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Continuity of intersection of analytic sets

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Dedicated to the memory of Jacek Szarski

Abstract. Let k be a positive integer and let $V_1, \ldots, V_k$ be purely k-dimensional analytic subsets of an open set $\Omega \subset C^n$. In this paper we present certain theorems on the continuity of the mapping

$$\cap: (V_1, \ldots, V_k) \to V_1 \cap \ldots \cap V_k.$$

As simple consequences we obtain the Hurwitz theorem and the theorem on injectivity of the limit of a sequence of injective holomorphic mappings f_ν: $\Omega \to C^n$.

1. Topology of local uniform convergence. Let X be a metric space. Let $\mathscr{F}_X$ be the family of all closed subsets of X. We endow $\mathscr{F}_X$ with the topology $\mathscr{T}_X$ generated by the sets

$$\mathscr{U}(S, K) = \{F \in \mathscr{F}_X\colon F \cap K = \emptyset,\ F \cap U \neq \emptyset \text{ for } U \in S\}$$

corresponding to all compact subsets $K \subset X$ and all finite families S of open subsets of X. We call this topology the *topology of local uniform convergence*.

A simple argument shows that if X, Y are metric spaces, then f: $X \to Y$ is a homeomorphism if and only if the mapping $\mathscr{F}_X \ni F \to f(F) \in \mathscr{F}_Y$ is a homeomorphism.

Now, $F_\nu \to F$ will denote that F is the limit of the sequence $\{F_\nu\}$ in the above topology. An immediate consequence of the definition is

LEMMA 1. *If $F, F_\nu \in \mathscr{F}_X$, $\nu = 1, 2, \ldots$, then the following statements are equivalent*:

(1) $F_\nu \to F$;

(2) *for every $x \in F$ there exists a sequence $x_\nu \in F_\nu$, $\nu = 1, 2, \ldots$, such that $x_\nu \to x$ (in the topology of X) and for every compact subset $K \subset X \setminus F$, $F_\nu \cap K \neq \emptyset$ for at most finitely many indices ν*;

(3) *for every $x \in F$ there exists a sequence $x_\nu \in F_\nu$, $\nu = 1, 2, \ldots$, such that $x_\nu \to x$, and for every $x \notin F$ there exists a neighbourhood U of x such that $F_\nu \cap U \neq \emptyset$ for at most finitely many indices ν*;

(4) *for every* $x \in F$, *and for every neighbourhood* V *of* x, $F_\nu \cap V = \emptyset$ *for at most finitely many integers* ν, *and for every* $x \notin F$ *there exists a neighbourhood* U *of* x *such that* $F_\nu \cap U \neq \emptyset$ *for at most finitely many indices* ν.

COROLLARY 1. *Let* X, Y *be two metric spaces and suppose that mappings* $f, f_\nu\colon X \to Y$, $\nu = 1, 2, \ldots$, *are continuous. If the sequence* $\{f_\nu\}$ *converges uniformly to* f *on compact subsets of* X, *then* $f_\nu \to f$ *in the topology* $\mathscr{T}_{X\times Y}$.

If X is a compact metric space, then the topology $\mathscr{T}_X$ can be introduced by the classical Hausdorff metric in $\mathscr{F}_X$ defined by

$$\operatorname{dist}(A, B) = \begin{cases} 0 & \text{for } A = B = \emptyset, \\ \max\{\max\limits_{x\in A}\operatorname{dist}(x, B), \max\limits_{x\in B}\operatorname{dist}(x, A)\} & \text{for } A \neq \emptyset,\ B \neq \emptyset, \\ \operatorname{diam} X + 1 & \text{in other cases.} \end{cases}$$

In this case $\mathscr{F}_X$ is a metrizable compact space (cf. [3], p. 58).

