \in R^m$ by

$$\|u\| = \sqrt{\sum_{i=1}^{m} u_i^2}.$$

This paper is based on [1] and [2] (cf. also [4]). In [1] Choczewski proved a theorem on the existence of infinitely many C^r solutions $(0 < r \leqslant \infty)$ of equation (1) for $m = 1$. The author supposed that the sets Γ_x and $\Omega_{f(x)}$ (the definition is given in Section 3) are identical. In our paper we assume that the symmetric difference of these sets is at most countable. For $r = 0$ the analogous results was obtained in [2].

2. The discussion of C^r solution of equation (1) is carried out with the aid of a certain theorem which is proved by means of some considerations of a topological nature.

THEOREM 1. *We assume that:*

1. *Functions* $u_n(x)$, $n = 1, 2, \ldots$, *are vector-valued functions defined in* $\langle 0, 1\rangle$, *and* $V_0^1(u_n) < \infty$, *where the symbol* $V_0^1(u_n)$ *denotes the variation of the function* u_n^i *in* $\langle 0, 1\rangle$, $i = 1, 2, \ldots, m$.

2. *For every* $x \in \langle 0, 1\rangle$ *we have* $\|u_n(x)\| \leqslant K$, $n = 1, 2, \ldots$, *where* $K > 0$.

3. *The vectors* $a_0 \in R^m$ *and* $a_1 \in R^m$ *fulfil the conditions*

$$\|a_0\| < K, \quad \|a_1\| < K,$$

and, moreover,

$$u_n(0) \neq a_0, \quad u_n(1) \neq a_1.$$

Then for every two systems of vectors $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r$ *and* $\boldsymbol{l}'_1, \ldots, \boldsymbol{l}'_r$, $0 < r \leqslant \infty$, *there exists a vector-valued function* $\boldsymbol{u}_0 \in C^r(\langle 0, 1\rangle)$ *such that*

$$\boldsymbol{u}_0(0) = \boldsymbol{a}_0, \quad \boldsymbol{u}_0(1) = \boldsymbol{a}_1,$$
$$\boldsymbol{u}_0^{(k)}(0) = \boldsymbol{l}_k, \quad \boldsymbol{u}_0^{(k)}(1) = \boldsymbol{l}'_k, \quad k = 1, 2, \ldots, r,$$
$$\|\boldsymbol{u}_0(x)\| \leqslant K \qquad \text{for every } x \in \langle 0, 1\rangle,$$

and

$$\boldsymbol{u}_0(x) \neq \boldsymbol{u}_n(x) \quad \text{for every } x \in \langle 0, 1\rangle, \ n = 1, 2, \ldots$$

Proof [1]. We take a function $\bar{\boldsymbol{u}}(x)$ such that $\bar{\boldsymbol{u}}(x) \in C^r(\langle 0, 1\rangle)$,

$$\bar{\boldsymbol{u}}(0) = \boldsymbol{a}_0, \quad \bar{\boldsymbol{u}}(1) = \boldsymbol{a}_1, \quad \bar{\boldsymbol{u}}^{(k)} = \boldsymbol{l}_k, \quad \bar{\boldsymbol{u}}^{(k)}(1) = \boldsymbol{l}'_k, \quad k = 1, 2, \ldots, r,$$
$$\|\bar{\boldsymbol{u}}(x)\| \leqslant \max(\|\boldsymbol{a}_0\|, \|\boldsymbol{a}_1\|) \overset{\mathrm{df}}{=} K_0 \quad \text{for every } x \in \langle 0, 1\rangle,$$

and a function $\lambda(x)$ such that $\lambda(x) \in C^r(\langle 0, 1\rangle)$, $\lambda^{(k)}(0) = \lambda^{(k)}(1) = 0$, $k = 0, 1, 2, \ldots, r$, $0 < \lambda(x) \leqslant (K - K_0)/K$ for $x \in (0, 1)$ and we consider the m-parametre family of functions

$$\mathscr{E} = \{\boldsymbol{u}(x, \boldsymbol{t}) \colon \boldsymbol{u}(x, \boldsymbol{t}) = \bar{\boldsymbol{u}}(x) + \boldsymbol{t}\lambda(x), \ x \in \langle 0, 1\rangle, \ \boldsymbol{t} \in R^m\}.$$

