ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

Dilability of sesquilinear form-valued kernels

by J. Stochel (Kraków)

Abstract. In the present paper we investigate the dilatability of sesquilinear form-valued kernels defined on any semigroup, not necessarily involutory and not necessarily unital. The main result, Theorem 2.2, gives necessary and sufficient conditions for the dilatability of such kernels in terms of new ones defined on the Cartesian product of the semigroup by itself. All the results that follow, concerning the problem of $*$-dilatability on $*$-semigroups, are deduced from the main result. As applications we give new versions of the dilation theorems of Arveson and Naimark. In the last section we present an example of a kernel which has a dilation, but has no minimal dilation.

0. Introduction. The general dilation theory of Hilbert space operator valued functions on unital $*$-semigroups has been developed by Sz.-Nagy in [26]. The study of dilations of operator functions in non-Hilbert spaces has been initiated in probability theory on Banach spaces (cf. [10], [28], [29] for an exhaustive list of references). Górniak and Weron (cf. [4], [6]) seem to be the first to formulate a purely algebraic analogue of Sz.-Nagy's dilation theorem for functions with values in the space $\bar{L}(X)$ of all antilinear operators from a linear space X to its algebraic dual X'. Within this general framework they were able to explain the role of the boundedness condition, which is one of two sufficient conditions for dilatability, in dilation theory. At that time Szafraniec (cf. [21], [23]–[25]) and Masani (cf. [10], [11]), both inspired by Arveson's dilation theorem, found simplified forms of the boundedness condition.

The study of dilations of Hilbert space operator valued functions on $*$-semigroups without unit has been initiated by Mlak and Szymański in [14] (cf. also [12]). Complete necessary and sufficient conditions for the dilatability of such functions have been found by Szafraniec (cf. [22]) and improved by the author (cf. [17]). Mlak and Weron (cf. [15]) have considered dilations of Banach space operator valued kernels defined on non-involutary and non-unital simigroups. Basing on [14] they gave some sufficient conditions for the dilatability of such kernels.

The purpose of this paper is to obtain necessary and sufficient conditions for a kernel defined on a semigroup S without unit to be dilatable. All the kernels we take here into consideration are assumed to have values in the space of all sesquilinear forms on a vector space X (cf. [13], [19], [25], [27] for a similar approach). Notice that although there exists a dilatability-preserving natural one-to-one correspondence between such kernels and kernels with values in the space $\bar{L}(X)$ (in other words, a kernel of the first type is dilatable if and only if its counterpart is), consideration of kernels of the second type is more natural and interesting from both the probabilistic and topological point of view (cf. [3], [4], [5], [7], [8]). Motivation for this general approach can be found in general dilation theory (cf. [13], [25]) as well as in probability theory (cf. [4], [10], [15], [27], [28]).

Section 2 deals with dilation of kernels defined on non-involutory and non-unital semigroups. Using the main result of the section (Theorem 2.2) we essentially simplify the criterion of dilatability given by Mlak and Weron (cf. [15], Theorem 1). In the next section $*$-dilations of positive-definite kernels are investigated (here the semigroup S is assumed to have an involution). This part of the paper is a continuation of our earlier studies on this kind of problems (cf. [17]). In particular, Theorem 3.7, which is an adaptation of Theorem 1 of [17] to our general algebraic setting, is deduced from Theorem 2.2. In Section 4 strictly algebraic versions of the well-known dilation theorems of Arveson and Naimark are presented. Moreover, we give a full (and correct) proof of Theorem 37.11 of [2]. In the last section we show that the minimality condition (15) plays an important role in Theorem 2.2. An example of an undilatable kernel which does have dilations is given.

In the Appendix we present a stronger version of Theorem 3.7.

1. Preliminaries. Let us fix some preliminaries. In all what follows $\boldsymbol{F}$ stands for either the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let X, Y be two vector spaces over $\boldsymbol{F}$. Denote by:

$L(X, Y)$ the space of all linear operators from X into Y,

$F(X)$ the space of all sesquilinear forms over X with values in $\boldsymbol{F}$.

If X and Y are topological vector spaces over $\boldsymbol{F}$, then we denote by:

$CL(X, Y)$ the space of all continuous linear operators from X into Y,

$CF(X)$ the space of all jointly continuous sesquilinear forms over X with values in $\boldsymbol{F}$.

For convenience we write $L(X)$ (resp. $CL(X)$) instead of $L(X, X)$ (resp. $CL(X, X)$), and if $A \in F(X)$ then we write $\langle Ax, y\rangle$ instead of the usual $A(x, y)$ $(x, y \in X)$. If $\{Z_s\colon s \in S\}$ is a family of subsets of a Hilbert space H over $\boldsymbol{F}$, then let we denote by $\bigvee\{Z_s\colon s\in S\}$ the closure of the linear span of the union $\bigcup\{Z_s\colon s\in S\}$.

Let T be any non-void set and let X be a vector space over $\boldsymbol{F}$. By an $F(X)$-valued kernel on the set T we mean a function C which maps the Cartesian product of T by itself into $F(X)$. Interesting examples of $F(X)$-valued kernels on $*$-semigroups have been given by Szafraniec in [25] (for the case of non-involuntary semigroups see [15]). Here we only mention that, similarly to [25], one can construct $F(X)$-valued kernels on T with the aid of suitable families of densely defined unbounded (linear) operators in a Hilbert space H all the domains of whose contain a dense linear submanifold X of H. Such kernels have values in the set $F(X)\setminus CF(X)$.

An $F(X)$-valued kernel C on T is said to be *positive-definite* (*PD*) if the following two conditions hold true:

$$(1)\qquad \sum_{j,k=1}^{n} \langle C(t_j, t_k)x_k, x_j\rangle \geqslant 0 \quad \text{for all finite sequences } t_1, \ldots, t_n \in T \text{ and } x_1, \ldots, x_n \in X,$$

$$(2)\qquad \langle C(s, t)x, y\rangle = \overline{\langle C(t, s)y, x\rangle} \quad \text{for all } s, t \in T \text{ and } x, y \in X.$$

Notice that if $\mathbf{F} = \mathbf{C}$ then condition (2) follows from condition (1) and is therefore redundant. This fact is a consequence of the polarization formula for sesquilinear forms over a complex vector space.

With every positive-definite $F(X)$-valued kernel C on T we can associate in the canonical fashion a Hilbert space H and a family $\{D(t)\colon t \in T\}$ of linear operators from X into H. Namely, we have the following Kolmogorov–Aronszajn type theorem (cf. [13], KMKA Lemma; [7], Remark, p. 239; [3], Theorem, p. 29).

1.1. THEOREM. *Let C be a positive-definite $F(X)$-valued kernel on T. Then there exists a Hilbert space H over $\mathbf{F}$ and an operator-valued function D: $T \to L(X, H)$ such that*

$$(3)\qquad \langle C(s, t)x, y\rangle = (D(t)x, D(s)y)_H \quad (x, y \in X;\ s, t \in T),$$

$$(4)\qquad H = \bigvee \{D(t)X\colon t \in T\}.$$

Moreover, if some other pair (H', D') consisting of a Hilbert space H' over $\mathbf{F}$ and an operator-valued function D': $T \to L(X, H')$ satisfies (3) *and* (4), *then there exists a unique unitary operator $U \in CL(H, H')$ such that*

$$(5)\qquad UD(t) = D'(t) \quad (t \in T).$$

Any pair (H, D) satisfying (3) and (4) will be called a *minimal factorization of C*. Thus the second part of Theorem 1.1 may be stated as follows: minimal factorizations of C are determined up to unitary equivalence. For the convenience of the reader and to make our exposition self-contained we sketch the proof of Theorem 1.1. The idea of the proof is similar to that of [28] (cf. also [7]).

Proof of Theorem 1.1. Denote by Λ the Cartesian product of the sets T and X. Define a new scalar kernel c on Λ by

$$(6)\qquad c\big((s, x), (t, y)\big) = \langle C(s, t)y, x\rangle \quad (x, y \in X,\ s, t \in T).$$

Then it is easy to check that the kernel c is *PD*. Using the Kolmogorov–

Aronszajn factorization theorem (cf. [12], Proposition 1, p. 20; [10], Theorem 2.10, p. 421) we obtain a Hilbert space H over $\boldsymbol{F}$ and a function $d\colon \Lambda \to H$ such that

$$c(\lambda, \mu) = (d(\mu), d(\lambda))_H \quad (\lambda, \mu \in \Lambda), \tag{7}$$

$$H = \bigvee \{d(\lambda)\colon \lambda \in \Lambda\}. \tag{8}$$

It follows from (6) and (7) that

$$(d(s, \alpha x + \beta y), d(t, z))_H = (\alpha d(s, x) + \beta d(s, y), d(t, z))_H$$
$$(s, t \in T,\ x, y \in X,\ \alpha, \beta \in \boldsymbol{F})$$

so, by (8), $d(t, \cdot) \in L(X, H)$ for each $t \in T$. Denote by D the function from T into $L(X, H)$ which sends t to $d(t, \cdot)$ for $t \in T$. Then the pair (H, D) satisfies conditions (3) and (4).

Let now (H', D') be another pair satisfying conditions (3) and (4). Then for all finite sequences $t_1, \ldots, t_n \in T$ and $x_1, \ldots, x_n \in X$ we have

$$\Big\| \sum_{k=1}^{n} D(t_k) x_k \Big\|_H^2 = \sum_{j,k=1}^{n} \langle C(t_j, t_k) x_k, x_j \rangle = \Big\| \sum_{k=1}^{n} D'(t_k) x_k \Big\|_{H'}^2.$$

Thus, by the minimality condition (4), there exists a unique unitary operator $U \in CL(H, H')$ which sends $D(t)x$ to $D'(t)x$ for $t \in T$ and $x \in X$. This completes the proof.

Now let S be a multiplicative semigroup which is not assumed to have a unit. The following definition originates from the boundedness condition in the Sz.-Nagy dilation theorem for involutory semigroups (cf. [26]). We say that an $F(X)$-valued kernel C on S satisfies the boundedness condition BC if there exists a non-negative function $M\colon S \to \boldsymbol{R}_+$ such that for each $t \in S$ the kernel $M(t)C - C^t$ is positive-definite, where C^t is an $F(X)$-valued kernel on S defined by $C^t(u, v) = C(tu, tv)$ for $u, v \in S$. As will be shown below BC is necessary and sufficient for a PD kernel to have a propagator in the sense of Masani (cf. [10], Definition 3.2, p. 424).

1.2. Theorem. *Let C be a positive-definite $F(X)$-valued kernel on S which satisfies the boundedness condition. Then there exists a Hilbert space H over $\boldsymbol{F}$, an operator-valued function $D\colon S \to L(X, H)$ and a representation π of S on H, i.e., a semigroup homomorphism $\pi\colon S \to CL(H)$, such that*

$$(H, D) \quad \text{is a minimal factorization of } C, \tag{9}$$

$$\pi(t) D(s) = D(ts) \quad (s, t \in S). \tag{10}$$

If (H', D', π') is another triple satisfying (9) *and* (10), *then there exists a unique unitary operator $U \in CL(H, H')$ such that*

$$UD(s) = D'(s) \quad (s \in S), \tag{11}$$

(12) $$U\pi(s) = \pi'(s)U \qquad (s \in S).$$

Any triple (H, D, π) satisfying (9) and (10) will be called a *minimal propagator of* C.

