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Analysis of parabolic difference schemes by Gerschgorin's method

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Dedicated to the memory of Jacek Szarski

Abstract. Gerschgorin's method of proving convergence of difference schemes consists in exploiting inverse isotonicity of a boundary-value problem and of its discretization by establishing an error bound via an error majorizing function. In 1930 Gerschgorin has shown that for elliptic problems it often suffices to find a majorizing function for the inhomogeneous problem with all right-hand sides equal to the constant 1, provided the boundary conditions are properly discretized. In the present paper an abstract version of the method is described and demonstrated to be also applicable to one-step discretizations of parabolic problems. As a model problem the linear second-order parabolic initial-boundary-value problem with mixed lateral boundary conditions is analyzed (just for simplicity of presentation with one space dimension and constant coefficients).

1. Introduction. The aim of this paper is to demonstrate the usefulness of Gerschgorin's method (first applied in [2] to elliptic boundary value problems) for proving stability and convergence of parabolic difference schemes. We give an abstract version of the method and then analyze the standard class of one-step schemes for a linear parabolic initial-boundary value problem in which, for ease of presentation, we assume the coefficients to be constant. The final results are not essentially new, but the method of proof, because of its perspicuity, may deserve popularization. It exploits a very essential property of the problem, namely inverse isotonicity, which also has a physical relevance in the behaviour of diffusion processes.

For the general background on inverse isotonicity (often called *inverse positivity* or *monotonic kind*) of parabolic problems (Nagumo–Westphal lemma) we quote the books of Protter and Weinberger [9], Szarski [11], and Walter [13]. For the abstract notions see also Collatz [1] and Schröder [10]. In Section 5.1 of Chapter II of [10] a generalization of Gerschgorin's method is described, simpler notions than those presented there suffice for our purpose. Schröder gives several important references which are not reproduced here, but he does not describe application to parabolic problems.

The essential tool of the method is an appropriate error majorizing function, which, however, is not used via maximum principles as is often done in the literature but directly via inverse isotonicity. The other essential aspect is that the suitability of a function as a comparison function is tested by verifying differential but not difference inequalities, the required difference inequalities being obtained subsequently via consistency conditions. One may compare the proofs of this paper with those of Isaacson in [8] or Gorenflo in [3], [4], [5]. Gorenflo [6] has described the method in functional-analytic terms and applied it to two-point(1) and elliptic problems; he and Niedack [7] have applied it to parabolic problems with conservation properties discretized in a way more complicated than here.

Generalizations are possible to the case of non-constant coefficients, to non-linear problems, and to some types of weakly coupled parabolic systems.

2. Description of Gerschgorin's method. As already remarked, we do not give the most general representation available but one sufficient for our purposes. For orientation on half-ordered vector spaces see, for example, [1] and [10].

Let U and Y be vector spaces coordinatized corresponding to some $\boldsymbol{R}^m$ or vector spaces of vectors of finitely many bounded real functions. U and Y are assumed to be naturally half-ordered by $\leqslant$ (meaning «less than or equal to» componentwise or component and pointwise). This relation $\leqslant$ is transitive, reflexive, antisymmetric, and compatible with the vector space structure. Without discrimination $\leqslant$ is used for all occurring natural half-orders (likewise $\geqslant$). We introduce supremum norms $\|\cdot\|_U$ and $\|\cdot\|_Y$ as maximum of absolute values of components or maximum of suprema of absolute values of component functions, and we define a mapping $|\cdot|$ as transforming into the vector of absolute values componentwise or component and pointwise (for $u \geqslant 0$ with $u \in U$ we have $|u| = u$). Note that $|\cdot|$ may lead outside the space if this consists of smooth enough functions, but that $|U|$ and $|Y|$ can again be naturally half-ordered (they are not vector spaces).

Let $A\colon U \to Y$ be a linear operator and let $x \in U$.

We call A isotonic if $x \geqslant 0 \Rightarrow Ax \geqslant 0$, we call A inverse-isotonic if $Ax \geqslant 0 \Rightarrow x \geqslant 0$.

If the operator A is inverse-isotonic, it is injective, and $|Au| \leqslant Ay$ for $u, y \in U$ implies $|u| \leqslant y$.