We assume that $\|\boldsymbol{t}\| \leqslant K$. For every $\boldsymbol{t}$, $\|\boldsymbol{t}\| \leqslant K$, the functions of the family $\mathscr{E}$ have the following conditions:

$$\boldsymbol{u}(x, \boldsymbol{t}) \in C^r(\langle 0, 1\rangle), \quad \|\boldsymbol{u}(x, \boldsymbol{t})\| \leqslant K \quad \text{for every } x \in \langle 0, 1\rangle,$$
$$\boldsymbol{u}(0, \boldsymbol{t}) = \boldsymbol{a}_0, \quad \boldsymbol{u}(1, \boldsymbol{t}) = \boldsymbol{a}_1, \quad \boldsymbol{u}^{(k)}(0, \boldsymbol{t}) = \bar{\boldsymbol{u}}^{(k)}(0) = \boldsymbol{l}_k,$$
$$\boldsymbol{u}^{(k)}(1, \boldsymbol{t}) = \bar{\boldsymbol{u}}^{(k)}(1) = \boldsymbol{l}'_k.$$

If we introduce in $\mathscr{E}$ the metric

$$\varrho\big(\boldsymbol{u}(x, \boldsymbol{t}_1), \ \boldsymbol{v}(x, \boldsymbol{t}_2)\big) = \max_{1 \leqslant i \leqslant m} \sup_{0 \leqslant x \leqslant 1} |u_i(x, \boldsymbol{t}_1) - v_i(x, \boldsymbol{t}_2)|,$$

the family $\mathscr{E}$ becomes a complete metric space.

Now we define the sets:

$$E_n = \big\{\boldsymbol{u}(x, \boldsymbol{t}) \in \mathscr{E} \colon \underset{x \in (0,1)}{\exists} \boldsymbol{u}(x, \boldsymbol{t}) = \boldsymbol{u}_n(x)\big\}, \qquad n = 1, 2, \ldots$$

We shall prove that the E_n are nowhere dense sets. In order to attain this we shall prove that the sets

$$Z_n = \big\{\boldsymbol{t} \in R^m \colon \exists\, x \in (0, 1)\, \boldsymbol{u}(x, \boldsymbol{t}) = \boldsymbol{u}_n(x), \ \|\boldsymbol{t}\| \leqslant K\big\},$$

are nowhere dense. We have

$$Z_n = \bigcup_{j=3}^{\infty} Z_{nj},$$

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[1] I should like to express my thanks to Prof. dr A. Pliś for his very valuable remarks concerning the proof of this theorem.

where

$$Z_{nj} = \Big\{ t \in R^m: \underset{x \in \langle \frac{1}{j}, 1-\frac{1}{j} \rangle}{\exists} u(x, t) = u_n(x) \Big\}.$$

The sets Z_{nj} are subsets of the projections of the curves

$$x = s, \quad t = \gamma_n(s),$$

where

$$\gamma_n(s) = \frac{u_n(s) - \bar{u}(s)}{\lambda(s)}, \quad s \in \left\langle \frac{1}{j}, 1 - \frac{1}{j} \right\rangle,$$

on the space R^m. From Hypotheses 1 it follows that these are rectifiable curves and consequently their projections onto the space R^m have finite one-dimensional measure. It follows that they are nowhere dense sets and consequently the sets E_n are nowhere dense. Thus $\bigcup_{n=1}^{\infty} E_n$ is a set of first category, so there exists a function $u_0 \in \mathscr{E} \setminus \bigcup_{n=1}^{\infty} E_n$.