Proof. Let (H, D) be a minimal factorization of C. Then for all finite sequences $x_1, \ldots, x_n \in X$ and $s_1, \ldots, s_n, t \in S$ we have

$$\Big\|\sum_{k=1}^{n} D(ts_k)x_k\Big\|_H^2 = \sum_{j,k=1}^{n} \langle C(ts_j, ts_k)x_k, x_j\rangle \leqslant M(t) \sum_{j,k=1}^{n} \langle C(s_j, s_k)x_k, x_j\rangle$$
$$= M(t)\Big\|\sum_{k=1}^{n} D(s_k)x_k\Big\|_H^2.$$

Thus by the minimality condition (4), for each $t \in T$ there exists a unique bounded operator $\pi(t) \in CL(H)$ which fulfils condition (10).

Now if (H', D', π') is another triple which satisfies (9) and (10) then, by Theorem 1.1, there exists a unitary operator $U \in CL(H, H')$ such that (11) holds true. Thus $U\pi(t)D(s)x = \pi'(t)UD(s)x$ for all $x \in X$ and $s, t \in S$. Since the set $\{D(s)x\colon x \in X \text{ and } s \in S\}$ is linearly dense in H, equality (12) follows. This completes the proof.

1.3. Remark. A more general algebraic version of Theorem 1.2 can be found in [13].

2. Dilatability of kernels on semigroups without unit. Through the whole section, S denotes a semigroup without unit. Unless specified otherwise, X always stands for a vector space over $\boldsymbol{F}$.

Modifying slightly the definition of R-dilation in [15] (cf. [12]) we introduce the notion of minimal dilation. A triple (H, R, π) is said to be a *minimal dilation* of a kernel $C\colon S \times S \to F(X)$ if

(13) H is a Hilbert space over $\boldsymbol{F}$, $R \in L(X, H)$ and π is a representation of S on H,

(14) $$\langle C(s, t)x, y\rangle = \big(\pi(t)Rx, \pi(s)Ry\big)_H \qquad (s, t \in S,\ x, y \in X),$$

(15) $$H = \bigvee \{\pi(s)RX\colon s \in S\}.$$

Two such minimal dilations (H, R, π) and (H', R', π') are said to be *unitary equivalent* if there exists a unitary operator $U \in CL(H, H')$ such that

(16) $$U\pi(s) = \pi'(s)U \qquad (s \in S);$$

(17) $$UR = R'.$$

We say that a kernel C is *dilatable* if it has a minimal dilation.

In this section we try to solve the following two problems:

1° Find necessary and sufficient conditions for a kernel $C\colon S\times S\to F(X)$ to be dilatable.

2° Are all minimal dilations of C unitary equivalent?

Now we shall sketch an idea of solving problem 1° and we shall explain why the answer to the question posed in problem 2° is negative in general. First observe that if (H, R, π) is a minimal dilation of C, then (H, D, π), where $D(s)=\pi(s)R$ $(s\in S)$, is a minimal propagator of C. Conversely, if (H, D, π) is a minimal propagator of C, then (H, R, π), where R is a solution of the system of operator equations

$$D(s)=\pi(s)R \quad (s\in S) \tag{18}$$

is a minimal dilation of C. Briefly speaking, in order to find the answer to problem 1° we have to solve system (18) for the unknown quantity R. For this purpose fix a vector $y\in X$ and rewrite equality (14) as follows:

$$(\pi(s)^* D(t)x,\ Ry)_H=\langle C(s, t)x, y\rangle \quad (s, t\in S,\ x, y\in X). \tag{19}$$

Then one can try to look at the vector Ry as a bounded linear functional φ_y on H which fulfils the condition

$$\varphi_y\big(\pi(s)^* D(t)x\big)=\langle C(s, t)x, y\rangle \quad (s, t\in S,\ x\in X). \tag{20}$$

Denote by H_π the Hilbert subspace of H spanned by the vectors $\{\pi(s)^* h\colon s\in S,\ h\in H\}$. Evidently, $H_\pi=H\ominus N_\pi=\bigvee\{\pi(s)^* D(t)x\colon s, t\in S, x\in X\}$, where N_π is the null space of the representation π (i.e., N_π is the set of all vectors $h\in H$ such that $\pi(s)h=0$ for each $s\in S$). It is easy to see that all bounded functionals φ_y which satisfy condition (20) coincide on H_π. Since H_π does not have to be equal to H in general, it may happen that there exists more than one such a functional φ_y. This is why the answer to the question posed in 2° is negative in general (exhaustive solution of problem 2° can be found in [20]).

Reverting to equality (20) one can say that the boundedness of all functionals $\{\varphi_y|_{H_\pi}\colon y\in X\}$ is a necessary and sufficient condition for system (18) to have a solution $R\in L(X, H)$. An easy calculation shows that the functional $\varphi_y|_{H_\pi}$ is bounded (and hence correctly defined) if and only if inequality (22) below holds true for all finite sequences $s_1, \ldots, s_n\in S$, $t_1, \ldots, t_n\in S$ and $x_1, \ldots, x_n\in X$, where $\tilde{C}$ is a new kernel associated with the former one via the following lemma.

2.1. Lemma. *Let C be a positive-definite $F(X)$-valued kernel on S which satisfies the boundedness condition. Then there exists a unique positive definite $F(X)$-valued kernel $\tilde{C}$ on $S\times S$ such that*

$$\langle\tilde{C}\big((s, t), (u, v)\big)x, y\rangle=\big(\pi(u)^* D(v)x,\ \pi(s)^* D(t)y\big)_H \tag{21}$$

$$(s, t, u, v\in S,\ x, y\in X)$$

for each minimal propagator (H, D, π) of C. Moreover, if X is a topological vector space over $\boldsymbol{F}$, then $C(s, t) \in CF(X)$ for all $s, t \in S$ implies that $\tilde{C}(\lambda, \mu) \in CF(X)$ for all $\lambda, \mu \in S \times S$.

For the proof let us define $\tilde{C}$ by equality (21). Using the second part of Theorem 1.2 one can show that the definition of $\tilde{C}$ does not depend on the choice of a minimal propagator of C (in particular, this means that $\tilde{C}$ is correctly defined). We leave the details to the reader. The second part of Lemma 2.1 follows from equality (3).

Now we are able to formulate our main result which solves problem 1° completely and problem 2° partially.

2.2. THEOREM. *Let C be a positive-definite $F(X)$-valued kernel on S which satisfies the boundedness condition. Then*

(I) *The following two conditions are equivalent*:

(i) *C is dilatable,*

(ii) *there exists a non-negative function q: $X \to \boldsymbol{R}_+$ such that for all finite sequences $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ and $x_1, \ldots, x_n$, $y \in X$*

$$(22) \qquad \Big|\sum_{k=1}^{n} \langle C(s_k, t_k) x_k, y\rangle\Big|^2 \leqslant q(y)^2 \sum_{j,k=1}^{n} \langle \tilde{C}\big((s_j, t_j), (s_k, t_k)\big) x_k, x_j\rangle.$$

(II) *If (H, R, π) and (H', R', π') are two minimal dilations of C, then there exists a unique unitary operator $U \in CL(H, H')$ such that*

$$(23) \qquad UPR = P'R',$$

$$(24) \qquad U\pi(s) = \pi'(s)U \quad (s \in S),$$

where P (resp. P') stands for the orthogonal projection of H (resp. H') onto $H_\pi = \bigvee \{\pi(s)^* H\colon s \in S\}$ (resp. $H_{\pi'} = \bigvee \{\pi'(s)^* H'\colon s \in S\}$).

Proof. (I) (i)$\Rightarrow$(ii) Let (H, R, π) be a minimal dilation of C. Then (H, D, π), where $D(s) = \pi(s) R$ $(s \in S)$, is a minimal propagator of C. Applying Lemma 2.1 we obtain

$$\sum_{j,k=1}^{n} \langle \tilde{C}\big((s_j, t_j), (s_k, t_k)\big) x_k, x_j\rangle = \Big\|\sum_{k=1}^{n} \pi(s_k)^* D(t_k) x_k\Big\|_H^2$$

so

$$\begin{aligned}\Big|\sum_{k=1}^{n} \langle C(s_k, t_k) x_k, y\rangle\Big|^2 &= \Big|\sum_{k=1}^{n} \big(\pi(t_k) R x_k, \pi(s_k) R y\big)_H\Big|^2 \\ &= \Big|\Big(\sum_{k=1}^{n} \pi(s_k)^* D(t_k) x_k, Ry\Big)_H\Big|^2 \\ &\leqslant \|Ry\|_H^2 \Big\|\sum_{k=1}^{n} \pi(s_k)^* D(t_k) x_k\Big\|_H^2\end{aligned}$$

for all finite sequences $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ and $x_1, \ldots, x_n$, $y \in X$.

(ii)$\Rightarrow$(i) Let (H, D, π) be a minimal propagator of C. Fix a vector $y \in X$ and denote by H_π the space

$$\bigvee \{\pi(s)^* H:\ s \in S\} = \bigvee \{\pi(s)^* D(t) X:\ s,\ t \in S\}.$$

Then by (ii) there exists a unique linear bounded functional φ_y on H_π which fulfils condition (20). The Riesz representation theorem implies that there exists a unique vector $Ry \in H_\pi$ such that $\varphi_y(h) = (h, Ry)_H$ for each $h \in H_\pi$. In particular, we have

$$\begin{aligned}(D(t)x,\ \pi(s)Ry)_H &= (\pi(s)^* D(t)x,\ Ry)_H = \varphi_y(\pi(s)^* D(t)x) \\ &= \langle C(s, t)x,\ y\rangle \quad (s,\ t \in S,\ x \in X).\end{aligned}$$

But we know that

$$\langle C(s, t)x,\ y\rangle = (D(t)x,\ D(s)y)_H \quad (s,\ t \in S,\ x \in X)$$

so

$$(D(t)x,\ \pi(s)Ry - D(s)y)_H = 0 \quad (s,\ t \in S,\ x \in X).$$

Thus by (4) we obtain

$$\pi(s)Ry = D(s)y, \quad s \in S \text{ and } y \in X$$

and

$$H = \bigvee \{D(s)X:\ s \in S\} = \bigvee \{\pi(s)RX:\ s \in S\}$$

so (14) and (15) hold true. Since the functional $\langle C(s, t)x, \cdot\rangle$ is antilinear for all s, $t \in S$ and $x \in X$, the uniqueness part of the Riesz representation theorem implies the linearity of R. Summing up, (H, R, π) is a minimal dilation of C.