From now on we assume $A\colon U \to Y$ to be linear and inverse-isotonic

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(1) Unfortunately the treatment given there of the Bramble–Hubbard discretization of a two-point boundary value problem is erroneous.

and $A(U) = Y$, so that A is bijective. For given $y \in Y$ we want to approximate the unique solution $u \in U$ of the equation $Au = y$. Let Λ be a directed set, half-ordered by a relation also written $\leqslant$, and let λ run through Λ decreasingly: we write $\lambda \to 0$. In the applications λ is usually a meshwidth or a vector of meshwidths. Let vector spaces V_λ (identifiable with $\boldsymbol{R}^{\nu(\lambda)}$, where $\boldsymbol{N} \ni \nu(\lambda) \to \infty$ as $\lambda \to 0$) be given, naturally half-ordered and maximum-normed by $\|\cdot\|_\lambda$, and assume linear operators

$$R_\lambda\colon U \to V_\lambda, \quad M_\lambda\colon Y \to V_\lambda, \quad A_\lambda\colon V_\lambda \to V_\lambda, \tag{2.1}$$

$$\|R_\lambda u\|_\lambda \to \|u\|_U \quad \text{for } u \in U \text{ as } \lambda \to 0. \tag{2.1'}$$

Usually R_λ is a restriction operator and M_λ is a local averaging operator: $(R_\lambda u)(z) = u(z)$ for $z \in G_\lambda$, where G_λ is a grid and V_λ is a space of grid functions. In this case we have $\|R_\lambda u\|_\lambda \leqslant \|u\|_U$. There are now problem (P) and a collection (called *difference scheme*) $D = (P_\lambda)_{\lambda \in \Lambda}$ of approximating problems (P_λ) as follows:

(P) For given $y \in Y$ determine $u \in U$ so that $Au = y$.

(P_λ) For given $y \in Y$ determine $u_\lambda \in V_\lambda$ so that $A_\lambda u_\lambda = M_\lambda y$.

DEFINITION 2.1. (a) We call D *consistent* with (P) if

$$\|A_\lambda R_\lambda u - M_\lambda A u\|_\lambda \to 0 \quad \forall u \in U. \tag{2.2}$$

(b) We call D *stable* if there exist $K \in \boldsymbol{R}^+ = \{t \in \boldsymbol{R} \mid t > 0\}$ and $\lambda_0 \in \Lambda$ so that $\lambda \leqslant \lambda_0$ implies

$$\|v\|_\lambda \leqslant K \|A_\lambda v\|_\lambda \quad \forall v \in V_\lambda. \tag{2.3}$$

(c) We call D *discrete-convergent* if there exists a $\lambda_0 \in \Lambda$ such that (P_λ) has a solution u_λ for $\lambda \leqslant \lambda_0$ and with u as solution of (P) we have

$$\|R_\lambda u - u_\lambda\|_\lambda \to 0 \quad \text{as } \lambda \to 0. \tag{2.4}$$

Note that stability of D implies unisolvency of (P_λ) for λ sufficiently small.

THEOREM 2.1. *If the difference scheme D is consistent with* (P) *and stable, then it is discrete-convergent, and for $\lambda \leqslant \lambda_0$ with suitable $\lambda_0 \in \Lambda$ and $K \in \boldsymbol{R}^+$ the solution u_λ of* (P_λ) *obeys the error estimate*

$$\|R_\lambda u - u_\lambda\|_\lambda \leqslant K \|A_\lambda R_\lambda u - M_\lambda y\|_\lambda = K \|A_\lambda R_\lambda u - M_\lambda A u\|_\lambda. \tag{2.5}$$

Proof. Put $v = R_\lambda u - u_\lambda$ in (2.3) and use $A_\lambda u_\lambda = M_\lambda y = M_\lambda A u$.

DEFINITION 2.2. We call D *weakly inverse-isotonic* if A_λ is inverse-isotonic for $\lambda \leqslant \lambda_0$, *strongly inverse-isotonic* if it is weakly inverse-isotonic and M_λ is isotonic for $\lambda \leqslant \lambda_0$ (with $\lambda_0 \in \Lambda$ suitably chosen).

DEFINITION 2.3 (*unit functions*). Let $\eta \in Y$ and $\eta_\lambda \in V_\lambda$ be functions everywhere (component and pointwise) equal to 1.