3. Let $\Omega \subset R^{m+1}$ ($m \geqslant 2$) be connected region and suppose that we are given a function

$$g: \Omega \to R^m.$$

For an arbitrary x we shall denote by Ω_x the x-section of the set Ω, i.e.

$$\Omega_x = \{ y: (x, y) \in \Omega \}.$$

We assume that $\langle a, b \rangle \subset \{x: \Omega_x \neq \emptyset\}$ and we put

$$\Gamma_x = g(x, \Omega_x), \quad \Omega' = \{(x, z): x \in \langle a, b \rangle, z \in \Gamma_x\}.$$

We make the following hypotheses:

(I) $g(x, y) \in C^r(\Omega)$, $0 < r \leqslant \infty$, and for every $x \in \langle a, b \rangle$, g is invertible with respect to y.

(II) $h(x, z) \in C^r(\Omega')$, where $h(x, z)$ is the inverse function to $g(x, y)$ with respect to y.

(III) $f(x) \in C^r(\langle a, b \rangle)$, where a and b are two consecutive roots of the equation $f(x) = x$, $f'(x) > 0$ in $\langle a, b \rangle$, and $f(x) > x$ for $x \in (a, b)$.

(IV) $\Gamma_x \div \Omega_{f(x)} = \bigcup_{i=1}^{\infty} \{\overset{(i)}{u}(x)\} \cup \bigcup_{j=1}^{\infty} \{\overset{(j)}{v}(x)\}$, where

$$\overset{(i)}{u}(x) \in \Gamma_x \setminus \Omega_{f(x)}, \quad \overset{(j)}{v}(x) \in \Omega_{f(x)} \setminus \Gamma_x,$$

and

$$\overset{(i)}{u}(x) \in C^0(\langle a, b \rangle), \quad \overset{(j)}{v}(x) \in C^0(\langle a, b \rangle), \quad \bigvee\nolimits_b^a(\overset{(i)}{u}) < \infty, \quad \bigvee\nolimits_b^a(\overset{(j)}{v}) < \infty.$$

(V) There exists a point $(x_0, \eta) \in \Omega$, $x_0 \in (a, b)$, such that $\eta = g(x_0, \eta)$.

We define functions $\boldsymbol{G}_k(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$ by the recurrent relations

$$\begin{aligned}\boldsymbol{G}_1(x, \boldsymbol{y}, \boldsymbol{y}_1) &= [f'(x)]^{-1}[\boldsymbol{g}'_x(x, \boldsymbol{y})+\boldsymbol{g}'_{\boldsymbol{y}}(x, \boldsymbol{y})\boldsymbol{y}_1],\\ \boldsymbol{G}_{k+1}(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k+1}) &= [f'(x)]^{-1}\left[\frac{\partial \boldsymbol{G}_k}{\partial x}+\frac{\partial \boldsymbol{G}_k}{\partial \boldsymbol{y}}\boldsymbol{y}_1+\ldots+\frac{\partial \boldsymbol{G}_k}{\partial \boldsymbol{y}_k}\boldsymbol{y}_{k+1}\right],\\ &\qquad k = 1, 2, \ldots, r-1.\end{aligned} \tag{6}$$

Similarly we define functions $\boldsymbol{H}_k(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$ by the relations

$$\begin{aligned}\boldsymbol{H}_1(x, \boldsymbol{y}, \boldsymbol{y}_1) &= \boldsymbol{h}'_x(x, \boldsymbol{y})+\boldsymbol{h}'_{\boldsymbol{y}}(x, \boldsymbol{y})f'(x)\boldsymbol{y}_1,\\ \boldsymbol{H}_{k+1}(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k+1}) &= \frac{\partial \boldsymbol{H}_k}{\partial x}+f'(x)\left(\frac{\partial \boldsymbol{H}_k}{\partial \boldsymbol{y}}\boldsymbol{y}_1+\ldots+\frac{\partial \boldsymbol{H}_k}{\partial \boldsymbol{y}_k}\boldsymbol{y}_{k+1}\right),\\ &\qquad k = 1, 2, \ldots, r-1.\end{aligned} \tag{6'}$$

Here the symbols $\partial \boldsymbol{G}_k/\partial \boldsymbol{y}$, $\partial \boldsymbol{G}_k/\partial \boldsymbol{y}_i$, $\partial \boldsymbol{H}_k/\partial \boldsymbol{y}$, $\partial \boldsymbol{H}_k/\partial \boldsymbol{y}_i$, $k = 1, 2, \ldots, r$ $i = 1, 2, \ldots, k$, denote matrices of order m.