(II) To prove the second part of Theorem 2.2 let us take two minimal dilations (H, R, π) and (H', R', π') of C. Then the triples (H, D, π) and (H', D', π'), where $D(s) = \pi(s)R$ and $D'(s) = \pi'(s)R'$ for $s \in S$, are minimal propagators of C, so, by Theorem 1.2, there exists a unitary operator $U \in CL(H, H')$ such that conditions (11) and (12) hold true. Notice now that $UH_\pi = H_{\pi'}$. Indeed, it follows from (11) and (12) that

$$U\pi(s)^* D(t) = UU^* \pi'(s)^* UD(t) = \pi'(s)^* D'(t) \quad (s,\ t \in S).$$

From the above we obtain

$$\begin{aligned}(UPRx,\ \pi'(s)^* D'(t)y)_{H'} &= (PRx,\ \pi(s)^* D(t)y)_H = (\pi(s)Rx,\ D(t)y)_H \\ &= \langle C(t, s)x,\ y\rangle = (\pi'(s)R'x,\ \pi'(t)R'y)_{H'} \\ &= (\pi'(s)R'x,\ D'(t)y)_{H'} = (R'x,\ \pi'(s)^* D'(t)y)_{H'} \\ &= (P'R'x,\ \pi'(s)^* D'(t)y)_{H'}\end{aligned}$$

for all $x, y \in X$ and $s, t \in S$, which is equivalent to equality (23). This completes the proof.

2.3. Remark. Notice that if (H, R, π) is a minimal dilation of the kernel C, then so is (H, PR, π), where P is the orthogonal projection of H onto H_π (this follows from the equality $H = H_\pi \oplus N_\pi$). Thus if C is dilatable, then there always exists a minimal dilation of C, say (H, R, π), such that $R \in L(X, H_\pi)$, and, by Theorem 2.2, all such minimal dilations of C are unitary equivalent. Moreover, if q_m is the minimal function satisfying (22), then for any minimal dilation (H, R, π) of C such that $R \in L(X, H_\pi)$ we have $q_m(y) = \|Ry\|_H$ for $y \in X$.

One can ask whether the space H_π is invariant for π (here (H, R, π) is any dilation of C). The answer is *yes* if and only if $H_\pi = H$. Indeed, if H_π is invariant for π, then it reduces π because $H \ominus H_\pi = N_\pi$ is always invariant for π. Thus $P\pi(s) = \pi(s)P$ for each $s \in S$ and

$$H_\pi = PH = \bigvee P\pi(S)RX = \bigvee \pi(S)PRX,$$

where P denotes the orthogonal projection of H onto H_π. Since $\pi(s)PR = \pi(s)R$ for each $s \in S$, $H = \bigvee \pi(S)RX = H_\pi$.

The above observation and Theorem 3.1 in [20] imply that all minimal dilations of C are unitary equivalent if and only if for some (equivalently: for each) minimal dilation (H, R, π) of C, H_π is invariant for π.

Notice also that if S has a unit e and C is a PD $F(X)$-valued kernel on S which satisfies BC, then for each minimal propagator (H, D, π) of C, $(H, D(e), \pi)$ is a minimal dilation of C. Thus condition (ii) of Theorem 2.2 holds with $q(y) = \|D(e)y\| = \langle C(e, e)y, y\rangle^{1/2}$, $y \in X$. Since, in this case, $H_\pi = H$ for each minimal dilation (H, R, π) of C, all minimal dilations of C are unitary equivalent.

Our next theorem reveals how the continuity of the function $q\colon X \to \boldsymbol{R}_+$ appearing in inequality (22) affects the continuity of the forms $\{C(s, t)\colon s, t \in S\}$.

2.4. Theorem. *Let X be a topological vector space over $\boldsymbol{F}$ and let C be a positive-definite $F(X)$-valued kernel on S which satisfies the boundedness condition and condition* (ii) *of Theorem* 2.2 *with some continuous function $q\colon X \to \boldsymbol{R}_+$ such that $q(0) = 0$. Then $C(s, t) \in CF(X)$ for all $s, t \in S$.*

For the proof observe that the continuity of q implies the continuity of q_m. Since there exists a minimal dilation (H, R, π) of C such that $q_m(y) = \|Ry\|_H$ $(y \in Y)$, the operator R is continuous. In view of equality (14), the conclusion of Theorem 2.4 follows.

The results we have obtained hitherto can be used to improve the sufficient condition for the dilatability of Banach space operator valued functions given in [15] (cf. Theorem 1).

Let Z be a complex Banach space. Denote by $C\bar{L}(Z)$ the space of all bounded antilinear operators from Z into its topological dual Z^*. Associate with every function A: $S \times S \to C\bar{L}(Z)$ a $CF(Z)$-valued kernel C_A on S via the formula

$$\langle C_A(s, t)x, y\rangle = \big(A(s, t)y\big)(x) \quad (x, y \in Z).$$

We say that the function A is *positive-definite* (*PD*) (resp. satisfies the boundedness condition (*BC*)) if C_A is (resp. C_A does). It is obvious that our definitions of positive-definiteness and of boundedness condition for A coincide with those of [15].

Now the Mlak–Weron theorem can be stated as follows:

2.5. THEOREM. *Let A: $S \times S \to C\bar{L}(Z)$ be a positive-definite function which satisfies the boundedness condition. Assume also that there exists a net $\{e_\theta\} \subset S$ such that*

$$\text{(i)} \qquad \delta = \sup_\theta \|A(e_\theta, e_\theta)\| < +\infty,$$

$$\text{(ii)} \qquad \lim\big(A(se_\theta, t)x\big)(y) = \big(A(s, t)x\big)(y) \quad (s, t \in S,\ x, y \in Z).$$

Then there exists a complex Hilbert space H, an operator $R \in CL(Z, H)$ and a representation π of S on H such that

$$\text{(iii)} \qquad A(s, t) = R^* \pi(s)^* \pi(t) R \quad (s, t \in S),$$

$$\text{(iv)} \qquad H = \bigvee \{\pi(s)RZ:\ s \in S\}.$$

Proof. We first show that the kernel C_A satisfies condition (ii) of Theorem 2.2. For this purpose fix finite sequences $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ and $x_1, \ldots, x_n, y \in Z$. Let (H, D, π) be any minimal propagator of the kernel C_A. Then (i) implies that

$$\begin{aligned} \Big|\sum_{k=1}^n \big(A(s_k e_\theta, t_k)y\big)(x_k)\Big|^2 &= \Big|\sum_{k=1}^n \big(D(t_k)x_k, D(s_k e_\theta)y\big)_H\Big|^2 \\ &= \Big|\Big(\sum_{k=1}^n \pi(s_k)^* D(t_k)x_k, D(e_\theta)y\Big)_H\Big|^2 \\ &\leqslant \big(A(e_\theta, e_\theta)y\big)(y) \sum_{j,k=1}^n \big(\pi(s_k)^* D(t_k)x_k, \pi(s_j)^* D(t_j)x_j\big)_H \\ &\leqslant \gamma \|y\|_Z^2 \sum_{j,k=1}^n \langle \tilde{C}_A\big((s_j, t_j), (s_k, t_k)\big)x_k, x_j\rangle \end{aligned}$$

for each index θ. Thus, by (ii), we have

$$\begin{aligned} &\Big|\sum_{k=1}^n \langle C_A(s_k, t_k)x_k, y\rangle\Big|^2 = \Big|\sum_{k=1}^n \big(A(s_k, t_k)y\big)(x_k)\Big|^2 \\ &= \lim_\theta \Big|\sum_{k=1}^n \big(A(s_k e_\theta, t_k)y\big)(x_k)\Big|^2 \leqslant \delta \|y\|^2 \sum_{j,k=1}^n \langle \tilde{C}_A\big((s_j, t_j), (s_k, t_k)\big)x_k, x_j\rangle. \end{aligned}$$

This means that the kernel C_A satisfies condition (ii) of Theorem 2.2 with $q(y) = \sqrt{\delta}\|y\|_Z$ $(y \in Z)$. In view of Theorem 2.2, C_A has a minimal dilation (H, R, π) such that $R \in L(Z, H_\pi)$. Since the function q is continuous, the operator R is bounded (see Theorem 2.4). Thus equality (iii) follows from (14), where $X = Z$ and $C = C_A$.

This completes the proof.

2.6. Remark. The original formulation of the Mlak–Weron theorem contains the additional assumption

(25) $\quad \lim_{\theta}(A(e_\theta, s)x)(y)$ exists for every $s \in S$ and every $x, y \in Z$.

Below we show how to prove Theorem 2.5 altering slightly the original proof of Theorem 1 of [15]. Moreover, we relate condition (25) to condition (i) of Theorem 2.5.

Let (H, D, π) be a minimal propagator of C_A. Then assumption (i) of Theorem 2.5 implies the boundedness of the net $\{D(e_\theta)\}$ in $CL(Z, H)$. Since the unit ball of $CL(Z, H)$ is compact in the weak operator topology, there exists a subnet $\{e_{\theta'}\}$ of $\{e_\theta\}$ such that the net $\{D(e_{\theta'})\}$ is weakly convergent to an operator $R \in CL(Z, H)$. Now we can continue the proof in the same way as in [15]. In particular, we have

(26) $$\lim_{\theta'}(A(e_{\theta'}, s)x)(y) = \lim_{\theta'}(D(s)y, D(e_{\theta'})x)_H = (D(s)y, Rx)_H$$

for every $s \in S$ and every $x, y \in Z$.

In short, condition (i) of Theorem 2.5 implies condition (26) which is weaker than condition (25). But if $H_\pi = H$, then conditions (i) and (ii) of Theorem 2.5 yield directly the weak convergence of the net $\{D(e_\theta)\}$. Finally, notice that the first proof of Theorem 2.5 does not appeal to any form of the Tychonov theorem.

3. Dilatability of kernels on $*$-semigroups without unit. Let S_* be an involution semigroup (in short: $*$-semigroup) without unit. Denote by $*$ the involution of S_*. As usual X stands for a vector space over $\boldsymbol{F}$. This section deals with those minimal dilations of a given kernel C: $S_* \times S_* \to F(X)$ which "preserve involution" in the sense of the following definition. A triple (H, R, π) is said to be a *minimal $*$-dilation* of a kernel C if it is a minimal dilation of C and π is a $*$-representation of S_* on H (i.e., π: $S_* \to CL(H)$ is an involution-preserving homomorphism). A kernel C is said to be *$*$-dilatable* if it has a minimal $*$-dilation.

The aim of the first part of the section is to give necessary and sufficient conditions for $*$-dilatability. First observe that if a kernel C is $*$-dilatable, then it has the so-called *transfer property* (cf. [10], p. 430):

(27) $$C(us, t) = C(s, u^* t) \quad (s, t, u \in S_*).$$

Masani has pointed out that if S_* has a unit then the transfer property together with the positive-definiteness and the boundedness condition are sufficient conditions for the $*$-dilatability of the kernel C (cf. [10], Theorem 4.10). For a non-unital $*$-semigroup we have only the following

3.1. LEMMA. *Let (H, D, π) be a minimal propagator of a kernel C: $S_* \times S_* \to F(X)$. Then π is a $*$-representation of S_* if and only if C has the transfer property.*

Proof. If π is a $*$-representation of S_*, then for all $s, t, u \in S_*$ and $x, y \in X$ we have

$$\begin{aligned}\langle C(us, t)x, y\rangle &= (D(t)x, D(us)y)_H = (\pi(u)^* D(t)x, D(s)y)_H \\ &= (D(u^*t)x, D(s)y)_H = \langle C(s, u^*t)x, y\rangle.\end{aligned}$$

Conversely, if the kernel C has the transfer property, then we have

$$\begin{aligned}(\pi(u^*)D(t)x, D(s)y)_H &= (D(u^*t)x, D(s)y)_H = \langle C(s, u^*t)x, y\rangle \\ &= \langle C(us, t)x, y\rangle = (D(t)x, D(us)y)_H \\ &= (D(t)x, \pi(u)D(s)y)_H \\ &= (\pi(u)^* D(t)x, D(s)y)_H\end{aligned}$$

for all $s, t, u \in S_*$ and $x, y \in X$. Since the vectors of the form $\{D(s)x\colon s \in S_*,\ x \in X\}$ generate the Hilbert space H, the equality $\pi(u)^* = \pi(u^*)$ $(u \in S_*)$ follows. This completes the proof.