In general, we have

LEMMA 2. *Let* X *be a locally compact, second-countable metric space. Then* $\mathscr{F}_X$ *is a metrizable compact space.*

Proof. Let Y be a compact metric space and $h\colon X \to Y$ be a mapping such that $h(X)$ is an open subset of Y and $h\colon X \to h(X)$ is a homeomorphism. (For example we can take as Y a one-point compactification of X and for h the identical embedding.) It is easy to see that the mapping

$$p\colon \mathscr{F}_Y \ni F \to h^{-1}\big(F \cap h(X)\big) \in \mathscr{F}_X \quad \text{is continuous}$$

and that the set $\mathscr{K} = \{F \in \mathscr{F}_Y\colon F \supset \big(Y\setminus h(X)\big)\} \subset \mathscr{F}_Y$ is compact. The restriction resp: $\mathscr{K} \to \mathscr{F}_X$ is a continuous bijection defined on a compact space onto the Hausdorff space $\mathscr{F}_X$. Hence $\mathscr{K}$ and $\mathscr{F}_X$ are homeomorphic. This concludes the proof of Lemma 2.

2. Continuity of intersection. Let Ω be an open subset of C^n and let $\mathscr{T}_\Omega$ be the topology in $\mathscr{F}_\Omega$ described in Section 1, for $X = \Omega$. By $\mathscr{A}_p(\Omega)$ we will denote the subset of $\mathscr{F}_\Omega$ consisting of all purely p-dimensional analytic subsets of Ω. We will suppose that $\emptyset \in \mathscr{A}_p(\Omega)$ for $p = 0, 1, \ldots, n$.

PROPOSITION 1. *Suppose that* L *is an affine* $(n-k)$*-dimensional subspace of* C^n, $0 \leqslant k \leqslant n$, $F \in \mathscr{F}_\Omega$, z_0 *is an isolated point of* $F \cap L$. *Let* U *be an open neighbourhood of* z_0 *such that* $\overline{U} \subset \Omega$, $\overline{U}$ *is compact,* $\overline{U} \cap L \cap F = \{z_0\}$. *Then there exists a neighbourhood* $\mathscr{U}_F \in \mathscr{T}_\Omega$ *of* F *such that*

$$1 \leqslant \#(L \cap U \cap V) < \infty$$

for every purely k*-dimensional analytic subset* V *of* Ω *belonging to* $\mathscr{U}_F$.

Proof. If $k = 0$, then $L = C^n$ and it suffices to take $\mathscr{U}_F = \mathscr{U}(\{U\}, \partial U)$.

If $k = n$, then $L = \{z_0\}$, $\mathscr{U}_F = \mathscr{U}(\{\Omega_0\}, \emptyset)$, where Ω_0 is the component of Ω such that $z_0 \in \Omega_0$.

Let us fix $0 < k < n$ and assume that $z_0 = 0$. Let X be a k-dimensional vector subspace of C^n such that $C^n = X+L$. Obviously, there exist two open connected neighbourhoods U_X, U_L of 0 in X and L, respectively, such that $(\overline{U}_X+\partial U_L)\cap F = \emptyset$, $U_L\cap F = \{0\}$, $\overline{U_X+U_L} \subset U$. Let us define

$$\mathscr{U}_F^1 := \mathscr{U}(\{U_X+U_L\}, \overline{U}_X+\partial U_L),$$

and let p be the restriction of the projection $X+L \ni x+y \to x \in X$ to the set U_X+U_L and let V be a purely k-dimensional analytic subset of Ω. If $V\in\mathscr{U}_F^1$, then $p|_V\colon V\to U_X$ is proper. Since $V\cap(U_X+U_L)$ is a purely k-dimensional analytic subset and $p|_V$ is a proper mapping to an open, connected subset of C^k, it follows that

$$p|_V\colon V\to U_X$$

is a finite-sheeted branched covering of U_X (cf. Chapter III, Section B of [1], especially Theorem 21). Hence

$$1 \leqslant \#(p|_V)^{-1}(0) = \#(L\cap U_L\cap V) \leqslant \#(L\cap U\cap V).$$

Since $(F\cap\partial U)\cap L = \emptyset$, $F\cap\partial U$ is compact and L is closed, there exists an open set $G\supset F\cap\partial U$ such that $G\cap L = \emptyset$. Let us write

$$\mathscr{U}_F^2 = \mathscr{U}\big(\{\emptyset\}, (\partial U\setminus G)\big)$$

and let V be a purely k-dimensional analytic subset of Ω. If $V\in\mathscr{U}_F^2$, then $L\cap\overline{U}\cap V = L\cap U\cap V$. Hence $L\cap U\cap V$ is a compact analytic subset of U. Then it must be a finite subset of U. Therefore, it suffices to take $\mathscr{U}_F = \mathscr{U}_F^1\cap\mathscr{U}_F^2$.

