ANNALES
POLONICI MATHEMATICI
XLVII (1986)

# Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equation of the first order *

by PRIMO BRANDI and RITA CEPPITELLI ** (Perugia, Italy)

**Abstract.** We consider a class of hereditary equations which contains many functional differential equations of retarded type.

This hereditary equation is defined as follows:

(A) $$z_x(x, y) = f(x, y; T^{(1)}(x, y; z), \ldots, T^{(r)}(x, y; z); z_y(x, y)), \quad (x, y) \in I = [0, a] \times \boldsymbol{R},$$

where $f$ is a given real function and $T^{(i)}$ $(i = 1, 2, \ldots, r)$ is a continuous operator with the property

(**) $$\sup_{(\xi, y) \in [0, x] \times \boldsymbol{R}} |T^{(i)}(\xi, y; z) - T^{(i)}(\xi, y; \bar{z})| \leqslant L \sup_{(\xi, y) \in [0, x] \times \boldsymbol{R}} |z(\xi, y) - \bar{z}(\xi, y)|.$$

Under condition (**) the operator is of the Volterra type and the equation has a hereditary structure.

We prove theorems of existence, uniqueness and continuous dependence for the Cauchy problem of equation (A).

A quantitative estimate on the domain strip of the solution is given.

**1. Introduction.** Continuing the study begun in [3], in this paper the following Cauchy problem is considered:

($\tau$) $$\begin{aligned} z_x(x, y) &= f(x, y; T^{(1)}(x, y; z), \ldots, T^{(r)}(x, y; z); z_y(x, y)) && \text{on } I, \\ z &= \varphi && \text{on } I_0, \end{aligned}$$

where $f$ and $\varphi$ are given real functions and $I_0 = \{(x, y): p_0 \leqslant x \leqslant 0, \ y \in \boldsymbol{R}\}$ with $-\infty \leqslant p_0 \leqslant 0$, $I = \{(x, y): 0 \leqslant x \leqslant a, \ y \in \boldsymbol{R}\}$.

The operators $T^{(i)}: I \times \tilde{C}(I_0 \cup I) \to C^0(E_i)$ are continuous functions with the property

(**) $$\sup_{(\xi, y) \in [0, x] \times \boldsymbol{R}} |T^{(i)}(\xi, y; z) - T^{(i)}(\xi, y; \bar{z})| \leqslant L \sup_{(\xi, y) \in [0, x] \times \boldsymbol{R}} |z(\xi, y) - \bar{z}(\xi, y)|$$

for every $x \in [0, a]$, $z, \bar{z} \in C(I_0 \cup I)$, $i = 1, 2, \ldots, r$, where $E_i$ is a compact

---

* This research has been carried out within the Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni del Consiglio Nazionale delle Ricerche.

** Dipartimento di Matematica Università degli Studi di Perugia, 06100 Perugia, Italy.

topological space and $\tilde{C}(I_0 \cup I)$ is a subspace of $C^0$ (with topology of uniform convergence on compact sets).

Under condition (**) the operators are of the Volterra type and problem $(\tau)$ is hereditary since the present behaviour of the system depends in some way on its past history.

Many authors have studied particular formulations of problem $(\tau)$ while conserving its hereditary structure. For ordinary functional equations we simply cite [6]–[9], while for partial functional equations [2], [3], [10]–[15], [17], [18] are cited.

The operator $T^{(i)}$ contains as particular case the functional $T(x, y; z) = z(\alpha(x, y), \beta(x, y))$ which has been considered in [2], [3], [10], [11] hence the unretarded case, $\alpha(x, y) = x$, $\beta(x, y) = y$, which has been widely studied in the literature. Moreover, the integral operators $T(x, y; z) = \int_{H(x,y)} f(x, y, s, t, z(\alpha(s, t)))\,ds\,dt$ (see [14], [15]) and operators with range contained in the subspace of $C^0(E_i)$ made up of constant functions [13] are particular formulations of the functional $T$ here considered.

In this paper existence, uniqueness and continuous dependence theorems are given for problem $(\tau)$.

As in [3] for a particular case, a sequence of Cauchy problems $(\tau_n)$ for the equation $p = f(x, y; q)$ without functional argument is associated with problem $(\tau)$. To resolve each problem $(\tau_n)$ in uniqueness solution hypotheses, we chose a theorem stated by Baiada [1] from among the numerous existence theorems available in literature, because it provides a quantitative estimate on the domain strip of the solution $z_n$ in relatively non stringent hypotheses.

In Section 4, we prove that, definitely with respect to $n$, the $z_n$ surfaces admit the same domain strip. Considering this, a solution of problem $(\tau)$ is determined as surface limit of the sequence $(z_n)_n$.

Our hypotheses, while continuing to assure continuous dependence and uniqueness of solution (see n° 5), are less stringent than those given in [10]–[14] (see n° 6), while the quantitative estimate on the domain strip of the solution is not generally comparable with the estimates given in the quoted notes.

**2. Assumptions and definition of the problem.** Let $G$ be a given topological space, and let $C^0(G)$ be the space of all continuous real functions on $G$ endowed with topology of uniform convergence on compact sets.

If $G$ is a subset of $\boldsymbol{R}^2$, let $C_l^1(G) \subset C^0(G)$ be the topological subspace of all surfaces $z(x, y)$ of class $C^1$ in $G$ whose partial derivatives $\partial z/\partial x$, $\partial z/\partial y$, are bounded and Lipschitzian.

In the following $\mathscr{E}(G)$ will denote the set of all classes of functions belonging to $C^1$ whose derivatives are uniformly bounded, and $\mathscr{E}_{\mathrm{Lip}}(G)$ denote the subset of $\mathscr{E}(G)$ of all classes of surfaces whose derivatives satisfy a uniform Lipschitz condition.