3.2. COROLLARY. *If an $F(X)$-valued kernel C on S_* is $*$-dilatable, then each minimal dilation of C is a minimal $*$-dilation of C.*

Now consider a PD $F(X)$-valued kernel C on S_* which satisfies BC and has the transfer property. Then one can show that the $F(X)$-valued kernel $\tilde{C}$ on $S_* \times S_*$ associated with C by Lemma 2.1 has the form

$$\tilde{C}((s, t), (u, v)) = C(s^*t, u^*v) \quad (s, t, u, v \in S_*). \tag{28}$$

Indeed, if (H, D, π) is any minimal propagator of C, then by Lemma 3.1 we have

$$\begin{aligned}\langle \tilde{C}((s, t), (u, v))x, y\rangle &= (\pi(u)^* D(v)x, \pi(s)^* D(t)y)_H \\ &= (\pi(u^*)D(v)x, \pi(s^*)D(t)y)_H \\ &= (D(u^*v)x, D(s^*t)y)_H = \langle C(s^*t, u^*v)x, y\rangle \\ &\qquad (s, t, u, v \in S_*,\ x, y \in X),\end{aligned}$$

which proves equality (28). Thus, in this case, inequality (22) can be rewritten as follows:

$$\Big|\sum_{k=1}^{n} \langle C(s_k, t_k)x_k, y\rangle\Big|^2 \leqslant q(y)^2 \sum_{j,k=1}^{n} \langle C(s_j^* t_j, s_k^* t_k)x_k, x_j\rangle. \tag{29}$$

Conversely, assuming only inequality (29) one can obtain some of the above-mentioned properties of the kernel C like the transfer property.

3.3. Lemma. *Let C be an $F(X)$-valued kernel on S_* which satisfies inequality* (29). *Then*

(i) *there exists a function B: $S_* \to F(X)$ such that*

$$C(s, t) = B(s^* t) \quad (s, t \in S_*), \tag{30}$$

(ii) *the kernel C has the transfer property.*

Proof. Since (ii) follows directly from (i), we only have to prove (i). Define the function B: $S_* \to F(X)$ by the formula

$$B(u) = \begin{cases} C(s, t) & \text{if } u \in S_* \cdot S_* \text{ and } u = s^* t, \\ 0 & \text{if } u \notin S_* \cdot S_*. \end{cases}$$

The correctness of the definition of B is an easy consequence of the following implication:

(31) if $u = s_1^* t_1 = s_2^* t_2$, then $C(s_1, t_1) = C(s_2, t_2)$.

To verify (31), fix arbitrary two vectors $x, y \in X$ and denote by x_1 (resp. x_2) the vector x (resp. $(-x)$). Then by (29) we have

$$|\langle C(s_1, t_1)x, y\rangle - \langle C(s_2, t_2)x, y\rangle|^2 = \Big|\sum_{k=1}^{2} \langle C(s_k, t_k)x_k, y\rangle\Big|^2$$

$$\leqslant q(y)^2 \sum_{j,k=1}^{2} \langle C(s_j^* t_j, s_k^* t_k)x_k, x_j\rangle = q(y)^2 \sum_{j,k=1}^{2} \langle C(u, u)x_k, x_j\rangle = 0$$

so $\langle C(s_1, t_1)x, y\rangle = \langle C(s_2, t_2)x, y\rangle$. This completes the proof.

Inequality (29) does not in general imply the positive-definiteness of C, but merely the positive-definiteness of the restricted kernel $C|_{T_* \times T_*}$, where $T_* = S_* \cdot S_*$. But assuming that C satisfies BC, one can infer the positive-definiteness of C from (29).

3.4. Lemma. *Let C be an $F(X)$-valued kernel on S_* which satisfies the boundedness condition and* (29). *Then the kernel C is positive-definite.*

Proof. Let T_n $(n \geqslant 1)$ be the set of all $2n$-tuples $\lambda_n = (s_1, \ldots, s_n, x_1, \ldots, x_n)$ such that $s_1, \ldots, s_n \in S_*$ and $x_1, \ldots, x_n \in X$. For each $t \in S_*$ and $y \in X$ define the functions $\delta_{t,y}^n$, Δ_t^n, Δ^n: $T_n \to F$ by

$$\delta_{t,y}^n(\lambda_n) = \Big|\sum_{k=1}^{n} \langle C(t^*, s_k)x_k, y\rangle\Big|^2,$$

$$\Delta_t^n(\lambda_n) = \sum_{j,k=1}^{n} \langle C(ts_j, ts_k)x_k, x_j\rangle,$$

$$\Delta^n(\lambda_n) = \sum_{j,k=1}^{n} \langle C(s_j, s_k)x_k, x_j\rangle.$$

It follows from (29) that

$$(32) \qquad 0 \leqslant \delta_{t,y}^n \leqslant q(y)^2 \Delta_t^n \quad (t \in S_*,\ y \in X,\ n \geqslant 1).$$

If $q = 0$ then, by (32), $C = 0$ and so C is PD. If $q \neq 0$ then there exists $y_0 \in X$ such that $q(y_0) > 0$. Since inequalities (32) hold true for $y = y_0$, $\Delta_t^n \geqslant 0$ for all n and t. Now BC yields $M(t)\Delta^n \geqslant M(t)\Delta^n - \Delta_t^n \geqslant 0$ for all n and t. If $M = 0$ then $\Delta_t^n = 0$ for all n, t, so by (32), $C = 0$. If $M \neq 0$, then there exists $t_0 \in S_*$ such that $M(t_0) > 0$. Since $M(t_0)\Delta^n \geqslant 0$, $\Delta^n \geqslant 0$. This completes the proof.

The following theorem gives necessary and sufficient conditions for $*$-dilatability.

3.5. THEOREM. *Let C be an $F(X)$-valued kernel. Then:*

(I) *If C satisfies the boundedness condition, then the following two conditions are equivalent:*

(i) *C is $*$-dilatable,*

(ii) *there exists a non-negative function $q\colon X \to \boldsymbol{R}_+$ such that inequality (29) holds true for all finite sequences $s_1, \ldots, s_n \in S_*$, $t_1, \ldots, t_n \in S_*$ and $x_1, \ldots, x_n$, $y \in X$.*

(II) *If the kernel C is $*$-dilatable, then:*

(iii) *for each minimal dilation (H, R, π) of C, $H = H_\pi = \bigvee \{\pi(s)RX\colon s \in S_* \cdot S_*\}$,*

(iv) *all minimal dilations of C are unitary equivalent.*

Proof. In view of Lemmas 3.1, 3.3 and 3.4, the first part of Theorem 3.5 follows from Theorem 2.2. By Remark 2.3, condition (iv) follows from condition (iii). Therefore we only have to prove condition (iii). Let us fix any minimal dilation (H, R, π) of C. By Corollary 3.2, (H, R, π) is a minimal $*$-dilation of C. Thus the null space N_π of π is equal to $H \ominus \bigvee \{\pi(s)RX\colon s \in S_* \cdot S_*\}$.

Let h be an arbitrary vector of N_π. Then $(h, \pi(s)Rx) = (\pi(s^*)h, Rx)_H = 0$ for all $s \in S_*$ and $x \in X$. In view of the minimality condition (15), h must be equal to 0. Thus $H = H_\pi$. This completes the proof.

3.6. Remark. Notice that using the standard induction procedure one can show that if the kernel C is $*$-dilatable, then for each minimal dilation (H, R, π) of C and for each natural number n we have

$$H = \bigvee \{\pi(s)RX\colon s \in S_*^n\},$$

where the sets $\{S_*^n\}$ are defined by induction as follows: $S_*^1 = S_*$, $S_*^{n+1} = S_*^n \cdot S_*$, $n \geqslant 0$.

It follows from the second part of Theorem 3.5 that if q_m is the minimal function satisfying (29), then for any minimal dilation (H, R, π) of C, $q_m(y) = \|Ry\|$ for each $y \in X$ (compare with Remark 2.3).

Lemma 3.3 justifies the following definition. We say that a function $B\colon S_* \to F(X)$ is positive-definite (PD) (resp. satisfies the boundedness condition (BC)) if the kernel C_B defined by

$$C_B(s, t) = B(s^* t) \quad (s, t \in S_*) \tag{33}$$

is positive-definite (resp. satisfies the boundedness condition). A triple (H, R, π) is said to be a *minimal $*$-dilation* of B if H is a Hilbert space over F, $R \in L(X, H)$, π is a $*$-representation of S_* on H and

$$(\pi(s) Rx, Ry)_H = \langle B(s) x, y\rangle \quad (s \in S_*,\ x, y \in X), \tag{34}$$

$$H = \bigvee \{\pi(s) RX\colon s \in S_*\}. \tag{35}$$

It follows from the definition that each minimal $*$-dilation of B is automatically a minimal $*$-dilation of C_B, but not conversely. A function $B\colon S_* \to F(X)$ is said to be *$*$-dilatable* if it has a minimal $*$-dilation.

The following theorem gives necessary and sufficient conditions for a function $B\colon S_* \to F(X)$ to be $*$-dilatable.

3.7. THEOREM. *Let B be an $F(X)$-valued function on S_*. Then:*

(I) *If B satisfies the boundedness condition, then the following two conditions are equivalent:*

(i) *B is $*$-dilatable,*

(ii) *there exists a function $q\colon X \to \mathbf{R}_+$ such that for all finite sequences $s_1, \ldots, s_n \in S_*$ and $x_1, \ldots, x_n, y \in X$*

$$\Big|\sum_{k=1}^{n} \langle B(s_k) x_k, y\rangle\Big|^2 \leqslant q(y)^2 \sum_{j,k=1}^{n} \langle B(s_j^* s_k) x_k, x_j\rangle. \tag{36}$$

(II) *If B is $*$-dilatable, then:*

(iii) *for each minimal $*$-dilation of B the null space N_π of π is trivial ($N_\pi = \{0\}$).*

(iv) *all minimal $*$-dilations of B are unitary equivalent.*

Proof. In view of Theorem 3.5 we only have to prove the first part of Theorem 3.7.

(i)$\Rightarrow$(ii) Let (H, R, π) be a minimal $*$-dilation of B. Then

$$\Big|\sum_{k=1}^{n} \langle B(s_k) x_k, y\rangle\Big|^2 = \Big|\Big(\sum_{k=1}^{n} \pi(s_k) Rx_k, Ry\Big)_H\Big|^2 \leqslant \|Ry\|_H^2 \sum_{j,k=1}^{n} \langle B(s_j^* s_k) x_k, x_j\rangle$$

for all finite sequences $s_1, \ldots, s_n \in S_*$ and $x_1, \ldots, x_n, y \in X$.