From assumptions (I) and (III) it follows that the functions $\boldsymbol{G}_k(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$, $k = 1, 2, \ldots, r$, are defined and are of class C^{r-k} for $(x, \boldsymbol{y}) \in \Omega$ and arbitrary $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k$.

Similarly, by assumptions (II) and (III), the functions $\boldsymbol{H}_k(x, \boldsymbol{y}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$, $k = 1, 2, \ldots, r$, are defined and are of class C^{r-k} for $(x, \boldsymbol{y}) \in \Omega'$ and arbitrary $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k$.

One can easily verify that if $\boldsymbol{\varphi}(x)$ is a C^r solution of equation (1) or, equivalently, of the equation

$$\boldsymbol{\varphi}(x) = \boldsymbol{h}\big(x, \boldsymbol{\varphi}(f(x))\big), \tag{7}$$

and if hypotheses (I), (II) and (III) are fulfilled, then the derivatives $\boldsymbol{\varphi}^{(k)}(x)$ satisfy the equations (see [4]):

$$\boldsymbol{\varphi}^{(k)}(x) = \boldsymbol{H}_k\big(x, \boldsymbol{\varphi}(f(x)), \boldsymbol{\varphi}'(f(x)), \ldots, \boldsymbol{\varphi}^{(k)}(f(x))\big), \quad k = 1, 2, \ldots, r. \tag{8}$$

or

$$\boldsymbol{\varphi}^{(k)}\big(f(x)\big) = \boldsymbol{G}_k\big(x, \boldsymbol{\varphi}(x), \boldsymbol{\varphi}'(x), \ldots, \boldsymbol{\varphi}^{(k)}(x)\big), \quad k = 1, 2, \ldots, r. \tag{9}$$

Now we define sequences of functions $\overset{(i)}{\boldsymbol{u}}_n(x)$ and $\overset{(j)}{\boldsymbol{v}}_n(x)$. We put

$$\begin{aligned}&\overset{(i)}{\boldsymbol{u}}_0(x) = \overset{(i)}{\boldsymbol{u}}\big(f^{-1}(x)\big), && \overset{(j)}{\boldsymbol{v}}_0(x) = \overset{(j)}{\boldsymbol{v}}\big(f^{-1}(x)\big),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\overset{(i)}{\boldsymbol{u}}_{n+1}(x) = \boldsymbol{h}\big(x, \overset{(i)}{\boldsymbol{u}}_n(f(x))\big), && \overset{(j)}{\boldsymbol{v}}_{n+1}(x) = \boldsymbol{g}\big(f^{-1}(x), \overset{(j)}{\boldsymbol{v}}_n(f^{-1}(x))\big),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

$n = 0, 1, \ldots;\ i = 1, 2, \ldots;\ j = 1, 2, \ldots$

We shall prove the following

THEOREM 2. *If Hypotheses* (I) - (V) *are fulfilled, then for every* $\varepsilon > 0$ *and for every system of vectors* $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r$ *there exists a vector-valued function* $\boldsymbol{\varphi}(x)$ *with the following properties:*

$$\boldsymbol{\varphi}(x) \in C^r\big((a, b)\big), \tag{10}$$

$$\boldsymbol{\varphi}(x) \text{ satisfies equation (1) in } (a, b), \tag{11}$$

$$\|\boldsymbol{\varphi}(x) - \boldsymbol{\eta}\| < \varepsilon \quad \text{for every } x \in \langle x_0, f(x_0)\rangle, \tag{12}$$

$$\boldsymbol{\varphi}^{(k)}(x_0) = \boldsymbol{l}_k, \quad k = 1, 2, \ldots, r. \tag{13}$$