Given $r$ compact topological spaces $E_i$, $i = 1, 2, \dots, r$, and denoted by $I = \{(x, y)\colon 0 \leqslant x \leqslant a,\ y \in \boldsymbol{R}\}$, $I_0 = \{(x, y)\colon p_0 \leqslant x \leqslant 0,\ y \in \boldsymbol{R}\}$, $-\infty \leqslant p_0 \leqslant 0$, let ([1]) $T^{(i)}\colon I \times C_l^1(I_0 \cup I) \to C^0(E_i)$, $i = 1, 2, \dots, r$, be $r$ operators satisfying the condition

(*) for every $\bar{x} \in [0, a]$, there exists a set $H_{\bar{x}} \subset (I_0 \cup I) \cap ((-\infty, \bar{x}] \times \boldsymbol{R})$ such that if $z_1 = z_2$ in $H_{\bar{x}}$ then $T^{(i)}(\bar{x}, y; z_1) = T^{(i)}(\bar{x}, y; z_2)$ for every $i = 1, 2, \dots, r$.

In the present paper we seek a solution $z\colon I_0 \cup I \to \boldsymbol{R}$ of class $C_l^1$ of the following Cauchy problem

$$(\tau)\qquad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f\left(x, y; T^{(1)}(x, y; z), \dots, T^{(r)}(x, y; z); \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I,\\ &z = \varphi && \text{on } I_0, \end{aligned}$$

where the functions $f$, $\varphi$ are given.

We denote by $\Omega$ the set

$$\{(x, y; z_1, \dots, z_r; q)\colon 0 \leqslant x \leqslant a,\ y \in \boldsymbol{R},\ z_i \in C^0(E_i),\ i = 1, 2, \dots, r,\ |q| \leqslant S\},$$

$S \leqslant +\infty$, and by $f\colon \Omega \to \boldsymbol{R}$ a function verifying these two requirements:

($f_1$) $f$ is continuous in $\Omega$ and Lipschitzian in $x$ of constant([2]) $H$;

($f_2$) the partial derivatives([3]) $f_y, f_q\colon \Omega \to \boldsymbol{R}$, $f_{z_i}\colon \Omega \to \mathrm{CL}(C^0(E_i), \boldsymbol{R})$, $i = 1, 2, \dots, r$, are Lipschitzian in $y$, $z = (z_1, \dots, z_r)$, $q$ of constant $K_w$ and bounded by a constant $L_w$ in $\Omega$, with $L_w \leqslant K_w$, $w = y, z, q$.

Moreover, we assume that

(i) the function $\varphi\colon I_0 \to R$ is of class $C^1$ in $I_0$; the partial derivatives $\varphi_x$, $\varphi_y$ are bounded by constants $G$, $N$, respectively, and Lipschitzian in $x$ of constants $C$, $D$ and in $y$ of constants $D$, $M$, respectively;

(2i) the operator $T^{(i)}\colon I \times C_l^1(I_0 \cup I) \to C^0(E_i)$, $i = 1, 2, \dots, r$, is continuous in $I \times C^1(I_0 \cup I)$, Lipschitzian in $x$ in each class $\mathscr{C}$ belonging to

---

([1]) We will denote the norm by $|\cdot|$ in every normed space.

([2]) For simplicity's sake, we suppose $f$ is bounded in $\Omega$ by a constant $G$.

([3]) Given two normed spaces $X$ and $Y$, we denote by $\mathrm{CL}(X, Y)$ the set of all continuous linear mappings from $X$ to $Y$, and by $f_w\colon \mathscr{F} \to \mathrm{CL}(X, Y)$ ($f\colon \mathscr{F} \to Y$, $w \in X$) the Fréchet derivative.

$\mathscr{E}(I_0 \cup I)$ and satisfies the property[4]

(**)
$$\sup_{(\xi,y)\in[0,x]\times R} |T^{(i)}(\xi, y; z_1) - T^{(i)}(\xi, y; z_2)| \leqslant L \sup_{(\xi,y)\in[0,x]\times R} |z_1(\xi, y) - z_2(\xi, y)|$$

for every $x \in [0, a]$ and $z_1, z_2 \in C_l^1(I_0 \cup I)$;

(3i) the function $T_y^{(i)}$: $(I \times C_l^1(I_0 \cup I)) \to \mathrm{CL}(R, C^0(E_i))$, $i = 1, 2, \ldots, r$, is bounded in every class belonging to $\mathscr{E}(I_0 \cup I)$ and Lipschitzian in $y$ in every class belonging to $\mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$;

(4i) the following consistency condition is satisfied

$$\frac{\partial}{\partial x}\varphi(0, y) = f\left(0, y; T^{(1)}(0, y; \varphi), \ldots, T^{(r)}(0, y, \varphi); \frac{\partial}{\partial y}\varphi(0, y)\right).$$

Remark 1. As we set out, to establish a local existence theorem of problem $(\tau)$, it is necessary to define the operator $T^{(i)}$, $i = 1, 2, \ldots, r$, in the domain $(J \times C_l^1(I_0 \cup J))$ for every $J = [0, \delta] \times \boldsymbol{R}$, $\delta \leqslant a$. With this aim, for every $(x, y; z) \in (J \times C_l^1(I_0 \cup J))$ we define $T(x, y; z) = T(x, y; \tilde{z})$, where $\tilde{z}$ is an extension of $z$ to the strip $I_0 \cup I$ in class $C_l^1$. This definition is valid in virtue of property (*) on the operators.

Moreover, extension operators $z \to \tilde{z}$ exist which map every subset $\mathscr{F}$, $\mathscr{F} \subset \bigcup_{J \subset I} \mathscr{C}_l^1(I_0 \cup J)$, made up of surfaces with uniformly bounded derivatives in a class belonging to $\mathscr{E}(I_0 \cup I)$ and which also map every subset of $\mathscr{F}$ of functions whose derivatives satisfy a uniform Lipschitz condition in a class belonging to $\mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$.

Let us consider, for example, the extension

$$(\Xi) \quad \tilde{z}(x, y) = \begin{cases} z(x, y) & \text{if } (x, y) \in I_0 \cup J, \\ \frac{1}{2} z(\delta, y + (x-\delta)/2) + \frac{1}{2} z(\delta, y - (x-\delta)/2) + \displaystyle\int_{y-(x-\delta)/2}^{y+(x+\delta)/2} z_x(\delta, t)\, dt & \\ & \text{if } (x, y) \in [\delta, a] \times \boldsymbol{R} \end{cases}$$

for every $z \in C_l^1(I_0 \cup J)$.