Remark. For $y \in Y$, $v \in V_\lambda$ we have

$$|y| \leqslant \|y\|_Y \eta, \qquad |v| \leqslant \|v\|_\lambda \eta_\lambda. \tag{2.6}$$

THEOREM 2.2 (majorization). *Let the difference scheme D be weakly inverse-isotonic and let there exist for $\lambda \leqslant \lambda_0$ (appropriately chosen) $\bar{u}_\lambda \in V_\lambda$, so that with a constant $K \in \boldsymbol{R}^+$ we have*

$$\|\bar{u}_\lambda\|_\lambda \leqslant K \quad \text{and} \quad A_\lambda \bar{u}_\lambda \geqslant \eta_\lambda \quad \text{for } \lambda \leqslant \lambda_0. \tag{2.7}$$

Then D is stable, and for the solutions u and u_λ of (P) *and* (P_λ) *we have the local inclusion*

$$|u_\lambda - R_\lambda u| \leqslant \|A_\lambda R_\lambda u - M_\lambda A u\|_\lambda \bar{u}_\lambda \quad \text{for } \lambda \leqslant \lambda_0. \tag{2.8}$$

Proof. By (2.7) and (2.6) $v \in V_\lambda$ and $z = \|A_\lambda v\|_\lambda \bar{u}_\lambda$ imply (for $\lambda \leqslant \lambda_0$)

$$A_\lambda z = \|A_\lambda v\|_\lambda A_\lambda \bar{u}_\lambda \geqslant \|A_\lambda v\|_\lambda \eta_\lambda \geqslant |A_\lambda v|;$$

hence, by inverse isotonicity of the operators A_λ,

$$|v| \leqslant z \leqslant \|z\|_\lambda \eta_\lambda, \qquad \|v\|_\lambda \leqslant \|z\|_\lambda \leqslant \|A_\lambda v\|_\lambda K,$$

that is stability. From $|v| \leqslant z$ we obtain (2.8) with $v = u_\lambda - R_\lambda u$, using $A_\lambda u_\lambda = M_\lambda A u$ from (P) and (P_λ).

DEFINITION 2.4. The family $(M_\lambda)_{\lambda \in \Lambda}$ is called *consistent with* 1 if

$$\|M_\lambda \eta - \eta_\lambda\|_\lambda \to 0 \quad \text{as } \lambda \to 0. \tag{2.9}$$

In this case we call D *equilibrated.*

THEOREM 2.3. *Let the difference scheme D be consistent with* (P), *strongly* (*alternatively*: *weakly*) *inverse-isotonic, and equilibrated. Let there exist $\bar{u} \in U$ such that*

$$A\bar{u} \geqslant \eta \quad (\textit{alternatively } A\bar{u} = \eta). \tag{2.10}$$

Then D is stable (*hence, by Theorem* 2.1 *discrete-convergent*), *and with $c \in (0, 1)$ fixed we have* (2.7) *and* (2.8) *with*

$$c\bar{u}_\lambda = R_\lambda \bar{u}, \qquad cK = \sup\{\|R_\lambda \bar{u}\|_\lambda \mid \lambda \in \Lambda\}, \tag{2.11}$$

where λ_0 may depend on the choice of c.

Proof. Put $\delta_\lambda = A_\lambda R_\lambda \bar{u} - M_\lambda A \bar{u}$, observe $\|\delta_\lambda\|_\lambda \to 0$ as $\lambda \to 0$, and fix $c \in (0, 1)$. For sufficiently small λ we then have

$$\begin{aligned} A_\lambda R_\lambda \bar{u} &= \delta_\lambda + M_\lambda A \bar{u} \geqslant \delta_\lambda + M_\lambda \eta = \eta_\lambda + M_\lambda \eta - \eta_\lambda + \delta_\lambda \\ &\geqslant (1 - \|M_\lambda \eta - \eta_\lambda\|_\lambda - \|\delta_\lambda\|_\lambda) \eta_\lambda \geqslant c\eta_\lambda, \end{aligned}$$

and (2.1′) implies the finiteness of K.