Proof. Let us write

$$x_n = f^n(x_0), \quad n = 0, \pm 1, \pm 2, \ldots,$$

where $f^n(x_0)$ denotes the n-th iterate of the function $f(x)$, i.e.

$$f^0(x) = x, \quad f^{n+1}(x) = f\big(f^n(x)\big), \quad f^{n-1}(x) = f^{-1}\big(f^n(x)\big),$$
$$n = 0, \pm 1, \pm 2, \ldots$$

From hypothesis (III) and from the lemmas and corollary proved in [3] it follows that

$$f(\langle a, b\rangle) = \langle a, b\rangle,$$

the sequence $f^n(x_0)$ is strictly increasing, and

$$\lim_{n\to\infty} f^n(x_0) = b.$$

Similarly, the sequence $f^{-n}(x_0)$ is strictly decreasing and

$$\lim_{n\to\infty} f^{-n}(x_0) = a.$$

Thus we may write

$$(a, b) = \bigcup_{-\infty}^{\infty} \langle x_{i-1}, x_i)$$

and we have

$$f\big(\langle x_{i-1}, x_i)\big) = \langle x_i, x_{i+1}).$$

We consider the interval $\langle x_0, f(x_0)\rangle \subset (a, b)$ and we define the sets $\overset{(i)}{F}_n$ of all points $x \in \langle x_0, f(x_0)\rangle$ for which the functions $\overset{(i)}{\bar{\boldsymbol{u}}}_n(x)$ are defined and the inequalities $\|\overset{(i)}{\bar{\boldsymbol{u}}}_n(x) - \boldsymbol{\eta}\| \leqslant \varepsilon$ hold for $i = 1, 2, \ldots, n = 0, 1, \ldots$; and similarly the sets $\overset{(j)}{F'_n}$ of all points $x \in \langle x_0, f(x_0)\rangle$ for which the functions $\overset{(j)}{\bar{\boldsymbol{v}}}_n(x)$ are defined and $\|\overset{(j)}{\bar{\boldsymbol{v}}}_n(x) - \boldsymbol{\eta}\| \leqslant \varepsilon$ for $j = 1, 2, \ldots, n = 0, 1, \ldots$, where ε is a fixed positive number.

From hypotheses (I), (II) and (IV) it follows that $\overset{(i)}{\bar{\boldsymbol{u}}}_n(x) \in C^0(\overset{(i)}{F}_n)$, $\overset{(j)}{\bar{\boldsymbol{v}}}_n(x) \in C^0(\overset{(j)}{F'_n})$. There are at most countably many sets $\overset{(i)}{F}_n$, $\overset{(j)}{F'_n}$.

Now we form sequences $\overset{(i)}{\boldsymbol{u}}_n(x)$ and $\overset{(j)}{\boldsymbol{v}}_n(x)$ such that $\overset{(i)}{\boldsymbol{u}}_n(x) \in C^0(\langle x_0, f(x_0)\rangle)$, $\overset{(j)}{\boldsymbol{v}}_n(x) \in C^0(\langle x_0, f(x_0)\rangle)$, $V_{x_0}^{f(x_0)}(\overset{(i)}{\boldsymbol{u}}_n) < \infty$, $V_{x_0}^{f(x_0)}(\overset{(j)}{\boldsymbol{v}}_n) < \infty$, $\overset{(i)}{\boldsymbol{u}}_{n|\overset{(i)}{\boldsymbol{F}}_n} = \overset{(i)}{\bar{\boldsymbol{u}}}_n$, $\overset{(j)}{\boldsymbol{v}}_{n|\overset{(j)}{\boldsymbol{F}}_n}' = \overset{(j)}{\bar{\boldsymbol{v}}}_n$, $n = 1, 2, \ldots$ The existence of sequences $\overset{(i)}{\boldsymbol{u}}_n$ and $\overset{(j)}{\boldsymbol{v}}_n$ fulfilling the above conditions is almost evident, therefore we omit the proof of this fact.