3. In the present section we apply Baiada's existence theorem [1] to the Cauchy problem for the equation $p = f(x, y; z_1, \ldots, z_r, q)$ and we give a continuous dependence of solution theorem for it.

THEOREM 1. *Let $f$ and $\varphi$ be under the same hypothesis as Section 2, and let $\bar{z}_i$: $I \to C^0(E_i)$ be $r$ given functions satisfying the conditions*:

---

(4) Property (**) is more stringent then requirement (*).

($b_1$) *each functions $\bar{z}_i$ ($i = 1, 2, \dots, r$) is continuous in $I$, Lipschitzian in $x$ of constant $\hat{P}$ and differentiable with respect to $y$ at every $(x, y) \in I$;*

($b_2$) *for every $i = 1, 2, \dots, r$, the Fréchet derivative $\partial z_i/\partial y$: $I \to \mathrm{CL}(R, C^0(E_i))$ is bounded by a constant $\hat{F}$ and Lipschitzian in $y$ of constant $\hat{W}$.*

*Under these hypotheses, there exists a unique solution $z$ of the Cauchy problem*

$$(\beta)\qquad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f\left(x, y; \bar{z}_1(x, y), \dots, \bar{z}_r(x, y); \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I,\\ &z = \varphi && \text{on } I_0. \end{aligned}$$

Through a direct method of approximation taken from E. Baiada [1], we established that such a solution exists and that it belongs to class $C^1$ in a suitable strip $\{(x, y): p_0 \leqslant x \leqslant \delta,\ 0 < \delta \leqslant a,\ y \in \boldsymbol{R}\}$ that we denote, for brevity's sake, again by $I_0 \cup I$.

Considering the substitution

$$x = mX, \qquad y = \lambda m Y,$$

where $\lambda$, $m$ are positive constants, $\lambda m \leqslant 1$, and putting

$$z^*(X, Y) = z(mX, \lambda mY) = z(x, y), \qquad \varphi^*(X, Y) = \varphi(mX, \lambda mY) = \varphi(x, y),$$

$$\bar{z}_i^*(X, Y) = \bar{z}_i(mX, \lambda mY) = \bar{z}_i(x, y) \qquad (i = 1, 2, \dots, r),$$

$$f^*(X, Y, \bar{z}_1, \dots, \bar{z}_r; q) = mf\left(mX, \lambda mY, \bar{z}_1, \dots, \bar{z}_r; \frac{1}{\lambda m} q\right)$$

we associate with ($\beta$) the problem

$$(\beta')\qquad \begin{aligned} &\frac{\partial}{\partial X} z^*(X, Y) = f^*\left(X, Y; z_1^*(X, Y), \dots, z_r^*(X, Y), \frac{\partial}{\partial Y} z^*(X, Y)\right)\\ &\qquad\qquad \text{on } I^* = \{(X, Y): 0 \leqslant X \leqslant a/m,\ Y \in \boldsymbol{R}\},\\ &z^* = \varphi^* \quad \text{on } I_0^* = \{(X, Y): p_0/m \leqslant m \leqslant X \leqslant 0,\ Y \in \boldsymbol{R}\}. \end{aligned}$$

Each solution of problem ($\beta$) gives a solution of problem ($\beta'$) and vice versa.

As a consequence of the substitution, a function on $I_0^* \cup I^*$ and constants $G^*$, $N^*$, $L_y^*$, ... correspond in a natural way to every function on $I_0 \cup I$ and relative constants $G$, $N$, $L_y$, ...

Studying problem ($\beta'$), it is possible to suppose that the following inequalities hold, by a suitable choice of the constants $\lambda$ and $m$ and by eventually reducing the constant $a$ and therefore the strip $I$,

(1) $rL_z^* \hat{W}^* \leq L_q^* = \frac{1}{2}, \quad (1+r\hat{F}^*+M^*+a^*)(K_y^*+rK_z^*\hat{F}^*+K_q^*(M^*+a^*)) \leq \frac{1}{2}.$

As shown in [3], under the further Lipschitz condition on the function $f$ with respect to $x$ with the constant $H$, the solution $z^*$ of problem ($\beta'$) satisfies the conditions

$$
(2)\quad
\begin{aligned}
&|z_x^*(X, Y)-z_x^*(\bar{X}, \bar{Y})| \\
&\quad \leq [\tfrac{1}{4}(M^*+a^*)+\tfrac{1}{2}(L_y^*+rL_z^*\hat{F}^*+L_q^*(M^*+a^*))+H^*+rL_z^*\hat{P}^*]\,|X-\bar{X}| \\
&\qquad\qquad +[\tfrac{1}{2}(M^*+a^*)+L_y^*+rL_z^*\hat{F}^*+L_q^*(M^*+a^*)]\,|Y-\bar{Y}|, \\
&|z_y^*(X, Y)-z_y^*(\bar{X}, \bar{Y})| \\
&\quad \leq [\tfrac{1}{2}(M^*+a^*)+L_y^*+rL_z^*\hat{F}^*+L_q^*(M^*+a^*)]\,|X-\bar{X}|+(M^*+a^*)\,|Y-\bar{Y}|, \\
&|z_x^*(X, Y)| \leq G^*, \quad |z_y^*(X, Y)| \leq N^*+a^*
\end{aligned}
$$

for every $(X, Y)$, $(\bar{X}, \bar{Y}) \in I^*$.

The surface $z$, a solution of problem ($\beta$), belongs, therefore, to class $C_l^1(I_0 \cup I)$.