3. Treatment of a parabolic problem. We first describe the *initial-boundary value problem.* With $T \in \boldsymbol{R}^+$ let $\bar{G} = [0, 1] \times [0, T]$, and let

$a, b, c \in \boldsymbol{R},\ a > 0,\ \varphi, \psi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi],\ U = C^{2,1}(\bar{G}),\ Y = A(U)$, where

$$Au = \begin{cases} (u_t - au_{xx} - bu_x - cu)(x, t) & \text{for } (x, t) \in \bar{G}, \\ u(x, 0) & \text{for } 0 \leqslant x \leqslant 1, \\ -u_x(0, t)\cos\varphi + u(0, t)\sin\varphi & \text{for } 0 \leqslant t \leqslant T, \\ u_x(1, t)\cos\psi + u(1, t)\sin\psi & \text{for } 0 \leqslant t \leqslant T. \end{cases}$$

Then $y \in Y$ can be decomposed as

$$y = \begin{pmatrix} r \\ g \\ \alpha \\ \beta \end{pmatrix} \quad \text{with } r \in C(\bar{G}),\ g \in C[0, 1],\ \alpha, \beta \in C[0, T].$$

For given $y \in Y$ we want to approximate $u \in U$ so that $Au = y$. The purpose of the parameters φ and ψ is to avoid distinguishing between the types of boundary conditions.

Note that Y (owing to *compatibility conditions* at the corners $(0, 0)$ and $(1, 0)$ of $\bar{G}$) is but a proper subspace of

$$C(\bar{G}) \times C[0, 1] \times C[0, T] \times C[0, T].$$

Now with $2 \leqslant J \in \boldsymbol{N},\ h = 1/J,\ \mu \in \boldsymbol{R}^+,\ \tau = \mu h^2,\ x_j = jh,\ t_n = n\tau$, $N =$ entier (T/τ), and $\lambda = (h, \tau)$ as vector of mesh-widths we discretize $\bar{G} = [0, 1] \times [0, T]$ by a grid

$$G_\lambda = \{(x_j, t_n) |\ j = 0, 1, \ldots, J;\ n = 0, 1, \ldots, N\}.$$

We introduce difference operators $\bar{\delta}$ and δ^2 according to

$$\bar{\delta} v_{j,n} = \frac{1}{2h}(v_{j+1,n} - 2v_{j-1,n}), \qquad \delta^2 v_{j,n} = \frac{1}{h^2}(v_{j+1,n} - 2v_{j,n} + v_{j-1,n}),$$

and, with a weight factor $\Theta \in [0, 1]$ as parameter of implicity an interpolation operator I_Θ by $I_\Theta \tilde{v}_n = \Theta \tilde{v}_{n+1} + \bar{\Theta} \tilde{v}_n$, where $\bar{\Theta} = 1 - \Theta$.

To approximate the solution u of $Au = y$ by values $w_{j,n} \doteq u_\lambda(x_j, t_n) \approx u(x_j, t_n)$ we start by

$$w_{j,0} = g(x_j) \quad \text{for } j = 0, 1, \ldots, J,$$

and proceed from time-level t_{n-1} to t_n by the *running scheme*

$$\frac{1}{\tau}(w_{j,n} - w_{j,n-1}) - I_\Theta(a\delta^2 + b\bar{\delta} + c)w_{j,n-1} = I_\Theta r_{j,n-1}$$

(where $r_{j,n} = r(x_j, t_n)$) supplemented by the *discrete boundary conditions*

$$I_\Theta(-\bar{\delta} w_{0,n-1}\cos\varphi + w_{0,n-1}\sin\varphi) = I_\Theta \alpha_{n-1},$$

$$I_\Theta(\bar{\delta} w_{J,n-1}\cos\psi + w_{J,n-1}\sin\psi) = I_\Theta \beta_{n-1},$$

where $\alpha_n = \alpha(t_n),\ \beta_n = \beta(t_n)$.

For our intended application of Gerschgorin's method we *eliminate* the *external values* with indices $j = 1$ and $j = J+1$ by computing $I_\Theta w_{-1,n-1}$ and $I_\Theta w_{J+1,n-1}$ from the running scheme and inserting into the discrete boundary conditions (thereby taking account of $\tau = \mu h^2$). With V_λ as the space of grid functions on G_λ we define operators R_λ, A_λ, M_λ. First $(R_\lambda u)_{j,n} = u(x_j, t_n)$. Then let u_λ be the grid function $G_\lambda \to \boldsymbol{R}$ with $u_\lambda(x_j, t_n) = w_{j,n}$. We start by

$$(A_\lambda u_\lambda)_{j,0} = w_{j,0}, \quad (M_\lambda y)_{j,0} = g(x_j) \quad \text{for } j = 0, 1, \ldots, J$$