Now, keeping Ω as before, we shall prove the following

THEOREM 1. *Let us suppose that*

(1) *$F_1, \dots, F_k$ are closed subsets of Ω and z_0 is an isolated point of $F_1\cap\dots\cap F_k$;*

(2) *U is an open neighbourhood of z_0 such that $\overline{U}\subset\Omega$, $\overline{U}$ is compact and $\overline{U}\cap(F_1\cap\dots\cap F_k) = \{z_0\}$.*

Then there exist neighbourhoods $\mathscr{U}_{F_1}, \dots, \mathscr{U}_{F_k}$ (in topology $\mathscr{T}_\Omega$) of the sets $F_1, \dots, F_k$, respectively, such that the condition $V_j\in\mathscr{A}_{d_j}(\Omega)\cap\mathscr{U}_{F_j}$, $j = 1, \dots, k$, and $\sum_{j=1}^{k} d_j = (k-1)n$ implies

$$1 \leqslant \#(V_1\cap\dots\cap V_k\cap U) < \infty.$$

Proof. Straightforward computation with use of Lemma 1 (3) yields that the mapping

$$P\colon \mathscr{F}_\Omega\times\dots\times\mathscr{F}_\Omega \ni (X_1, \dots, X_k) \to X_1\times\dots\times X_k \in \mathscr{F}_{\Omega\times\dots\times\Omega}$$

is continuous.

If we set $\Delta_k = \{(z, \ldots, z) \in (C^n)^k\}$, then $z \in X_1 \cap \ldots \cap X_k \Leftrightarrow (z, \ldots, z) \in (X_1 \times \ldots \times X_k) \cap \Delta_k$. Since the mapping

$$\delta_k\colon C^n \ni z \to (z, \ldots, z) \in \Delta_k$$

is a homeomorphism, the point $(z_0, \ldots, z_0)$ is an isolated point of $(F_1 \times \ldots \times F_k) \cap \Delta_k = F \cap \Delta_k$. Let U_k be an open neighbourhood of $(z_0, \ldots, z_0)$ in Ω^k such that

1° $\overline{U}_k \subset \Omega \times \ldots \times \Omega$;

2° $\overline{U}_k$ is compact;

3° $\overline{U}_k \cap \Delta_k = \overline{\delta_k(U)}$ and $U_k \cap \Delta_k = \delta_k(U)$.

Then $\overline{U}_k \cap \Delta_k \cap F = \overline{\delta_k(U)} \cap F = \delta_k(\overline{U} \cap F_1 \cap \ldots \cap F_k) = \delta_k(z_0)$. It follows from Proposition 1 that there exists a neighbourhood $\mathcal{U}_F$ of $F_1 \times \ldots \times F_k$ (in the topology $\mathcal{T}_{\Omega \times \ldots \times \Omega}$) such that

$$1 \leqslant \#(V \cap \Delta_k \cap U_k) < \infty$$

for every $V \in \mathcal{A}_{(k-1)n}(\Omega^k) \cap \mathcal{U}_F$. Since the mapping P is continuous, there exist open neighbourhoods $\mathcal{U}_{F_1}, \ldots, \mathcal{U}_{F_k}$ of $F_1, \ldots, F_k$, respectively, such that

$$P(\mathcal{U}_{F_1} \times \ldots \times \mathcal{U}_{F_k}) \subset \mathcal{U}_F.$$

Let us now suppose that $V_j \in \mathcal{A}_{d_j}(\Omega) \cap \mathcal{U}_{F_j}$ for $j = 1, \ldots, k$ and $\sum_{j=1}^{k} d_j = (k-1)n$. Then $V = V_1 \times \ldots \times V_k$ is a purely $(k-1)n$-dimensional analytic subset of Ω^k and $V \in \mathcal{U}_F$. Hence

$$\begin{aligned}
&\#\big((V_1 \times \ldots \times V_k) \cap \Delta_k \cap U_k\big) \\
&= \#\big(\delta_k^{-1}((V_1 \times \ldots \times V_k) \cap \Delta_k \cap U_k)\big) = \#\big(V_1 \cap \ldots \cap V_k \cap \delta_k^{-1}(U_k)\big) \\
&= \#(V_1 \cap \ldots \cap V_k \cap U).
\end{aligned}$$

Therefore $1 \leqslant \#(V_1 \cap \ldots \cap V_k \cap U) < \infty$.