Proposition 1. *Having fixed an integer $n$, $n = 0, 1, \dots$, let $\varphi_n$: $I_0 \to R$ and $f_n$: $\Omega_0 \to R$, $\Omega_0 = \{(x, y, z, q)$: $x \in [0, a]$, $y, z \in \mathbf{R}$, $|q| \leq S\}$, be functions satisfying conditions* (i), (4i) *and* ($f$), *respectively.*

*If $\varphi_n \to \varphi_0$ and $f_n \to f_0$ in the topology of uniform convergence on compact sets, then the sequence $(z_n)_n$, where $z_n$, $n = 0, 1, \dots$, denotes the solution of the problem*

$$
(\alpha_n)\quad
\begin{aligned}
&\frac{\partial}{\partial x} z(x, y) = f_n\left(x, y; z(x, y); \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I, \\
&z = \varphi_n && \text{on } I_0;\ z \in C_l^1(I_0 \cup I),
\end{aligned}
$$

*converges in $I$ to the function $z_0$ in the topology of uniform convergence on compact sets.*

Proposition 1 is a direct application of the well-known Haar's lemma.

Corollary 1. *Under the same hypotheses as Theorem 1, the solution of problem* ($\beta$) *depends continuously upon the initial function $\varphi$ and upon the function $f$.*

## 4. Existence of solutions

**4a.** Given a function $z_0$: $I_0 \cup I \to R$ of class $C_l^1$ such that $z_0 = \varphi$ in $I_0$, we consider the following Cauchy problem

$$(\tau_1)\quad \begin{aligned}&\frac{\partial}{\partial x}z(x, y) = f\left(x, y; T^{(1)}(x, y; z_0), \ldots, T^{(r)}(x, y; z_0); \frac{\partial}{\partial y}z(x, y)\right) && \text{on } I,\\ &z = \varphi && \text{on } I_0;\ z \in C_l^1(I_0 \cup I).\end{aligned}$$

Let $\mathscr{C}_0$ be a class of functions belonging to $\mathscr{E}(I_0 \cup I)$ that contains $z_0$. Connected with $\mathscr{C}_0$, let (see (2i), (3i))

$P_0$ be the Lipschitz constant of the operators $T^{(i)}$, $i = 1, 2, \ldots, r$, with respect to $x$;

$F_0$ be the boundedness constant of functions $T_y^{(i)}$: $(I \times C_l^1(I_0 \cup I)) \to \mathrm{CL}(R, C^0(E_i))$, $i = 1, 2, \ldots, r$.

Moreover, let $W_0$ be the Lipschitz constant of Fréchet derivatives $T_y^{(i)}$ with respect to $y$ in a class of functions belonging to $\mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$, contained in $\mathscr{C}_0$ and containing the surface $z_0$.

If we put $\bar{z}_i(x, y) = T^{(i)}(x, y; z_0)$ $(i = 1, 2, \ldots, r)$ we equate problem $(\tau_1)$ with problem ($\beta$). The functions $\bar{z}_i$ satisfy, in fact, hypotheses (b) with the constants $\hat{P} = P_0$; $\hat{F} = F_0$; $\hat{W} = W_0$.

From Theorem 1, a unique solution $z_1$ of problem $(\tau_1)$ exists in a suitable strip that we denote by $I_0 \cup I_1$, where $I_1 = \{(x, y)\colon 0 \leqslant x \leqslant \delta_1,\ 0 < \delta_1 \leqslant a,\ y \in \boldsymbol{R}\}$.

**4b.** The function $z_1$: $I_0 \cup I_1 \to \boldsymbol{R}$ belongs to class $C_l^1$; the boundedness and Lipschitz constants of its derivatives follow from (2).

Let $\tilde{z}_1$ be extension ($\Xi$) of the function $z_1$ (see Remark 1) and let $\mathscr{C}_1$ be a class of functions belonging to the class $\mathscr{E}(I_0 \cup I)$ containing $\tilde{z}_1$.

Let $P_1$, $F_1$ be the constants (analogous to $P_0$ and $F_0$) of the operators $T^{(i)}$ and $T_y^{(i)}$, relative to class $\mathscr{C}_1$. Moreover, let $W_1$ be the constant (analogous to $W_0$) corresponding to class $\hat{\mathscr{C}}_1$ which belongs to $\mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$, is contained in $\mathscr{C}_1$ and contains the surface $\tilde{z}_1$.

Given $\bar{z}_i(x, y) = T^{(i)}(x, y; z_1)$ $(i = 1, 2, \ldots, r)$, let us consider the following Cauchy problem

$$(\tau_2)\quad \begin{aligned}&\frac{\partial}{\partial x}z(x, y) = f\left(x, y; \bar{z}_1(x, y), \ldots, \bar{z}_r(x, y); \frac{\partial}{\partial y}z(x, y)\right) && \text{on } I_1,\\ &z = \varphi && \text{on } I_0;\ z \in C_l^1(I_0 \cup I_1).\end{aligned}$$

As in problem $(\tau_1)$, we prove the existence of a unique solution $z_2$ of problem $(\tau_2)$ in a suitable strip, that we denote as $I_0 \cup I_2$, with $I_2 = \{(x, y)\colon 0 \leqslant x \leqslant \delta_2,\ 0 < \delta_2 \leqslant \delta_1,\ y \in \mathbf{R}\}$. Moreover, from (2) we deduce that the functions $\partial z_2/\partial x$, $\partial z_2/\partial y$ admit, respectively, the same boundedness constants as $\partial z_1/\partial x$, $\partial z_1/\partial y$, but it cannot be said that they have the same Lipschitz constants.

Therefore, put $\tilde{z}_2$ extension $(\Xi)$ of $z_2$. the function $\tilde{z}_2$ belongs to $\mathscr{C}_1$, but not necessarily to $\hat{\mathscr{C}}_1$, so that we can suppose (not being restrictive) $P_2 = P_1$ and $F_2 = F_1$. However, there would be loss in generality supposing that $W_2 = W_1$ (with the obvious meaning of the constant $P_2$, $F_2$, $W_2$).

Let $W_2$ then be the Lipschitz constant in $y$ of the functions $T_y^{(i)}$ ($i = 1, 2, \ldots, r$), relative to a class $\hat{\mathscr{C}}_2 \in \mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$, contained in $\mathscr{C}_1$ and containing $\tilde{z}_2$.

Having studied the Cauchy problem,

$$(\tau_3) \qquad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f\left(x, y;\ T^{(1)}(x, y; z_2), \ldots, T^{(r)}(x, y; z_2);\ \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I_2, \\ &z = \varphi && \text{on } I_0;\ z \in C_l^1(I_0 \cup I_2), \end{aligned}$$

its solution $z_3$ is defined in a suitable strip $I_0 \cup I_3$ contained in $I_0 \cup I_2$.