The vector $\boldsymbol{\eta}$ is a fixed point of the transformation $\bar{\boldsymbol{y}} = \boldsymbol{g}(x_0, \boldsymbol{y})$, since $\boldsymbol{g}(x_0, \boldsymbol{\eta}) = \boldsymbol{\eta}$. Thus there exists a neighbourhood U_η of the point $\boldsymbol{\eta}$, $U_\eta \subset \{\boldsymbol{y}, \|\boldsymbol{y}-\boldsymbol{\eta}\| < \varepsilon\}$, such that

$$\boldsymbol{g}(x_0, U_\eta) \subset \{\boldsymbol{y}, \|\boldsymbol{y}-\boldsymbol{\eta}\| < \varepsilon\}.$$

Since there are only at most countably many values $\bar{\boldsymbol{\eta}} \in U_\eta$ such that one of the equalities

$$\begin{gathered}\overset{(i)}{\boldsymbol{u}}_n(x_0) = \bar{\boldsymbol{\eta}}, \quad \overset{(i)}{\boldsymbol{u}}_n(f(x_0)) = \boldsymbol{g}(x_0, \bar{\boldsymbol{\eta}}), \\ \overset{(j)}{\boldsymbol{v}}_n(x_0) = \bar{\boldsymbol{\eta}}, \quad \overset{(j)}{\boldsymbol{v}}_n(f(x_0)) = \boldsymbol{g}(x_0, \bar{\boldsymbol{\eta}}), \\ i = 1, 2, \ldots, \; j = 1, 2, \ldots, \; n = 0, 1, 2, \ldots\end{gathered}$$

holds, there exists a value $\boldsymbol{\eta}^* \in U_\eta$ such that $\boldsymbol{g}(x_0, \boldsymbol{\eta}^*) \in \boldsymbol{g}(x_0, U_\eta)$ and

$$\begin{gathered}\overset{(i)}{\boldsymbol{u}}_n(x_0) \neq \boldsymbol{\eta}^*, \quad \overset{(i)}{\boldsymbol{u}}_n(f(x_0)) \neq \boldsymbol{g}(x_0, \boldsymbol{\eta}^*), \\ \overset{(j)}{\boldsymbol{v}}_n(x_0) \neq \boldsymbol{\eta}^*, \quad \overset{(j)}{\boldsymbol{v}}_n(f(x_0)) \neq \boldsymbol{g}(x_0, \boldsymbol{\eta}^*), \\ i = 1, 2, \ldots, \; j = 1, 2, \ldots, \; n = 0, 1, 2, \ldots\end{gathered}$$

As the sequence $\{\boldsymbol{u}_n(x)\}$ in Theorem 1 we take the sequence of all functions $\{\overset{(i)}{\boldsymbol{u}}_n(x)\}$ and $\{\overset{(j)}{\boldsymbol{v}}_n(x)\}$, as the points $(0, \boldsymbol{a}_0)$ and $(1, \boldsymbol{a}_1)$ we take the points $(x_0, \boldsymbol{\eta}^*)$ and $(f(x_0), \boldsymbol{g}(x_0, \boldsymbol{\eta}^*))$. Since $\boldsymbol{\eta}^* \in U_\eta$, we have $\|\boldsymbol{\eta}^* - \boldsymbol{\eta}\| < \varepsilon$ and, similarly, $\boldsymbol{g}(x_0, \boldsymbol{\eta}^*) \in \boldsymbol{g}(x_0, U_\eta)$ implies $\|\boldsymbol{g}(x_0, \boldsymbol{\eta}^*) - \boldsymbol{\eta}\| < \varepsilon$. Further, we take an arbitrary system of vectors $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r$ and then as the system $\boldsymbol{l}_1', \ldots, \boldsymbol{l}_r'$ we take respectively $\boldsymbol{G}_1(x_0, \boldsymbol{\eta}^*, \boldsymbol{l}_1)$, $\boldsymbol{G}_2(x_0, \boldsymbol{\eta}^*, \boldsymbol{l}_1, \boldsymbol{l}_2), \ldots$, $\boldsymbol{G}_r(x_0, \boldsymbol{\eta}^*, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_r)$.