Let Ω be an open subset of C^n and let F be a closed subset of Ω. Let $z_0 \in F$. We call z_0 a *t-proper point of F*, $t \in \{0, 1, \ldots, n\}$ if there exists an affine $(n-t)$-dimensional subspace of C^n such that z_0 is an isolated point of $L \cap F$.

THEOREM 2. *Let us suppose that*

(1) *$F_1, \ldots, F_k$ are closed subsets of Ω;*

(2) *z_0 is a t-proper point of $\bigcap_{j=1}^{k} F_j$, and L is an affine $(n-t)$-dimensional subspace of C^n such that z_0 is an isolated point of $F_1 \cap \ldots \cap F_k \cap L$;*

(3) *U is an open neighbourhood of z_0 such that $\overline{U} \subset \Omega$, $\overline{U}$ is compact and $\overline{U} \cap F_1 \cap \ldots \cap F_k \cap L = \{z_0\}$.*

Then there exist neighbourhoods $\mathscr{U}_{F_1}, \ldots, \mathscr{U}_{F_k}$ of the sets $F_1, \ldots, F_k$, respectively, such that if $V_j \in \mathscr{A}_{d_j}(\Omega) \cap \mathscr{U}_{F_j}$ for $j = 1, \ldots, k$, then the equality $t = \sum_{j=1}^{k} d_j - (k-1)n$ implies that

$$1 \leqslant \#(V_1 \cap \ldots \cap V_k \cap U \cap L) < \infty.$$

Proof. Applying Theorem 1 to $F_1, \ldots, F_k, F_{k+1} = L$ and $V_1, \ldots, V_k$, $V_{k+1} = L$ we get the required result.

Now we state the theorem on the continuity of intersection.

THEOREM 3. *Let $V_0 \in \mathscr{A}_p(\Omega)$, $W_0 \in \mathscr{A}_q(\Omega)$ and $p+q \geqslant n$. If $V_0 \cap W_0 \in \mathscr{A}_{p+q-n}(\Omega)$, then the mapping*

$$\cap: \mathscr{A}_p(\Omega) \times \mathscr{A}_q(\Omega) \ni (V, W) \to V \cap W \in \mathscr{F}_\Omega$$

is continuous at the point (V_0, W_0).

Proof. Let $V_\nu \in \mathscr{A}_p(\Omega)$, $W_\nu \in \mathscr{A}_q(\Omega)$ be two sequences such that $V_\nu \to V_0$ and $W_\nu \to W_0$. It has to be proved that $V_\nu \cap W_\nu \to V_0 \cap W_0$.

Let us fix $x \notin V_0 \cap W_0$. Then $x \notin V_0$ or $x \notin W_0$. By Lemma 1 (4) there exists an open neighbourhood U of x such that $V_\nu \cap U \neq \emptyset$ or $W_\nu \cap U \neq \emptyset$ for at most finitely many indices ν.

Then $(V_\nu \cap W_\nu) \cap U \neq \emptyset$ for at most finitely many indices ν.

If $x \in V_0 \cap W_0$, then it follows from the local analysis of analytic sets (see e.g. [1], [4]) that x is a $(p+q-n)$-proper point of $V_0 \cap W_0$. Let U be an open neighbourhood of x. It follows from Theorem 2 that there exists ν_0 such that $U \cap V_\nu \cap W_\nu \neq \emptyset$ for every $\nu \geqslant \nu_0$.

Therefore by Lemma 1 (4) we get $V_\nu \cap W_\nu \to V_0 \cap W_0$.

As an immediate consequence of Theorem 3 we obtain

COROLLARY 2. *Let W be a purely q-dimensional analytic subset of Ω. If $V_0 \in \mathscr{A}_p(\Omega)$, $p+q \geqslant n$ and $V_0 \cap W \in \mathscr{A}_{p+q-n}(\Omega)$, then the mapping*

$$\mathscr{A}_p(\Omega) \ni V \to V \cap W \in \mathscr{F}_W$$

is continuous at the point V_0.