Let $\tilde{z}_3$ be extention $(\Xi)$ of $z_3$.

Such function $\tilde{z}_3$ belongs to class $\hat{\mathscr{C}}_2$, because the Lipschitz constants of the $z_3$ derivatives coincide with the Lipschitz constants of the $z_2$ derivatives in that they depend, as shown by (2), only on $P_2$, $F_2$ and on the constants of $f$ and $\varphi$, and do not depend on $W_2$.

Therefore, it is not restrictive to suppose $W_3 = W_2$, obviously beyond $P_3 = P_2$, $F_3 = F_2$.

To iterate, let us consider the following sequence of Cauchy problem

$$(\tau_n) \qquad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f\left(x, y;\ \bar{z}_1(x, y), \ldots, \bar{z}_r(x, y), \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I_{n-1}, \\ &z = \varphi && \text{on } I_0;\ z \in C_l^1(I_0 \cup I_{n-1}), \end{aligned}$$

where $\bar{z}_i(x, y) = T^{(i)}(x, y; z_{n-1})$ ($i = 1, 2, \ldots, r$) and $z_{n-1}$ is the solution of problem $(\tau_{n-1})$.

Considering the above-stated, as it is not restrictive to suppose $P_{n-1} = P_2$, $F_{n-1} = F_2$, $W_{n-1} = W_2$, $n \geqslant 4$, the $z_n$ surface is defined in the strip $I_0 \cup I_3$ (therefore $I_n = I_3$) because the width of the domain strip depends only on $F_{n-1}$, $W_{n-1}$ and on the constants of $f$ and $\varphi$, as shown in (1).

Moreover, its extension $\tilde{z}_n$ belongs to the class $\hat{\mathscr{C}}_2$, as, in virtue of (2), the

$z_n$ derivatives admit the same boundedness and Lipschitz constants as $z_3$ derivatives.

To sum up, every function of $(z_n)_n$ sequence, where $z_n$ is problem $(\tau_n)$ solution, is, for $n \geq 4$, defined in the same strip $I_0 \cup I_3$ and it belongs to the same class $\hat{\mathscr{C}}_2$.

We denote $I_0 \cup I_3$ with $I_0 \cup I$, for brevity's sake.

With a procedure analogous to Kamont's used in the linear case in [10], we prove that the sequence $(z_n)_n$ is uniformly convergent.

PROPOSITION 2. *For every* $n \in \mathbf{N}$ *we have*

$$(3) \qquad |z_n(x, y) - z_{n-1}(x, y)| \leq U(Vx)^n/n! \quad \text{for every } (x, y) \in I_0 \cup I,$$

*where* $U = 2G/rL_z$, $V = rL_z Le^{L_z a}$.

Proof. Fixed $y_0$ in $\mathbf{R}$, let us indicate with $T$ the triangle bounded by the straight lines $x = 0$, $y - y_0 = L_z(x-a)$, $y - y_0 = -L_z(x-a)$.

Let us prove (3) by induction.

For $n = 1$, taking into account (f), (2) we have

$$(4) \qquad \left|\frac{\partial}{\partial x} z_1(x, y) - \frac{\partial}{\partial x} z_0(x, y)\right|$$

$$= \left|f\left(x, y; \ldots, T^{(i)}(x, y; z_0), \ldots; \frac{\partial}{\partial y} z_1(x, y)\right) - \frac{\partial}{\partial x} z_0(x, y)\right|$$

$$\leq \left|f\left(x, y; \ldots, T^{(i)}(x, y; z_0), \ldots; \frac{\partial}{\partial y} z_1(x, y)\right)\right.$$

$$\left. - f\left(x, y; \ldots, T^{(i)}(x, y; z_0), \ldots; \frac{\partial}{\partial y} z_0(x, y)\right)\right|$$

$$+ \left|f\left(x, y; \ldots, T^{(i)}(x, y; z_0), \ldots; \frac{\partial}{\partial y} z_0(x, y)\right) - \frac{\partial}{\partial x} z_0(x, y)\right|$$

$$\leq L_q \left|\frac{\partial}{\partial y} z_1(x, y) - \frac{\partial}{\partial y} z_0(x, y)\right| + 2LG + L_z |z_1(x, y) - z_0(x, y)|$$

for every $(x, y) \in I$.

Moreover,

$$(5) \qquad |z_1(0, y) - z_0(0, y)| = 0$$

holds for every $y \in \mathbf{R}$. From (4), (5) and Theorem 37.1 of Szarski [17] it follows that

$$|z_1(x, y) - z_0(x, y)| \leq v_1(x) \quad \text{for every } (x, y) \in T,$$

where $v_1(x) = 2LG(e^{L_z x} - 1)/L_z$ is the solution of the following Cauchy problem $du/dx = 2LG + L_z u$; $u(0) = 0$. As $e^x - 1 \leq xe^x$ for every $x \geq 0$, it follows that

$$v_1(x) = 2LG(e^{L_z x} - 1)/L_z \leqslant 2LGe^{L_z a}x = U(Vx)/1!.$$

In this way, we have obtained

$$(6) \qquad |z_1(x, y) - z_0(x, y)| \leqslant U\,(Vx)/1! \quad \text{for every } (x, y) \in T.$$

Because of the arbitrary choice of $y_0$, (6) holds for every $(x, y) \in I$.

Now supposing that (3) is true for $n \geqslant 1$, let us prove that it holds for $n+1$.