From Theorem 1 it follows that there exists a vector-valued function

$$\boldsymbol{u}(x) \in C^r(\langle x_0, f(x_0)\rangle) \tag{13'}$$

such that

$$\boldsymbol{u}(x_0) = \boldsymbol{\eta}^*, \quad \boldsymbol{u}(f(x_0)) = \boldsymbol{g}(x_0, \boldsymbol{\eta}^*), \tag{14'}$$

$$\boldsymbol{u}^{(k)}(x_0) = \boldsymbol{l}_k, \quad \boldsymbol{u}^{(k)}(f(x_0)) = \boldsymbol{G}_k(x_0, \boldsymbol{\eta}^*, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_k), \quad k = 1, 2, \ldots, r, \tag{15'}$$

$$\begin{gathered}\boldsymbol{u}(x) \neq \overset{(i)}{\boldsymbol{u}}_n(x), \quad \boldsymbol{u}(x) \neq \overset{(j)}{\boldsymbol{v}}_n(x), \\ i = 1, 2, \ldots, \; j = 1, 2, \ldots, \; n = 0, 1, 2, \ldots,\end{gathered} \tag{16'}$$

and

$$\|\boldsymbol{u}(x) - \boldsymbol{\eta}\| < \varepsilon \quad \text{for every } x \in \langle x_0, f(x_0)\rangle. \tag{17'}$$

Thus the function $\boldsymbol{u}(x)$ fulfils the following conditions:

$$\boldsymbol{u}(x) \in C^r(\langle x_0, f(x_0)\rangle), \tag{13}$$

$$\boldsymbol{u}(f(x_0)) = \boldsymbol{g}(x_0, \boldsymbol{u}(x_0)), \tag{14}$$

$$\boldsymbol{u}^{(k)}(f(x_0)) = \boldsymbol{G}_k(x_0, \boldsymbol{u}(x_0), \boldsymbol{u}'(x_0), \ldots, \boldsymbol{u}^{(k)}(x_0)), \quad k = 1, 2, \ldots, r, \ 0 < r \leqslant \infty. \tag{15}$$

$$\boldsymbol{u}(x) \neq \overset{(i)}{\boldsymbol{u}}_n(x) \quad \boldsymbol{u}(x) \neq \overset{(j)}{\boldsymbol{v}}_n(x), \quad i = 1, 2, \ldots, \ j = 1, 2, \ldots, \ n = 0, 1, 2, \ldots \tag{16}$$

$$\|\boldsymbol{u}(x) - \boldsymbol{\eta}\| < \varepsilon \quad \text{for every } x \in \langle x_0, f(x_0)\rangle. \tag{17}$$

Now we can define the function $\varphi(x)$, the existence of which is asserted by the present theorem. We put

$$\varphi(x) = \begin{cases} \boldsymbol{u}(x) & \text{for } x \in \langle x_0, x_1), \\ \boldsymbol{g}(f^{-1}(x), \varphi(f^{-1}(x))) & \text{for } x \in \langle x_n, x_{n+1}), \ n = 1, 2, \ldots, \\ \boldsymbol{h}(x, \varphi(f(x))) & \text{for } x \in \langle x_{-n}, x_{-n+1}), \ n = 1, 2, \ldots \end{cases} \tag{18}$$

From the theorem proved in [2] it follows that the function $\varphi(x)$ defined by (18) is a continuous solution of equation (1).

We shall prove that under hypotheses of the present theorem formula (18) defines a function of class C^r in the interval (a, b).