THEOREM 4. *Let T be a topological space and let Ω be an open subset of C^n. If $t_0 \in T$ and*

$$g: T \times \Omega \ni (t, z) \to g(t, z) = g_t(z) \in C^m \quad (m \leqslant n)$$

is a continuous mapping such that

(1) *for every $t \in T$, $g_t: \Omega \to C^m$ is holomorphic,*

(2) $g_{t_0}^{-1}(0) \in \mathscr{A}_{n-m}(\Omega)$,

then the mapping $T \ni t \to g_t^{-1}(0) \in \mathscr{F}_\Omega$ is continuous at the point t_0.

Proof. It is easy to see that the mapping $\varphi: T \ni t \to g_t \in \mathscr{F}_{\Omega \times C^m}$ is continuous.

By Corollary 2 the mapping

$$\psi\colon \mathscr{A}_n(\Omega\times C^m)\ni V\to V\cap(\Omega\times\{0\})\in\mathscr{F}_{\Omega\times\{0\}}=\mathscr{F}_\Omega$$

is continuous at the point g_{t_0}. Therefore $\psi\circ\varphi\colon T\ni t\to g_t^{-1}(0)\in\mathscr{F}_\Omega$ is continuous at the point t_0.

Let us end with three corollaries.

COROLLARY 3. *Let T be a topological space and let Ω be an open subset of C^n. If $t_0\in T$, $a_0\in\Omega$ and $g\colon T\times\Omega\ni(t,z)\to g_t(z)\in C^m$ $(m\leqslant n)$ is a continuous mapping such that*

(1) *for every $t\in T$, g_t is holomorphic in Ω,*

(2) $g_{t_0}^{-1}\big(g_{t_0}(a_0)\big)\in\mathscr{A}_{n-m}(\Omega)$,

then the mapping $T\times\Omega\ni(t,a)\to g_t^{-1}\big(g_t(a)\big)$ is continuous at (t_0,a_0).

Proof. To prove this let us define the mapping

$$(T\times\Omega)\times\Omega\ni\big((t,a),z\big)\to g_t(z)-g_t(a)\in C^m$$

and observe that all assumptions of Theorem 4 are satisfied.

COROLLARY 4 (Hurwitz). *Let Ω be an open subset of C^n. Suppose that $f_\nu, f\colon \Omega\to C^m$ $(m\leqslant n)$, $\nu=1,2,\ldots$, are holomorphic mappings. If the sequence $\{f_\nu\}$ converges to f, uniformly on compact subsets of Ω, and if $f^{-1}(0)\in\mathscr{A}_{n-m}(\Omega)$, then $f_\nu^{-1}(0)\to f^{-1}(0)$.*

Proof. If we set $T=\{0\}\cup\bigcup_{\nu=1}^{\infty}\{1/\nu\}$ and

$$g\colon T\times\Omega\ni(t,z)\to\begin{cases} f_\nu(z) & \text{for } t=1/\nu,\\ f(z) & \text{for } t=0,\end{cases}$$

then we see at once that Corollary 4 is a simple consequence of Theorem 4.

COROLLARY 5. *Let Ω be a domain in C^n. Let $\{f_\nu\}$ be a sequence of holomorphic injective mappings. Suppose that f converges uniformly on compact subsets of Ω to a holomorphic mapping $f\colon\Omega\to C^n$. Then the following three properties are equivalent:*

(1) *f is injective;*

(2) *f is open;*

(3) $\operatorname{Int} f(\Omega)\neq\emptyset$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). By Sard's theorem, there exists $z_0\in\Omega$ such that $\det f'(z_0)\neq 0$. Since f_ν are injective for $\nu=1,2,\ldots$, then by Osgood's theorem $\det f'_\nu(z)\neq 0$ for all ν and all $z\in\Omega$. Hence, using Corollary 4 (see also [2], p. 80), we obtain that $\det f'(z)\neq 0$ for all $z\in\Omega$. Therefore $f^{-1}\big(f(a)\big)\in\mathscr{A}_0(\Omega)$ for all $a\in\Omega$. This enables us to use Corollary 4.

Let us suppose that there exist two points $a, b\in\Omega$, $a\neq b$, such that $f(a)=f(b)$. Applying Corollary 4 to the sequence $\{f_\nu-f(a)\}$ we get that f_ν is not injective for sufficiently large ν. This contradicts our assumption.

References

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