In the first place, considering hypothesis (2i) relative to the operator $T^{(i)}$, $i = 1, 2, \ldots, r$, we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial x} z_{n+1}(x, y) - \frac{\partial}{\partial x} z_n(x, y)\right| \\
&= \left| f\left(x, y; \ldots, T^{(i)}(x, y; z_n), \ldots; \frac{\partial}{\partial y} z_{n+1}(x, y)\right)\right. \\
&\qquad\qquad \left. - f\left(x, y; \ldots, T^{(i)}(x, y; z_{n-1}), \ldots; \frac{\partial}{\partial y} z_n(x, y)\right)\right| \\
&\leqslant \left| f\left(x, y; \ldots, T^{(i)}(x, y; z_n), \ldots; \frac{\partial}{\partial y} z_{n+1}(x, y)\right)\right. \\
&\qquad\qquad \left. - f\left(x, y; \ldots, T^{(i)}(x, y; z_n), \ldots; \frac{\partial}{\partial y} z_n(x, y)\right)\right| \\
&\qquad\qquad + \left| f\left(x, y; \ldots, T^{(i)}(x, y; z_n), \ldots; \frac{\partial}{\partial y} z_n(x, y)\right)\right. \\
&\qquad\qquad \left. - f\left(x, y; \ldots, T^{(i)}(x, y; z_{n-1}), \ldots; \frac{\partial}{\partial y} z_n(x, y)\right)\right| \\
&\leqslant L_q \left|\frac{\partial}{\partial y} z_{n+1}(x, y) - \frac{\partial}{\partial y} z_n(x, y)\right| + L_z \sum_{i=1}^{r} |T^{(i)}(x, y; z_n) - T^{(i)}(x, y; z_{n-1})| \\
&\leqslant L_q \left|\frac{\partial}{\partial y} z_{n+1}(x, y) - \frac{\partial}{\partial y} z_n(x, y)\right| + rL_z L \sup_{(\xi, y) \in [0, x] \times \boldsymbol{R}} |z_n(\xi, y) - z_{n-1}(\xi, y)| \\
&\leqslant L_q \left|\frac{\partial}{\partial y} z_{n+1}(x, y) - \frac{\partial}{\partial y} z_n(x, y)\right| + (rL_z LU(Vx)^n/n!) + L_z |z_{n+1}(x, y) \\
&\qquad\qquad - z_n(x, y)|
\end{aligned}$$

for every $(x, y) \in I$.

On the other hand, $|z_{n+1}(0, y) - z_n(0, y)| = 0$ holds for every $y \in \boldsymbol{R}$.

In virtue of Szarski's theorem we deduce

$$|z_{n+1}(x, y) - z_n(x, y)| \leqslant v_{n+1}(x) \quad \text{for every } (x, y) \in T,$$

where $v_{n+1}(x)$ is the solution of the Cauchy problem $du/dx = rL_z LU(Vx)^n/n! + L_z u$; $u(0) = 0$. Since

$$v_{n+1}(x) = rLUV^n(e^{L_z x} - 1 - L_z x/1! - \ldots - (L_z x)^n/n!)/L_z n$$

and

$$e^x - 1 - x/1! - \ldots - x^n/n! \leqslant x^{n+1} e^x/(n+1)! \quad \text{for } x \geqslant 0,$$

it follows that

$$v_{n+1}(x) \leqslant rLUV^n(L_z x)^{n+1} e^{L_z a}/L_z^n (n+1)! = U(Vx)^{n+1}/(n+1)!$$

and so

$$(7) \qquad |z_{n+1}(x, y) - z_n(x, y)| \leqslant U(Vx)^{n+1}/(n+1)! \quad \text{for every } (x, y) \in T.$$

Because of the arbitrary choice of $y_0$, (7) is true for every $(x, y) \in I$.

As a consequence of Proposition 2, $(z_n)_n$ is a uniformly Cauchy sequence in $I$ and therefore it converges uniformly to a surface $z$.

We appreciate the difference $|z - z_n|$.

In virtue of (3) we have, for every $(x, y) \in I$,

$$(8) \qquad |z(x, y) - z_n(x, y)| \leqslant \sum_{k=n}^{\infty} |z_{k+1}(x, y) - z_k(x, y)| \leqslant \sum_{k=n+1}^{\infty} U(Vx)^k/k!$$
$$< U(Va)^{n+1}/(n+1)! \sum_{k=0}^{\infty} (Va)^k/k! = U(Va)^{n+1} e^{Va}/(n+1)!$$

By construction it results that $z = \varphi$ in $I_0$.

The sequences of partial derivatives $(\partial z_n/\partial x)_n$ and $(\partial z_n/\partial y)_n$, being uniformly bounded and uniformly Lipschitzian whit respect to $x$ and $y$ in $I$, satisfy the hypotheses of the well-known Ascoli–Arzelà lemma in each compact set in $I$.

Having arbitrarily fixed a point $P = (x, y) \in I$, let $R \subset I$ be a rectangle containing $P$. In virtue of the Ascoli–Arzelà lemma, from each subsequence $(\partial z_{n_k}/\partial x)_k$, $[(\partial z_{n_k}/\partial y)_k]$, a further subsequence converging in $R$ to $\partial z/\partial x$ $[\partial z/\partial y]$ can be extracted. From this, because of the arbitrary choice of rectangle $R$, we deduce that the sequences $(\partial z_n/\partial x)_n$ and $(\partial z_n/\partial y)_n$ uniformly converge in $I$ to $\partial z/\partial x$ and $\partial z/\partial y$ respectively. Therefore, by passing to the limit in the first equation $(\tau_n)$ we have

$$\frac{\partial}{\partial x} z(x, y) = f\left(x, y; \ldots, T^{(i)}(x, y; z), \ldots; \frac{\partial}{\partial y}(x; y)\right).$$

As shown in the preceding sections it follows that

THEOREM 2. *Under hypotheses* (f), (i)–(4i) *there exists a solution belonging to class* $C_l^1(I_0 \cup I)$ *of the Cauchy problem* $(\tau)$.