(ii)$\Rightarrow$(i) Observe that (36) implies (29) (put $C = C_B$). Thus, in virtue of Theorem 3.5, the kernel C_B has a minimal $*$-dilation (H, R, π) such that $H = H_\pi$. In particular, we have

$$\langle B(s) x, y\rangle = (\pi(s) Rx, Ry)_H \quad (s \in S_*^2,\ x, y \in X).$$

To prove equality (34), fix $\varepsilon > 0$, $s \in S_*$ and $x, y \in X$. Since $H = H_\pi$, there exist finite sequences $s_1, \ldots, s_m \in S_* \cdot S_*$ and $x_1, \ldots, x_m \in X$ such that

$$\Delta = \|\pi(s) Rx - h\| < \varepsilon, \quad \text{where } h = \sum_{k=1}^{m} \pi(s_k) Rx_k.$$

Put $s_{m+1} = s$ and $x_{m+1} = -x$. It follows from (36) that

$$\begin{aligned}
|(\pi(s) Rx, Ry)_H - \langle B(s) x, y\rangle| &\leqslant |(\pi(s) Rx - h, Ry)_H| + \Big|\sum_{k=1}^{m+1} \langle B(s_k) x_k, y\rangle\Big| \\
&\leqslant \varepsilon \|Ry\|_H + q(y) \Big(\sum_{j,k=1}^{m+1} \langle B(s_j^* s_k) x_k, x_j\rangle\Big)^{1/2} \\
&= \varepsilon \|Ry\|_H + q(y) \Big\|\sum_{k=1}^{m+1} \pi(s_k) Rx_k\Big\|_H \\
&= \varepsilon \|Ry\|_H + q(y) \Delta \leqslant \varepsilon (\|Ry\|_H + q(y)).
\end{aligned}$$

Since ε is an arbitrary positive real number, equality (34) follows. Thus all we have just proved can be stated as follows: if the function B satisfies BC and (36), then each minimal $*$-dilation of C_B is a minimal $*$-dilation of B. This completes the proof.

3.8. Remark. A careful examination of the proof of Theorem 3.7 shows that condition (ii) of the theorem can be replaced by the following one: there exists a function q: $X \to \boldsymbol{R}_+$ for which inequality (36) holds true for all finite sequences $x_1, \ldots, x_n, y \in X$, $s_1, \ldots, s_{n-1} \in S_* \cdot S_*$ and $s_n \in S_*$, where $n \geqslant 2$ (see Appendix for a more general version of Theorem 3.7).

3.9. Remark. Notice that if q_{m}^B: $X \to \boldsymbol{R}_+$ is the minimal function satisfying (36), then for each minimal $*$-dilation (H, R, π) of B, $q_{\mathrm{m}}^B(y) = \|Ry\|_H$ ($y \in X$). Moreover, if q_{m} stands for the minimal function q which satisfies (29) with $C = C_B$, then $q_{\mathrm{m}} = q_{\mathrm{m}}^B$. Thus Theorem 1 of [17] follows from Theorem 3.7.

The next result extends Theorem 3.7 to a little more general algebraic context.

3.10. THEOREM. *Let J be a subsemigroup of S_* such that $J^* = J$. Let B: $S_* \to F(X)$ be a positive-definite function which satisfies the boundedness condition. Then*:

(I) *The following two conditions are equivalent*:

(i) *there exists a Hilbert space H ever $\boldsymbol{F}$, a function D: $S_* \to L(X, H)$, a $*$-representation π of S_* on H and an operator $R \in L(X, H)$ such that*

(37) *(H, D, π) is a minimal propagator of C_B,*

$$D(s) = \pi(s)R \quad (s \in J), \tag{38}$$

$$\langle B(s)x, y\rangle = (\pi(s)Rx, Ry)_H \quad (s \in J,\ x, y \in X); \tag{39}$$

(ii) *there exists a function* $q\colon X \to \boldsymbol{R}_+$ *such that for all finite sequences* $s_1, \ldots, s_n \in J$ *and* $x_1, \ldots, x_n$, $y \in X$ *we have*

$$\Big|\sum_{k=1}^{n} \langle B(s_k)x_k, y\rangle\Big|^2 \leqslant q(y)^2 \sum_{j,k=1}^{n} \langle B(s_j^* s_k)x_k, x_j\rangle. \tag{40}$$

(II) *If* (H, D, π, R) *and* (H', D', π', R') *are two quadruples which satisfy conditions* (37), (38), *and* (39), *then there exists a unique unitary operator* $U \in CL(H, H')$ *which fulfils conditions* (11), (12) *and*

$$UP_J R = P'_J R', \tag{41}$$

where P *(resp.* P'*) is the orthogonal projection of* H *(resp.* H'*) onto* $H_J = \bigvee \{D(s)X\colon s \in J\}$ *(resp.* $H'_J = \bigvee \{D'(s)X\colon s \in J\}$*).*

Using arguments similar to those utilized in the proof of Lemma 3.4 one can show the following:

If J *is a* $*$*-ideal of* S_* *(i.e.,* $JS_* \subset J$ *and* $J^* = J$*) and a function* $\boldsymbol{B}\colon S_* \to F(X)$ *satisfies condition* (ii) *of Theorem* 3.10 *as well as the boundedness condition with a scalar function* M *not identically vanishing on* J*, then* $\boldsymbol{B}$ *is positive-definite.*

Proof of Theorem 3.10. (I) The proof of the implication (i)$\Rightarrow$(ii) is similar to that of Theorem 3.7, so we only have to prove the converse implication.

(ii)$\Rightarrow$(i) It follows from Theorem 1.2 that the kernel C_B has a minimal propagator (H, D, π). By Lemma 3.1, π is a $*$-representation of S_*. Since $J = J^*$ is a subsemigroup of S_*, the space $H_J = \bigvee \{D(s)X\colon s \in J\}$ reduces all the operators $\{\pi(s)\colon s \in J\}$. This implies that the function $\pi_J\colon J \to CL(H_J)$ defined by

$$\pi_J(s)h = \pi(s)h \quad (s \in J,\ h \in H_J)$$

is a $*$-representation of J on H_J. In the sequel D_J stands for the function from J into $L(X, H_J)$ defined by

$$D_J(s)x = D(s)x \quad (s \in J,\ x \in X).$$

Applying the first part of Theorem 3.7 to the restriction B_J of B to J, we obtain a minimal $*$-dilation (H', R', π') of B_J. Since (H', D', π') and (H_J, D_J, π_J) are minimal propagators of C_{B_J} $(D'(s) = \pi'(s)R',\ s \in J)$, Theorem 1.2 gives a unitary operator $U \in CL(H', H_J)$ such that $UD'(s) = D_J(s)$ and $U\pi'(s) = \pi_J(s)U$ for each $s \in J$. Denote by R the operator from X into H_J

defined by $Rx = UR'x$, $x \in X$. Then we have

$$\langle B(s)x, y\rangle = \langle B_J(s)x, y\rangle = (\pi'(s)R'x, R'y)_{H'} = (U\pi'(s)R'x, UR'y)_{H_J}$$
$$= (\pi_J(s)UR'x, Ry)_{H_J} = (\pi(s)Rx, Ry)_H \quad (s \in J,\ x, y \in X).$$

Similarly we show that condition (38) is fulfilled.

Notice that the implication (ii) $\Rightarrow$ (i) of Theorem 3.10 can also be proved in the way we have used to solve problem 1° of Section 2 (define the functional φ_y on H_J by $\varphi_y(D(s)x) = \langle B(s)x, y\rangle$, $s \in J$, $x \in X$).

The second part of Theorem 3.10 follows from the second part of Theorem 3.7 via Theorem 1.2. We leave the details to the reader. This completes the proof.

Observe now that using Lemma 3.3(i), Lemma 3.4 and Proposition 5.5 one can directly deduce Theorem 3.5 from Theorem 3.10 (put $J = S_* \cdot S_*$), without appealing to Theorem 2.2 (recall that Theorem 2.2 is not utilized in the proof of condition (iii) of Theorem 3.5). Summing up, Theorems 3.5, 3.7 and 3.10 are logically equivalent.

In the second part of this section we shall make some comments on the kernel $\tilde{C}$ canonically associated with a given $F(X)$-valued kernel defined on an arbitrary semigroup S without unit. To make our considerations more pellucid we shall replace the kernel $\tilde{C}$ by a new one C_T defined on a supersemigroup T of S. As in Lemma 2.1, C is a PD kernel which satisfies BC. Denote by S_1 the unitization of S (i.e., $S_1 = S \cup \{1\}$, $s \cdot 1 = 1 \cdot s = s$, $s \in S$, and $1 \cdot 1 = 1$). Then the set $T = S_1 \times S_1$ with the multiplication

$$(s, t)(u, v) = (us, tv) \quad (s, t, u, v \in S_1)$$

and the involution

$$(s, t)^* = (t, s) \quad (s, t \in S_1)$$

become a $*$-semigroup with the unit (1.1). Moreover, the map $(-)$: $S \to T$ defined by

$$\bar{s} = (1, s) \quad (s \in S) \tag{42}$$

is an injective semigroup homomorphism. Now let us fix a minimal propagator (H, D, π) of C and put $\pi(1)$ = the identity operator on H and $D(1) = D(s_0)$, where s_0 is an arbitrary element of S. Then we define a new kernel C_T: $T \times T \to F(X)$ by the same formula as in Lemma 2.1:

$$\langle C_T((s, t), (u, v))x, y\rangle = (\pi(u)^* D(v)x, \pi(s)^* D(t)y)_H$$
$$(s, t, u, v \in S_1,\ x, y \in X).$$

It follows from the definition that C_T is a PD kernel which has the following two properties:

$$C(s, t) = C_T(\bar{s}, \bar{t}) \quad (s, t \in S), \tag{43}$$

$$C_T(\bar{u}\bar{s}, \bar{t}) = C_T(\bar{s}, \bar{u}^* \bar{t}) \quad (s, t, u \in S). \tag{44}$$

Notice that condition (44) is the restriction of the transfer property of C_T to the subsemigroup $\bar{S}$ of T.

The existence of a PD kernel C_T which satisfies conditions (43) and (44) is closely related to the problem of the existence of closed densely defined propagators. Namely, we have the following generalization of the Masani theorem (cf. [10], Theorem 4.7, p. 430).

3.11. THEOREM. *Let (H, D) be a minimal factorization of a positive-definite $F(X)$-valued kernel C on S. Then the following two conditions are equivalent:*

(i) *there exists a family $\{\pi(s)\colon s \in S\}$ of closed densely defined operators in H which satisfies the following two conditions:*

(45) *the set $\bigcup \{D(s)X\colon s \in S\}$ is contained in the domain of each operator of the family $\{\pi(s)\colon s \in S\} \cup \{\pi(t)^*\colon t \in S\}$,*

$$\pi(s) D(t) x = D(st) x \quad (s, t \in S,\ x \in X); \tag{46}$$

(ii) *there exists a $*$-semigroup T, an injective semigroup homomorphism $(-)\colon S \to T$ and a positive-definite $F(X)$-valued kernel C_T on T which fulfils conditions* (43) *and* (44).