and get the *running scheme* and the *discrete boundary conditions* for $n = 1, 2, \ldots, N$ according to

$$(A_\lambda u_\lambda)_{j,n} = \frac{1}{\tau}(w_{j,n} - w_{j,n-1}) - I_\Theta(a\delta^2 + b\bar{\delta} + c)w_{j,n-1},$$

$$(M_\lambda y)_{j,n} = I_\Theta r_{j,n-1}, \text{ both for } j = 1, 2, \ldots, J-1,$$

$$(A_\lambda u_\lambda)_{0,n} = \left(a - \frac{bh}{2}\right)^{-1} \left\{\frac{h}{2\tau}(w_{0,n} - w_{0,n-1}) + \frac{a}{h} I_\Theta(w_{0,n-1} - w_{1,n-1}) - \right.$$
$$\left. - \frac{ch}{2} I_\Theta w_{0,n-1}\right\}\cos\varphi + I_\Theta w_{0,n-1}\sin\varphi,$$

$$(M_\lambda y)_{0,n} = I_\Theta \alpha_{n-1} + \frac{h\cos\varphi}{2a - bh} I_\Theta r_{n-1}.$$

Analogously to $(A_\lambda u_\lambda)_{0,n}$ and $(M_\lambda y)_{0,n}$ define $(A_\lambda u_\lambda)_{J,n}$ and $(M_\lambda y)_{J,n}$: simply replace first indices 0 by J, 1 by $J-1$, and φ, α, b by $\psi, \beta, -b$.

Close inspection shows that $(M_\lambda)_{\lambda\in\Lambda}$ is consistent with 1 and that M_λ is isotonic if $|b|h < 2a$. If $u \in C^{4,2}(\bar{G})$, then the whole scheme is $(h^2+\tau)$-consistent, to be verified by the usual ugly Taylor type calculations.

The inverse isotonicity of the operators A_λ can be examined time-step-wise. With the spatial *tridiagonal discretization matrix* H write

$$\text{(3.1)} \qquad \tilde{u}_n - \tilde{u}_{n-1} = \mu H(\Theta\tilde{u}_n + \bar{\Theta}\tilde{u}_{n-1}) + \tilde{s}_{n-1} \quad \text{for } n = 1, 2, \ldots, N,$$

where in the case of $\varphi \neq \frac{1}{2}\pi$, $\psi \neq \frac{1}{2}\pi$ the vectors $\tilde{u}_n$ and $\tilde{s}_n$ are the transposes of

$$(w_{0,n}, w_{1,n}, \ldots, w_{J,n})$$

and

$$I_\Theta\left(\tau r_{0,n} + \frac{\mu h}{\cos\varphi}(2a - bh)\alpha_n, \tau r_{1,n}, \ldots, \tau r_{J-1,n}, \tau r_{J,n} + \frac{\mu h}{\cos\psi}(2a + bh)\beta_n\right)$$

whereas $H = (h_{j,k})_{j,k=0,1,\dots,J}$ with

$$h_{j,j-1} = a - \frac{bh}{2}, \quad h_{j,j} = -2a + ch^2, \quad h_{j,j+1} = a + \frac{bh}{2} \quad \text{for } 1 \leqslant j \leqslant J-1,$$

$$h_{0,0} = -2a + ch^2 - h(2a - bh)\tan\varphi, \quad h_{0,1} = 2a,$$
$$h_{J,J-1} = 2a, \quad h_{J,J} = -2a + ch^2 - h(2a + bh)\tan\psi.$$

In the case of *Dirichlet boundary conditions* the vectors $\tilde{u}_n$ and $\tilde{s}_n$ must be shortened. If, for example, $\varphi = \psi = \frac{1}{2}\pi$ take $\tilde{u}_n$ as the transpose of $(w_{1,n}, \dots, w_{J-1,n})$ and $\tilde{s}_n$ as the transpose of

$$I_\Theta\big(\tau r_{1,n} + \mu(a - \tfrac{1}{2}bh)\alpha_n, \tau r_{2,n}, \dots, \tau r_{J-2,n}, \tau r_{J-1,n} + \mu(a + \tfrac{1}{2}bh)\beta_n\big).$$

Correspondingly delete the first and last rows and columns of the just defined matrix H.