## 5. Uniqueness and continuous dependence

THEOREM 3. *Under the hypotheses of Theorem 2, the solution of problem* $(\tau)$ *is unique.*

Proof. Let $z_1$ and $z_2$ be two solutions of problem $(\tau)$ belonging to class $C_l^1(I_0 \cup I)$. In virtue of hypotheses (f) and (**) we have

$$(9) \quad \left|\frac{\partial}{\partial x} z_1(x, y) - \frac{\partial}{\partial x} z_2(x, y)\right|$$
$$\leqslant L_z \sum_{i=1}^{r} |T^{(i)}(x, y; z_1) - T^{(i)}(x, y; z_2)| + L_q \left|\frac{\partial}{\partial y} z_1(x, y) - \frac{\partial}{\partial y} z_2(x, y)\right|$$
$$\leqslant rL_z L \sup_{(\xi, y) \in [0, x] \times \mathbf{R}} |z_1(\xi, y) - z_2(\xi, y)| + L_q \left|\frac{\partial}{\partial y} z_1(x, y) - \frac{\partial}{\partial y} z_2(x, y)\right|.$$

Being $z_1, z_2 \in C_l^1(I_0 \cup I)$, there exists a constant $\hat{M}$ such that

$$\sup_{(x,y) \in I} |z_1(x, y) - z_2(x, y)| \leqslant \hat{M}.$$

In virtue of the well-known Haar lemma it follows that

$$(10) \quad |z_1(x, y) - z_2(x, y)| \leqslant rL_z \hat{M} L x \quad \text{for every } x \in [0, a],\ y \in \mathbf{R},$$

from which

$$(11) \quad \sup_{(\xi, y) \in [0, x] \times \mathbf{R}} |z_1(\xi, y) - z_2(\xi, y)| \leqslant rL_z \hat{M} L a \quad \text{for every } x \in [0, a].$$

In virtue of (11) and applying the Haar lemma again, we deduce

$$(10_2) \quad |z_1(x, y) - z_2(x, y)| \leqslant \hat{M}(rL_z L)^2 a x \leqslant \hat{M}(rL_z La)^2 \quad \text{for } (x, y) \in I.$$

By iterating we shown that

$$(10_n) \quad |z_1(x, y) - z_2(x, y)| \leqslant \hat{M}(rL_z La)^n, \quad n \in N,\ (x, y) \in I,$$

and so the theorem is proved.

THEOREM 4. *Under the hypotheses of Theorem 2, the solution of problem* $(\tau)$ *depends continuously upon the initial function* $\varphi$ *as well as upon the function* $f$.

Proof. Let $(\varphi_n)_n$ and $(f_n)_n$ $(n = 0, 1, \ldots)$ be two sequences of functions with properties (i), (4i) and (f), respectively, and such that $\varphi_n \to \varphi_0$ uniformly in compact sets in $I_0$ and $f_n \to f_0$ uniformly in compact sets in $\Omega$.

Moreover, let $z_{n,0}\colon I_0 \cup I \to \mathbf{R}$, $n \in N$, a sequence of functions belonging to a class $\mathscr{C}_0$ contained in $\mathscr{E}_{\mathrm{Lip}}(I_0 \cup I)$ such that $z_{n,0} \to z_{0,0}$ uniformly in compact sets in $I_0 \cup I$ and $z_{n,0} = \varphi_n$ in $I_0$.

Fixed $n = 0, 1, \dots,$ let $z_{n,1}$ be the solution of the problem

$$(\tau_{n,1})\quad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f_n\left(x, y; T^{(1)}(x, y; z_{n,0}), \dots, T^{(r)}(x, y; z_{n,0}); \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I, \\ &z = \varphi_n && \text{on } I_0;\ z \in C_l^1(I_0 \cup I) \end{aligned}$$

whose existence derives from Theorem 1.

Given $f_{n,1}(x, y; q) = f_n(x, y; T^{(1)}(x, y; z_{n,0}), \dots, T^{(r)}(x, y; z_{n,0}); q)$, and having arbitrarily fixed a compact set $H \subset I$ and a real positive constant $S_1 \leqslant S$, it follows that

$$\begin{aligned} &\max_{(x,y)\in H, |q| \leqslant S} |f_{n,1}(x, y; q) - f_{0,1}(x, y; q)| = |f_{n,1}(\bar{x}, \bar{y}; \bar{q}) - f_{0,1}(\bar{x}, \bar{y}; \bar{q})| \\ &\leqslant \left|f_n(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}; z_{n,0}), \dots; \bar{q}) - f_n(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}; z_{0,0}), \dots; \bar{q})\right| \\ &+ \left|f_n(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}; z_{0,0}), \dots; \bar{q}) - f_0(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}; z_{0,0}), \dots; \bar{q})\right| \\ &\leqslant L_z \sum_{i=1}^{r} |T^{(i)}(\bar{x}, \bar{y}; z_{n,0}) - T^{(i)}(\bar{x}, \bar{y}; z_{0,0})| \\ &+ \left|f_n(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}; z_{0,0}), \dots; \bar{q}) - f_0(\bar{x}, \bar{y}; \dots, T^{(i)}(\bar{x}, \bar{y}, z_{0,0}), \dots; \bar{q})\right| \end{aligned}$$

with $\bar{x}$, $\bar{y}$, $\bar{q}$ suitably chosen.

Consequently, considering the hypotheses on the sequences $(z_{n,0})_n$ and $(f_n)_n$ and the continuity of the operators $T^{(i)}$, it follows from Proposition 1 that the sequence $(z_{n,1})_n$ converges to $z_{0,1}$ uniformly in compact sets in $I_0 \cup I$.

For every $k \in N$, let $z_{n,k}$ be the solution of the problem

$$(\tau_{n,k})\quad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f_n\left(x, y; T^{(1)}(x, y; z_{n,k-1}), \dots, T^{(r)}(x, y; z_{n,k-1}); \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I, \\ &z = \varphi_n && \text{on } I_0;\ z \in C_l^1(I_0 \cup I). \end{aligned}$$

Given $f_{n,k}(x, y; q) = f_n(x, y; T^{(1)}(x, y; z_{n,k-1}), \dots, T^{(r)}(x, y; z_{n,k-1}); q)$, it results that, as in case $k = 1$,

$$\begin{aligned} &\max_{(x,y)\in H, |q| \leqslant S_1} |f_{n,k}(x, y; q) - f_{0,k}(x, y; q)| \\ &\qquad = |f_{n,k}(\tilde{x}, \tilde{y}; \tilde{q}) - f_{0,k}(\tilde{x}, \tilde{y}; \tilde{q})| \\ &\qquad \leqslant L_z \sum_{i=1}^{r} |T^{(i)}(\tilde{x}, \tilde{y}; z_{n,k-1}) - T^{(i)}(\tilde{x}, \tilde{y}; z_{0,k-1})| \end{aligned}$$

$+|f_n(\tilde{x}, \tilde{y}; \ldots, T^{(i)}(\tilde{x}, \tilde{y}; z_{0,k-1}), \ldots; \tilde{q}) - f_0(\tilde{x}, \tilde{y}; \ldots, T^{(i)}(\tilde{x}, \tilde{y}; z_{0,k-1}), \ldots; \tilde{q})|$

for $\tilde{x}$, $\tilde{y}$, $\tilde{q}$ suitably chosen.