Proof. We only have to prove the implication (ii) $\Rightarrow$ (i). Our proof is an adaptation of the proof of the Proposition from [25], p. 253. For each $s \in S$ we define an operator $\pi_0(s)$ on the linear span of the set $\bigcup \{D(t)X\colon t \in S\}$ by

$$\pi_0(s) \sum_{k=1}^{n} D(t_k) x_k = \sum_{k=1}^{n} D(st_k) x_k \quad (t_1, \ldots, t_n \in S,\ x_1, \ldots, x_n \in X).$$

All we have to prove now is the correctness of the definition of $\pi_0(s)$ and the validity of condition (45). Let (H_T, D_T) be a minimal factorization of C_T. Denote by H' the Hilbert space $\bigvee \{D_T(\bar{s})X\colon s \in S\}$ and by $D'(s)$ the operator from X into H' which maps the vector x to $D_T(\bar{s})x$ for $s \in S$ and $x \in X$. Then by (43), (H', D') is a minimal factorization of C. It follows from Theorem 1.1 that there exists a unitary operator $U \in CL(H, H')$ which fulfils condition (5). Thus by (43) and (44) we have

$$\begin{aligned}
(47)\quad \Big(\sum_{k=1}^{n} D(st_k) x_k, \sum_{j=1}^{m} D(u_j) y_j\Big)_H &= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle C(u_j, st_k) x_k, y_j \rangle \\
&= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle C_T(\bar{s}^* \bar{u}_j, \bar{t}_k) x_k, y_j \rangle = \Big(\sum_{k=1}^{n} D_T(\bar{t}_k) x_k, \sum_{j=1}^{m} D_T(\bar{s}^* \bar{u}_j) y_j\Big)_{H_T} \\
&= \Big(\sum_{k=1}^{n} D'(t_k) x_k, P_{H'}\Big(\sum_{j=1}^{m} D_T(\bar{s}^* \bar{u}_j) y_j\Big)\Big)_{H'} \\
&= \Big(\sum_{k=1}^{n} D(t_k) x_k, U^* P_{H'}\Big(\sum_{j=1}^{m} D_T(\bar{s}^* \bar{u}_j) y_j\Big)\Big)_H
\end{aligned}$$

for all finite sequences $t_1, \ldots, t_n \in S$, $u_1, \ldots, u_m \in S$, $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in X$, where $P_{H'}$ stands for the orthogonal projection of H_T onto H'. Now it is easy to see that both the correctness of the definition of $\pi_0(s)$ and condition (45) follows from equalities (47) and the minimality condition (4). In particular, we have

$$\pi(s)^* D(t) x = U^* P_{H'} D_T(\bar{s}^* \bar{t}) x \quad (s, t \in S,\ x \in X),$$

where $\pi(s)$ stands for the closure of $\pi_0(s)$. This completes the proof.

4. Applications. This section is complementary to the previous one. As usual, through the whole section X stands for a vector space over F. We start with the following generalization of the Arveson theorem (cf. [21], Corollary, p. 880; [17], Corollary 1, p. 151).

4.1. THEOREM. *Let S_* be a $*$-semigroup without unit and let B be an $F(X)$-valued function on S_* which satisfies condition* (ii) *of Theorem* 3.7. *If there exists a function $p\colon X \to \boldsymbol{R}_+$ such that*

$$\langle B(s^* s) x, x\rangle \leqslant p(x)^2 \quad (x \in X,\ s \in S_*),$$

then B is $*$-dilatable.

Proof. In virtue of Theorem 3.7 we only have to show that B satisfies BC. Observe that

$$\liminf_{k\to\infty} \langle B(t^*(s^* s)^{2^{k+1}} t) x, x\rangle^{2^{-(k+1)}} \leqslant \liminf_{k\to\infty} p(x)^{2^{-k}} \leqslant 1 \quad (s, t \in S_*,\ x \in X).$$

Thus BC follows from Theorem 1 (iii) of [19]. This completes the proof.

Representation theory of star algebras is the next area of our applications. First, recall some indispensable notions and definitions. Let $\mathscr{A}$ be a Banach star algebra without unit. A linear functional φ on $\mathscr{A}$ is said to be *positive* if

$$\varphi(a^* a) \geqslant 0 \quad \text{for each } a \in \mathscr{A}.$$

A non-zero linear functional φ on $\mathscr{A}$ is said to be *representable* if there exists a complex Hilbert space H, a $*$-representation π of $\mathscr{A}$ on H and a vector $h \in H$ such that

$$\varphi(a) = (\pi(a) h, h)_H \quad (a \in \mathscr{A}), \tag{48}$$

$$H = \bigvee \{\pi(a) h\colon\ a \in \mathscr{A}\}. \tag{49}$$

The following theorem gives necessary and sufficient conditions for linear functionals on $\mathscr{A}$ to be representable.

4.2. THEOREM. *A non-zero linear functional φ on a Banach star algebra $\mathscr{A}$ without unit is representable if and only if there exists $\delta > 0$ such that*

$$|\varphi(a)|^2 \leqslant \delta \varphi(a^* a) \quad (a \in \mathscr{A}). \tag{50}$$

Theorem 4.2 can be found in the book of Bonsall and Duncan [2] (cf. Theorem 37.11, p. 199). In their proof, the authors make essential use of the self-adjointness of the functional φ, i.e.,

$$\varphi(a^*) = \overline{\varphi(a)} \quad (a \in \mathscr{A}) \tag{51}$$

without any substantiation of that fact. Our proof, contrary to theirs, does not utilize equality (51) at all. In other words, to prove the implication (50) $\Rightarrow$ (51) we must use Theorem 4.2 (condition (51) then follows directly from the definition of the representability of φ). As will be shown below, the automatic validity of BC for positive linear functionals on Banach star algebras guarantees the truthfulness of Theorem 4.2 and the implication (50) $\Rightarrow$ (51). Unfortunately, we do not know any direct proof of the implication (50) $\Rightarrow$ (51).

Proof of Theorem 4.2. Let φ be a linear functional on $\mathscr{A}$ which satisfies inequality (50). Then the functional φ is positive and, by Lemma 37.6 (iv) of [2], p. 197, fulfils

$$\varphi(b^* a^* ab) \leqslant r\big((a^* a)^2\big)^{1/2} \varphi(b^* b) \quad (a, b \in \mathscr{A}), \tag{52}$$

where $r(a)$ stands for the spectral radius of $a \in \mathscr{A}$. Now treating $\mathscr{A}$ as a multiplicative $*$-semigroup and φ as a linear function from $\mathscr{A}$ into F one can infer condition (ii) of Theorem 3.7 and BC from inequalities (50) and (52), respectively. Thus, in virtue of Theorem 3.7 (or Theorem 1 of [17], p. 150), there exists a minimal $*$-dilation (H, R, π) of φ. It follows from the Sz.-Nagy dilation theorem (cf. [26]; [12], Proposition 2 (a), p. 29) that π is a linear map. Take $h = R1$. Then the triple (H, h, π) satisfies conditions (48) and (49). The converse implication is obvious. This completes the proof.

The next theorem is related to the Naimark dilation theorem (cf. [1]; [12], Theorem 4, p. 30; [16]; [29], Proposition 3; see also [4]; [5]; [13], Proposition 2.1 and [8], Theorem 3.4). Here we consider set functions which are defined on rings of sets. Recall that a ring $\mathscr{R}$ of subsets of a set Ω is not assumed to contain Ω (for the definitions of all related notions see [9]).

4.3. THEOREM. *Let $\mathscr{R}$ be a ring of subsets of a set Ω and let $G\colon \mathscr{R} \to F(X)$ be a set function which satisfies the following condition:*

(53) *for each $x \in X$, $\langle G(\cdot)x, x\rangle$ is a finitely additive (resp. σ-additive) non-negative set function.*

Then the following two conditions are equivalent:

(i) *there exists a Hilbert space H over F, an orthogonal projection-valued function $E\colon \mathscr{R} \to CL(H)$ and an operator $R \in L(X, H)$ such that*

(54) *for each $h \in H$, $(E(\cdot)h, h)_H$ is a finitely additive (resp. a σ-additive) set function,*

$$E(A \cap B) = E(A)E(B) \quad (A, B \in \mathscr{R}), \tag{55}$$

$$\langle G(\cdot)x, y\rangle = (E(\cdot)Rx, Ry)_H \quad (x, y \in X), \tag{56}$$

$$H = \bigvee \{E(A)RX\colon A \in \mathscr{R}\}; \tag{57}$$

(ii) *there exists a function* $q\colon X \to \boldsymbol{R}_+$ *such that*

$$\langle G(\cdot)x, x\rangle \leqslant q(x)^2 \quad (x \in X).$$

Proof. (i)$\Rightarrow$(ii) Since for each $A \in \boldsymbol{R}$, $E(A)$ is an orthogonal projection, we can write

$$\langle G(A)x, x\rangle = (E(A)Rx, Rx)_H \leqslant \|Rx\|_H^2 \quad (x \in X).$$

(ii)$\Rightarrow$(i) $\mathscr{R}$ with intersection as semigroup operation and the identity mapping as involution becomes a $*$-semigroup. Condition (53) implies that G is a *PD* function (cf. [26]; [10], Lemma 5.6, p. 442; [12], p. 30), so it has a minimal factorization (H, D). Now it is plain that for all finite sequences $A_1, \ldots, A_n \in \mathscr{R}$ and $x_1, \ldots, x_n, y \in X$ we have

$$\begin{aligned}
\Big|\sum_{k=1}^{n} \langle G(A_k)x_k, y\rangle\Big|^2 &= \Big|\sum_{k=1}^{n} \langle G(A \cap A_k)x_k, y\rangle\Big|^2 = \Big|\Big(\sum_{k=1}^{n} D(A_k)x_k, D(A)y\Big)_H\Big|^2 \\
&\leqslant \langle G(A)y, y\rangle \sum_{j,k=1}^{n} \langle G(A_j \cap A_k)x_k, x_j\rangle \\
&\leqslant q(y)^2 \sum_{j,k=1}^{n} \langle G(A_j \cap A_k)x_k, x_j\rangle,
\end{aligned}$$

where $A = \bigcup_{k=1}^{n} A_k$. Moreover, for all $A, B \in \mathscr{R}$ and $x \in X$

$$\langle G(B \cap A \cap B)x, x\rangle \leqslant \langle G(B \cap B)x, x\rangle.$$

Thus, by Theorem 1 (ii) of [25] (cf. also [19], Theorem 1 (ii)), G satisfies *BC*. Now Theorem 3.7 implies that there is a minimal $*$-dilation (H, R, E) of G. It is easy to see that E satisfies (55), (56) and (57). Condition (54) follows from (53) via (57) (for this see [12], Proposition 2 (b), p. 29; [6], Theorem 2). This completes the proof.

Notice that Theorem 4.3 can also be deduced from the Naimark dilation theorem by extending the measure G to a new one defined on a suitable algebra of sets (cf. [1], Theorem 1, p. 172).

5. Comments on minimality conditions. In this section we try to construct minimal-dilation-type objects (such as factorizations, propagators and so on) from those which are not assumed to satisfy a suitable minimality condition (such as (4) or (15)). First we explain what we mean by factorization, propagator and so on. Since all these notions come from the corresponding former ones simply by dropping the minimality condition, we shall present only the definition of dilation. By a *dilation* of a kernel $C\colon S\times S\to F(X)$, S being a semigroup without unit, we mean a triple (H, R, π) which satisfies all the requirements of the definition of minimal dilation except the minimality condition (15).