Remark. For investigation of A_λ on inverse isotonicity we should forget the specific form of the vectors $\tilde{s}_n$ and instead consider them as arbitrary vectors, because inverse isotonicity of A_λ is equivalent to all $g(x_j) \geqslant 0$ and all $\tilde{s}_n \geqslant 0$ implying all $\tilde{u}_n \geqslant 0$ (regardless of the $\tilde{s}_n$ stemming from $M_\lambda y$).

From (3.1) we get, for $n = 1, 2, \dots, N-1$,

$$\tilde{u}_n = (I - \mu\Theta H)^{-1}\{(I + \mu\bar{\Theta}H)\tilde{u}_{n-1} + \tilde{s}_{n-1}\}, \tag{3.2}$$

and as *sufficient conditions for strong inverse isotonicity* of D we recognize that:

(i) *$\tilde{s}_n$ should depend isotonically on y,*

(ii) *$I - \mu\Theta H$ should be an M-matrix,*

(iii) *We should have $I + \mu\bar{\Theta}H \geqslant 0$ (each entry $\geqslant 0$).*

We have already observed that M_λ is isotonic if $|b|h < 2a$, and this implies (i). For (ii) and (iii) we use the theory of non-negative matrices and of M-matrices as developed in [12] and [1]. (iii) can be analyzed by close inspection, whereas (ii) is treated by a well-known lemma.

LEMMA. *Let the real square matrix B be irreducible and have the following properties: all diagonal entries are positive, all non-diagonal entries are non-positive, all row-sums are non-negative, at least one row-sum is positive. Then B is an M-matrix (so that B^{-1} exists and is non-negative entry-wise).*

As a result we obtain the following theorem.

THEOREM 3.1. *Conditions* (a) *and* (b) *imply strong inverse isotonicity of the scheme D of this section*

(a) $|b|h < 2a$,

(b) 1° *If $\varphi = \psi = \frac{1}{2}\pi$, then* $(2a + |c|h^2)\bar{\Theta}\mu \leqslant 1$;

2° *If* $\varphi, \psi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, *then*

$$\mu\bar{\Theta}\big(2a+|c|h^2+h(2a+|b|h)\max(|\tan\varphi|, |\tan\psi|)\big) \leqslant 1,$$

$$\mu\Theta h\big(|c|h+(2a+|b|h)\max(|\tan\varphi|, |\tan\psi|)\big) \leqslant 1;$$

3° *If* $\varphi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ *and* $\psi = \frac{1}{2}\pi$ *or if* $\varphi = \frac{1}{2}\pi$ *and* $\psi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, *then as in* 2° *but with* $\max(|\tan\varphi|, |\tan\psi|)$ *replaced by* $|\tan\varphi|$ *or* $|\tan\psi|$, *respectively*.

Now, in order to apply Theorem 2.3 for proving stability, convergence, and local inclusion we have to display a majorizing function $\bar{u} \in U$ with $A\bar{u} \geqslant \eta$.

THEOREM 3.2. *We have* $A\bar{u} \geqslant \eta$ *with*

$$\bar{u}(x, t) = \exp(Bt)\cosh\big(C(x-\tfrac{1}{2})\big),$$

where $C = \max(C_\varphi, C_\psi)$, $B = aC^2+|b|C+|c|+1$. *For* $\omega \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ *we have to take* C_ω *as a positive solution* C^* *of the equation* $(\cos\omega)\{C^*\tanh(C^*/2)+\tan\omega\} = 1$, *and* $C_{\pi/2}$ *as a positive solution* C^* *of* $\cosh(C^*/2) = 1$.

The proof is straightforward.

For simpler particular problems majorizing functions of simpler structure may suffice. For the problem

$$u_t(x, t)-au_{xx}(x, t) = r(x, t), \quad u(x, 0) = g(x), \quad u(0, t) = \alpha(t),$$
$$u(1, t) = \beta(t), \quad a > 0,\ 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T$$

one may take

$$\bar{u}(x, t) = 1+\frac{1}{2a}(1-x^2).$$

Final remark. We should point to the fact that for $\Theta = \frac{1}{2}$ (Crank–Nicolson-scheme) inverse isotonicity requires boundedness of μ in contrast to the L_2-convergence theory. However, for $\Theta = 0$ (explicit scheme) and $\Theta = 1$ (fully implicit scheme) our requirements on μ are essentially the same as in the L^2-theory.

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