First we shall prove that the function $\varphi(x)$ is of class C^r in the interval (x_0, x_n) for every n. For $n = 1$ it is true on account of (18) and (13). If $n = 2$, then $\varphi(x)$ is of class C^r in $(x_0, x_1) \cup (x_1, x_2)$ on account of (18) and (13) and by assumptions (I) and (III).

Further, from (9) it follows that

$$\varphi^{(k)}(x) = \boldsymbol{G}_k(f^{-1}(x), \boldsymbol{u}(f^{-1}(x)), \boldsymbol{u}'(f^{-1}(x)), \ldots, \boldsymbol{u}^{(k)}(f^{-1}(x))) \quad \text{for } x \in (x_1, x_2).$$

As $x \to x_1 + 0$, $f^{-1}(x) \to x_0 + 0$, then by the continuity of thefunctions $\boldsymbol{G}_k$ and by (15) we have

$$\lim_{x \to x_1 + 0} \varphi^{(k)}(x) = \lim_{x \to x_1 + 0} \boldsymbol{G}_k(f^{-1}(x), \boldsymbol{u}(f^{-1}(x)), \ldots, \boldsymbol{u}^{(k)}(f^{-1}(x))) = \boldsymbol{G}_k(x_0, \boldsymbol{u}(x_0), \boldsymbol{u}'(x_0), \ldots, \boldsymbol{u}^{(k)}(x_0)) = \boldsymbol{u}^{(k)}(f(x_0)), \tag{19}$$

$$\lim_{x \to x_1 - 0} \varphi^{(k)}(x) = \lim_{x \to x_1 - 0} \boldsymbol{u}^{(k)}(x) = \boldsymbol{u}^{(k)}(x_1) = \boldsymbol{u}^{(k)}(f(x_0)). \tag{20}$$

From (19) and (20) it follows that the derivatives $\varphi^{(k)}(x)$, $k = 1, 2, \ldots, r$ exist and are continuous at the point x_1, thus $\varphi(x) \in C^r((x_0, x_2))$.

Now we suppose $\varphi(x) \in C^r(x_0, x_n)$ for a certain index $n \geqslant 2$. For $x \in (x_n, x_{n+1})\, \varphi(x) = g\big(f^{-1}(x), \varphi(f^{-1}(x))\big)$ is evidently of class C^r. Further we have for $x \in (x_1, x_n) \cup (x_n, x_{n+1})$

$$\varphi^{(k)}(x) = G_k\Big(f^{-1}(x), u\big(f^{-1}(x)\big), u'\big(f^{-1}(x)\big), \ldots, u^{(k)}\big(f^{-1}(x)\big)\Big), \qquad k = 1, 2, \ldots, r,$$

and since all the derivatives $\varphi', \varphi'', \ldots, \varphi^{(k)}$ exist and are continuous at the point x_{n-1}, there exist the limits $\lim\limits_{x \to x_n} \varphi^{(k)}(x)$. Thus $\varphi(x) \in C^r\big((x_0, x_{n+1})\big)$, and, by induction, we see that $\varphi(x) \in C^r\big(\langle x_0, x_n)\big)$ for every n. It is known that $\lim\limits_{n \to \infty} x_n = b$, thus $\varphi(x) \in C^r\big(\langle x_0, b)\big)$.

Further we prove that $\varphi(x) \in C^r\big((a, x_1\rangle\big)$. According to formulas (7) and (8) we have

$$\begin{gathered} u(x_0) = h\Big(x_0, u\big(f(x_0)\big)\Big), \\ u^{(k)}(x_0) = H_k\Big(x_0, u\big(f(x_0)\big), u'\big(f(x_0)\big), \ldots, u^{(k)}\big(f(x_0)\big)\Big), \quad k = 1, 2, \ldots, r. \end{gathered} \tag{21}$$

Similarly we prove, with use of (18) and (21) and of assumptions (II) and (III) that $\varphi(x)$ is of class C^r in every interval (x_{-n}, x_1). Thus finally $\varphi(x) \in C^r\big((a, b)\big)$.

Properties (12) and (13) results immediately from (15) and (17).

This completes the proof.

References

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