The following proposition shows how to obtain minimal factorizations and minimal propagators from given factorizations and propagators, respectively.

5.1. PROPOSITION. (I) *Let (H, D) be a factorization of a kernel $C\colon T\times T\to F(X)$ defined on a set T. Denote by H_0 the Hilbert space $\bigvee\{D(t)X\colon t\in T\}$ and by $D_0\colon T\to L(X, H_0)$ the function defined by $D_0(t)x = D(t)x$ for $t\in T$ and $x\in X$. Then (H_0, D_0) is a minimal factorization of C.*

(II) *Let (H, D, π) be a propagator of a kernel $C\colon S\times S\to F(X)$. Denote by $\pi_0\colon S\to CL(H_0)$ the representation of S defined by $\pi_0(s)h = \pi(s)h$ for $s\in S$ and $h\in H_0$. Then (H_0, D_0, π_0) is a minimal propagator of C.*

Suppose we are given a kernel $C\colon S\times S\to F(X)$. Let (H, R, π) be a dilation of C which is not minimal. Then (H, D, π), where $D(s) = \pi(s)R$ for $s\in S$, is a propagator of C, and (H_0, D_0, π_0) defined in the same way as in Proposition 5.1 is a minimal propagator of C. Denote by H_π and H_{π_0} the Hilbert spaces $\bigvee\{\pi(s)^* H_0\colon s\in S\}$ and $\bigvee\{\pi_0(s)^* H_0\colon s\in S\}$, respectively. In the sequel P_0, P_π and P_{π_0} stand for the orthogonal projections of H onto H_0, H_π and H_{π_0}, respectively. Observe now that if there exists an operator $\tilde{R}\in L(X, H_0)$ which satisfies

$$\pi_0(s)\tilde{R}x = \pi(s)Rx \quad (s\in S,\ x\in X), \tag{58}$$

then $(H_0, \tilde{R}, \pi_0)$ is a minimal dilation of C and so C is dilatable. The natural question arises whether a kernel C which has a dilation is always dilatable. The answer is negative in general (see Example 5.4).

Suppose now that a kernel C is dilatable. Then (see the proof of Theorem 2.2) there exists a unique operator $R_0\in L(X, H_{\pi_0})$ such that

$$\big(\pi_0(s)^*\pi(t)Rx, R_0 y\big)_{H_{\pi_0}} = \langle C(s, t)x, y\rangle \quad (s, t\in S,\ x, y\in X) \tag{59}$$

and

$$\pi_0(s)R_0 x = D_0(s)x = \pi(s)Rx \quad (s\in S,\ x\in X). \tag{60}$$

This means that in this case there always exists an operator $\tilde{R}\in L(X, H_0)$ which fulfils condition (58).

The next proposition gives necessary and sufficient conditions for an operator $\tilde{R}\in L(X, H_0)$ to satisfy equalities (58).

5.2. PROPOSITION. *Suppose that a kernel C is dilatable. If $\tilde{R}\in L(X, H_0)$, then the following three conditions are equivalent*:

(i) *the equality* (58) *holds true,*

(ii) $P_\pi \tilde{R}x = P_\pi Rx$ $(x \in X)$,

(iii) $P_{\pi_0} \tilde{R}x = R_0 x$ $(x \in X)$.

Proof. (i) $\Rightarrow$ (ii) Condition (58) implies that

$$\begin{aligned}(P_\pi \tilde{R}x - P_\pi Rx,\ &\pi(s)^* \pi(t) Ry)_{H_\pi}\\ &= (\tilde{R}x,\ \pi(s)^* \pi(t) Ry)_H - (Rx,\ \pi(s)^* \pi(t) Ry)_H\\ &= (\pi_0(s) \tilde{R}x,\ \pi(t) Ry)_{H_0} - (\pi(s) Rx,\ \pi(t) Ry)_H = 0\end{aligned}$$

for all $s, t \in S$ and $x, y \in X$. This means that (ii) holds true.

(ii) $\Rightarrow$ (iii) Condition (ii) yields

$$\begin{aligned}(61)\quad (\pi_0(s)^* \pi(t) Rx,\ P_{\pi_0} \tilde{R}y)_{H_{\pi_0}} &= (\pi_0(s)^* \pi(t) Rx,\ \tilde{R}y)_{H_0}\\ &= (P_0 \pi(s)^* \pi(t) Rx,\ \tilde{R}y)_{H_0}\\ &= (\pi(s)^* \pi(t) Rx,\ \tilde{R}y)_H\\ &= (\pi(s)^* \pi(t) Rx,\ Ry)_H = \langle C(s, t) x, y\rangle\end{aligned}$$

for all $s, t \in S$ and $x, y \in X$. Comparing equations (61) and (59), we obtain condition (iii).

(iii) $\Rightarrow$ (i) Since the null space of the representation π_0 is equal to $H_0 \ominus H_{\pi_0}$, conditions (60) and (iii) yield

$$\pi_0(s) \tilde{R}x = \pi_0(s) P_{\pi_0} \tilde{R}x = \pi_0(s) R_0 x = \pi(s) Rx \qquad (s \in S,\ x \in X).$$

This completes the proof.

Proposition 5.2 can be rewritten in a slightly modified version.

5.3. Proposition. *Let* (H, R, π) *be a dilation of* C. *Then the following three conditions are equivalent*:

(i) *there is* $\tilde{R} \in L(X, H_0)$ *such that equality* (58) *holds,*

(ii) *there is* $\tilde{R} \in L(X, H_0)$ *such that* $P_\pi \tilde{R} = P_\pi R$,

(iii) C *is dilatable.*

Proof. We only have to prove (ii) $\Rightarrow$ (iii). It follows from (ii), via (61), that

$$(62)\qquad \langle C(s, t) x, y\rangle = (\pi_0(s)^* D_0(t) x,\ P_{\pi_0} \tilde{R}y)_{H_{\pi_0}} \qquad (s, t \in S,\ x, y \in X).$$

Since (H_0, D_0, π_0) is a minimal propagator of C, one can show that equality (62) implies inequality (22) with $q(y) = \|P_{\pi_0} \tilde{R}y\|$, $y \in X$. In virtue of Theorem 2.2 the kernel C is dilatable. This completes the proof.

Now we give an example of a non-dilatable kernel which has a dilation.

5.4. Example. Let S be the additive semigroup of all natural numbers n

$\geqslant 1$. Let H be an inifinite-dimensional separable complex Hilbert space with an orthogonal bases $\{h_n\}_{n=1}^{\infty}$. Denote by V the unilateral shift of multiplicity one on H related to $\{h_n\}_{n=1}^{\infty}$ (i.e., $Vh_n = h_{n+1}$ for $n \geqslant 1$). Now let us define an $F(C)$-valued kernel C on S, an operator $R \in L(C, H)$ and a representation π of S on H by

$$\langle C(n, m)\alpha, \beta\rangle = \begin{cases} 0 & \text{if } n \neq m \\ \alpha\bar{\beta} & \text{if } n = m \end{cases} \quad (m, n \in S,\ \alpha, \beta \in C),$$

$$R\alpha = \alpha h_1 \quad (\alpha \in C), \qquad \pi(n) = V^n \quad (n \in S).$$

Then (H, R, π) is a dilation of C (C is a PD kernel which satisfies BC). Since $V^* h_{n+1} = h_n$ for $n \geqslant 1$ and $V^* h_1 = 0$, we obtain $H_0 = H_{\pi_0} = H \ominus Ch_1$ and $H_\pi = H$. Therefore $P_{\pi_0} = P_0$ and $P_\pi =$ the identity operator on H. Suppose for a while that the kernel C is dilatable. Then there exists an operator $\tilde{R} \in L(C, H_0)$ which fulfils condition (58) with $X = C$. Thus, by Proposition 5.2 (ii), $h_1 = R1 = \tilde{R}1 \in H_0$, which contradicts the conditions: $h_1 \perp H_0$ and $\|h_1\| = 1$. This means that C is not dilatable.

Let (H, R, π) be a dilation of an $F(X)$-valued kernel C on S. Denote by $\mathscr{D}_\pi$ the class of all operators $\tilde{R} \in L(X, H_0)$ which fulfil condition (58). The next proposition describes that class with the aid of the space $L(X, H_0 \ominus H_{\pi_0})$.

5.5. PROPOSITION. *Suppose that* $\mathscr{D}_\pi \neq \emptyset$. *Then the function* $\psi\colon L(X, H_0 \ominus H_{\pi_0}) \to \mathscr{D}_\pi$ *which maps an operator* W *to* $R_0 + W$ *for* $W \in L(X, H_0 \ominus H_{\pi_0})$ *is correctly defined, one-to-one and onto.*

Proof. We first show that ψ is correctly defined. Indeed, since $H_0 \ominus H_{\pi_0} \perp H_\pi$, we have for each $W \in L(X, H_0 \ominus H_{\pi_0})$

$$P_\pi(R_0 x + Wx) = P_\pi R_0 x = P_\pi Rx \quad (x \in X).$$

In virtue of Proposition 5.2, $\psi(W) \in \mathscr{D}_\pi$ for each $W \in L(X, H_0 \ominus H_{\pi_0})$.

It is plain that ψ is one-to-one. To prove that ψ is onto, take $\tilde{R} \in \mathscr{D}_\pi$. Then, by Proposition 5.2, we have

$$\tilde{R}x = P_{\pi_0}\tilde{R}x + P_1\tilde{R}x = R_0 x + P_1\tilde{R}x \quad (x \in X),$$

where P_1 is the orthogonal projection of H onto $H \ominus H_{\pi_0}$. Thus $\tilde{R} = \psi(W)$ with a suitable $W \in L(X, H_0 \ominus H_{\pi_0})$.

This completes the proof.

In general, Propositions 5.2 and 5.5 do not give any simple method of constructing operators of class $\mathscr{D}_\pi$. In the sequel we shall consider the following two cases when these propositions allow us to describe $\mathscr{D}_\pi$ with the aid of the operators R and P_0.

(a) S has a unit e and $\pi(e) =$ the identity operator on H.

(b) S is a $*$-semigroup (with or without unit) and π is a $*$-representation (equivalently, (H, R, π) is a $*$-dilation of C).

Notice that in both these case the kernel C is dilatable and so $\mathcal{S}_\pi \neq \emptyset$. It is obvious that in the first case $H_{\pi_0} = H_0 \subset H_\pi$, hence $\mathcal{S}_\pi \doteq R$. In the second case $H_{\pi_0} = H_\pi \subset H_0$. Since $H_0 = \bigvee \{\pi(s)RX\colon s \in S_*\}$, one can show that $H_0 = \bigvee \{\pi(s)RX\colon s \in S_*^2\} = H_\pi$ (see the proof of condition (iii) of Theorem 3.5). Thus by Proposition 5.2, $\mathcal{S}_\pi = \{P_0 R\}$.

The next proposition can also be proved independently on Proposition 5.2 (cf. [18], Proposition 1).

5.6. Proposition. *Let (H, R, π) be a $*$-dilation of an $F(X)$-valued function B defined on a $*$-semigroup S_* without unit. Then the space H_0 reduces π and $(H_0, \tilde{R}_0, \pi_0)$ is a minimal $*$-dilation of B, where $\tilde{R}_0$ is defined by $\tilde{R}_0 x = P_0 Rx$ for $x \in X$.*

We can sum up the results of Section 5 as follows:

1° If a kernel has a factorization (resp. propagator, $*$-dilation), then it always has a minimal factorization (resp. minimal propagator, minimal $*$-dilation).

2° If a function has a $*$-dilation, then it always has a minimal $*$-dilation.

3° Unlike 1° and 2°, there exist non-dilatable kernels which have dilations.

Appendix. We state here a version of Theorem 3.7 whose the generalization of (I) has been done in Remark 3.8.

Theorem A. *Let B be an $F(X)$-valued function defined on a $*$-semigroup S_*. Suppose that B satisfies BC. Then for each $N \geqslant 1$ the following two conditions are equivalent:*

(i) *B is $*$-dilatable,*

(ii)$_N$ *there is a function $q\colon X \to \mathbf{R}_+$ such that inequality* (36) *holds for each $n \geqslant 2$ and for all finite sequences $x_1, \ldots, x_n, y \in X$, $s_1, \ldots, s_{n-1} \in S_*^N$ and $s_n \in S_* \setminus (S_*^N)$.*

In order to prove Theorem A we need two lemmas.

Lemma A. *Let (H, D, π, R) be a system which satisfies conditions* (37), (38) *and* (39) *of Theorem* 3.10 *with $J := S_*^2$. If $RX \subset H_J := \bigvee D(J)X$, then*

(i) $H = \bigvee \pi(S_*)RX = H_J$,

(ii) $D(s) = \pi(s)R$ $(s \in S_*)$.

Proof. It follows from Lemma 3.1 that π is a $*$-representation of S_*. Therefore the Hilbert spaces $H_0 := \bigvee \pi(S_*)RX$ and H_J reduce π to the $*$-representations $\pi_0\colon S_* \to CL(H_0)$ and $\pi'\colon S_* \to CL(H_J)$, respectively. This and equality (38) imply $H_J = \bigvee \{D(st)X\colon s, t \in S_*\} = \bigvee \{\pi(st)RX\colon s, t \in S_*\} \subset H_0$. Since $RX \subset H_J$ and H_J reduces π, $H_0 = \bigvee \pi(S_*)RX \subset H_J$. Thus

(63) $$H_0 = H_J.$$

Denote by D' the function from S_* into $L(X, H_J)$ defined by

(64) $$D'(s) = \pi(s)R \quad (s \in S_*).$$

It follows from (39) and (63) that (H_J, D') is a minimal factorization of C_B. Since (H, D) is another minimal factorization of C_B, Theorem 1.1 implies the existence of a unitary operator $U \in CL(H_J, H)$ such that

(65) $$UD'(s) = D(s) \quad (s \in S_*).$$

Using (38), (64) and (65) we obtain

(66) $$UD(s) = U\pi(s)R = UD'(s) = D(s) \quad (s \in J).$$

Since the set $\{D(s)x\colon s \in J,\ x \in X\}$ is total in H_J, equalities (66) give $Uf = f$ for each $f \in H_J$. Thus $H = UH_J = H_J$ and so $U = I_H =$ the identity operator on H. In particular, (64) and (65) imply (ii). This completes the proof.

LEMMA B. *For each $N \geqslant 1$, if B satisfies BC and* $(\text{ii})_N$, *then B is positive-definite.*

Proof. It follows from $(\text{ii})_N$ that

(67) $$|\langle B(s)x, y\rangle|^2 \leqslant q(y)^2 \langle B(s^* s)x, x\rangle \quad (s \in S_*,\ x, y \in X).$$

Thus if $q = 0$ then $B = 0$. If $q \neq 0$ then there is $y_0 \in X$ such that $q(y_0) > 0$. Substituting $y = y_0$ into $(\text{ii})_N$, one can show that the restriction B_N of B to S_*^N is *PD*. Let M be the scalar function appearing in the definition of *BC*. If $M = 0$ then $\langle B(s^* s)x, x\rangle \leqslant 0$ for all $s \in S_*^2$ and $x \in X$. Thus, by (67), $B(s) = 0$ for each $s \in S_*^2$. Using again (67) we infer $B = 0$. If $M \neq 0$ then there is $t_0 \in S_*$ such that $M(t_0) > 0$ and

$$M(t_0)\sum_{k,l} \langle B(s_k^* s_l)x_l, x_k\rangle - \sum_{k,l} \langle B(s_k^* t_0^* t_0 s_l)x_l, x_k\rangle \geqslant 0$$

for all finite sequences $s_1, \ldots, s_n \in S_*$ and $x_1, \ldots, x_n \in X$. The above inequality can be used to obtain the following implication:

(68) If B_m is *PD* then B_{m-1} is *PD* $(m \geqslant 2)$.

Since B_N is *PD*, an application of (68) gives the positive-definiteness of $B = B_1$. This completes the proof.

Proof of Theorem A. In virtue of Lemma B we can assume that B is *PD*. Let $N_m := 2^m$ for $m \geqslant 0$. It is enough to prove the implication $(\text{ii})_{N_m} \Rightarrow (\text{i})$ for each $m \geqslant 0$. The case $m = 0$ is covered by Theorem 3.7. Suppose now that the implication $(\text{ii})_{N_m} \Rightarrow (\text{i})$ holds for every *PD* function which satisfies *BC*. We have to prove the implication $(\text{ii})_{N_{m+1}} \Rightarrow (\text{i})$. Let B be a *PD* function which satisfies *BC* and condition $(\text{ii})_{N_{m+1}}$. Denote by J the $*$-ideal S_*^2. Then B satisfies inequality (40) for all finite sequences $x_1, \ldots, x_n,\ y \in X$,

$s_1, \ldots, s_{n-1} \in J^{N_m}$ and $s_n \in J$. Applying the implication $(\mathrm{ii})_{N_m} \Rightarrow (\mathrm{i})$ to the restriction B_J of B to J we infer the $*$-dilatability of B_J. Now we can use Theorem 3.10 to obtain a system (H, D, π, R) which satisfies all the assumptions of Lemma A (the operator R constructed in the proof of Theorem 3.10 always satisfies the condition $RX \subset H_J$). Thus (H, R, π) is a minimal $*$-dilation of C_B. To prove equality (34) we can proceed in the same way as in the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 3.7. But now we have to use the equality $H = \bigvee \pi(S_*^{N_m+1}) RX$ and condition $(\mathrm{ii})_{N_{m+1}}$. This completes the proof.

Acknowledgment. The author is grateful to Professor A. Weron for many helpful suggestions improving the final version of the paper.

References

[1] S. K. Berberian, *Naimark's moment theorem*, Michigan Math. J. 13 (2) (1966), 171–184.
[2] F. F. Bonsall, J. Duncan, *Complete normed algebras*, Springer–Verlag (1973).
[3] S. D. Chatterji, *Factorization of positive definite operator-valued kernels*; *Prediction theory and harmonic analysis*, The Pesi Masani Volume, editors: V. Mandrekar and H. Salehi, North-Holland Publishing Company (1983), 23–36.
[4] J. Górniak, *Dilations of positive definite operator valued functions in linear spaces* (in Polish), Komunikaty Inst. Mat. Polit. Wrocł. Nr 73 (1977) (see also: *Locally convex spaces with the factorization property*, Colloq. Math. 48 (1984), 69–79).
[5] –, *Remarks on positive definite operator valued functions in linear spaces*, Lecture Notes in Math. vol. 656, Springer-Verlag (1978), 37–44.
[6] –, A. Weron, *An analogue of Sz.-Nagy's dilation theorem*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 24 (10) (1976), 867–872.
[7] –, –, *Aronszajn–Kolmogorov type theorems for positive definite kernels in locally convex spaces*, Studia Math. 69 (1981), 235–246.
[8] –, A. Makagon, A. Weron, *An explicit form of dilation theorems for semispectral measures*, in the Pesi Masani volume: *Prediction theory and harmonic analysis*, V. Mandrekar and H. Salehi, eds., North-Holland Publishing Company (1983), 85–111.
[9] P. R. Halmos, *Measure theory*, New York 1950.
[10] P. Masani, *Dilations as propagators of Hilbertian varieties*, SIAM J. Math. Anal. Vol. 9, No. 3 (1978), 414–456.
[11] –, *Propagators and dilations*, Lecture Notes in Math., vol. 656, Springer-Verlag (1978), 95–117.
[12] W. Mlak, *Dilations of Hilbert space operators (General theory)*, Dissertationes Math., Warszawa 1978, 1–65.
[13] –, *Conditionally positive definite functions on linear spaces*, Ann. Polon. Math. 42 (1983), 187–239.
[14] –, W. Szymański, *Dilations theorems for $*$-semigroups without unit*, preprint (1974).
[15] –, A. Weron, *Dilations of Banach space operator valued functions*, Ann. Polon. Math. 38 (1980), 295–303.

[16] M. A. Naimark, *On a representation of additive operator-valued set functions*, Dokl. Akad. Nauk SSSR 41 (1943), 359–361.

[17] J. Stochel, *A note on general operator dilations over $*$-semigroups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 28 No 3–4 (1980), 149–153.

[18] –, *Dilations of inductive limits of operator functions*, ibidem 28 No. 3–4 (1980), 143–148.

[19] –, *Individual boundedness condition for positive definite sesquilinear form valued kernels*, Studia Math. 74 (1982), 293–302.

[20] –, W. Szymański, *On the uniqueness of minimal R-dilations for unstarred semigroups*, Indiana University Math. J. Vol. 32, No. 6 (1983), 793–800.

[21] F. H. Szafraniec, *On the boundedness condition involved in dilation theory*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 24 (10) (1976), 877–881.

[22] –, *A general dilation theorem*, ibidem 25 (3) (1977), 263–267.

[23] –, *Dilations on involution semigroups*, Proc. Amer. Math. Soc. 66 (1977), 30–32.

[24] –, *A propos of Professor Masani's talk, Probability theory on vector spaces*, Lecture Notes in Math., vol. 656, Springer-Verlag (1978), 249–299.

[25] –, *Boundedness of the shift operator related to positive definite forms: an application to moment problems*, Arkiv för matematik, Vol. 19, No. 2 (1981), 251–259.

[26] B. Sz.-Nagy, *Fortsetzung linearer Transformationen des Hilbertschen Raumes mit Austritt aus dem Raum*, Appendix to F. Riesz., B. Sz.-Nagy, *Vorlesung über Funktionalanalyse*, Deutscher Verlag der Wissenschaften, Berlin 1950.

[27] B. Truong-Van-Manh, *Une généralisation du Théorème de Kolmogorov–Aronszajn; processus V-Bornés q-dimensionnelles: domaine spectral-dilatations stationnaires*, Ann. Inst. H. Poincaré, Sect. B, 17 (1981), 31–49.

[28] A. Weron, *Prediction theory in Banach spaces*, Probability-Winter School, Karpacz, Poland, 1975, Lecture Notes in Math., Springer 472 (1975), 207–228.

[29] –, *Remarks on positive definite operator valued functions in Banach spaces*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 24 (10) (1976), 873–876.

INSTYTUT MATEMATYCZNY PAN
ODDZIAŁ W KRAKOWIE, SOLSKIEGO 30
31-027 KRAKÓW, POLAND

Reçu par la Rédaction le 1981.02.16