In virtue of Proposition 1 it follows that the sequence $(z_{n,k})_n$ converges to $z_{0,k}$.

On the other hand, for every fixed $n$, the sequence $(z_{n,k})_k$ converges uniformly in $I_0 \cup I$ to a function $z_n$ which is the solution of the following Cauchy problem

$$(\tau_n) \qquad \begin{aligned} &\frac{\partial}{\partial x} z(x, y) = f_n\left(x, y; T^{(1)}(x, y; z), \ldots, T^{(r)}(x, y; z), \frac{\partial}{\partial y} z(x, y)\right) && \text{on } I, \\ &z = \varphi_n && \text{on } I_0;\ z \in C_l^1(I_0 \cup I). \end{aligned}$$

This is a result of what is proved in Section 4b.

From (8) of Section 4b it follows that the convergence of $(z_{n,k})_k$ to $z_n$ is uniform also in respect to $n$.

Moreover, we can suppose that all functions $z_{n,k}$, with $n, k \in N$, have a common domain which we indicate again with $I_0 \cup I$. In fact, fixed $n$, in virtue of the proof given in Section 4, the functions $z_{n,k}$, for every $k = 0, 1, \ldots$, are defined in the same strip. Moreover, this strip is independent from $n$ because its width depends, by virtue of (1), only on the constants relative to $\varphi_n$, $f_n$ and $z_{n,0}$ (which are, by hypothesis, invariable with respect to $n$).

Fixed $\varepsilon > 0$ and a compact set $H \subset I$ and given $|u|_H = \max_{(x,y)\in H} |u(x, y)|$ for every $u \in C^0(H)$, let $k = k(\varepsilon, H)$ be a integer for which it follows that

$$(12) \qquad |z_{n,k} - z_n|_H < \varepsilon/3 \quad \text{for every } n = 0, 1, \ldots$$

Finally, let $\bar{n} = \bar{n}(\varepsilon, k, H) = \bar{n}(\varepsilon, H)$ be an integer such that for every $n > \bar{n}$ we have

$$(13) \qquad |z_{n,k} - z_{0,k}|_H < \varepsilon/3.$$

From (12) and (13) it follows, then, that

$$|z_n - z_0|_H \leqslant |z_n - z_{n,k}|_H + |z_{n,k} - z_{0,k}|_H + |z_{0,k} - z_0|_H < \varepsilon,$$

which provides the theorem, by the arbitrariness of $H$.

The following diagram therefore exists

$$\begin{array}{ccc} (\varphi_n, f_n) & & z_{n,k} \xrightarrow[k]{} z_n \\ \downarrow^n & & \downarrow^n \qquad \downarrow^n \\ (\varphi_0, f_0) & & z_{0,k} \xrightarrow[k]{} z_0 \end{array}$$

**6.** Problem ($\tau$) is reduced to the problem studied by Kamont in [13], whenever the range of the operator $T^{(i)}$ is contained in the subspace of the constant functions of $C^0(E_i)$.

We show that our hypotheses on the operator $T^{(i)}$ are less restrictive then Kamont's, in particular the hypothesis which gives a hereditary structure to the problem.

Fixed a point $\bar{y} \in \boldsymbol{R}$, we consider the operator $T\colon I \times C_l^1(I_0 \cup I) \to \boldsymbol{R}$ defined by $T(x, z) = \max_{\xi \in [0, x]} z(\xi, \bar{y})$.

This operator verifies hypotheses (2i), (3i) (as is easily proved) but does not satisfy the following condition (cf. [13]):

(&) there exist two real functions $\alpha$, $\beta$ of class $C^1$ in $I$ and a constant $L$ with the properties

$$(\&_0) \qquad p_0 < \alpha(x, y) \leqslant x,$$

$$(\&_1) \qquad |T(x, z) - T(x, \bar{z})| \leqslant L\big|z\big(\alpha(x, \bar{y}), \beta(x, \bar{y})\big) - \bar{z}\big(\alpha(x, \bar{y}), \beta(x, \bar{y})\big)\big|$$

for every $z, \bar{z} \in C_l^1(I_0 \cup I)$ and for every $x \in [0, a]$.

Having arbitrarily fixed a function $\alpha$ and a point $\bar{x} \in ]0, a]$, let $z$ be a surface of class $C_l^1(I_0 \cup I)$ such that

$$z\big(\alpha(\bar{x}, \bar{y}), \bar{y}\big) + \bar{x} - \alpha(\bar{x}, \bar{y}) \neq \max_{\xi \in [0, \bar{x}]} z(\xi, \bar{y}).$$

Given $\bar{z}(x, y) = z\big(\alpha(\bar{x}, \bar{y}), y\big) + x - \alpha(\bar{x}, \bar{y})$, it follows that $z\big(\alpha(\bar{x}, \bar{y}), \beta(\bar{x}, \bar{y})\big) - \bar{z}\big(\alpha(\bar{x}, \bar{y}), \beta(\bar{x}, \bar{y})\big) = 0$ for every function $\beta$, while

$$|T(\bar{x}, z) - T(\bar{x}, \bar{z})| = |\max_{\xi \in [0, \bar{x}]} z(\xi, \bar{y}) - \max_{\xi \in [0, \bar{x}]} \bar{z}(\xi, \bar{y})|$$

is strictly positive. This proves that operator $T$ does not satisfy condition (&).

## References

[1] E. Baiada, *Sul teorema di esistenza per le equazioni alle derivate parziali del primo ordine*, Ann. Scuola Normale Sup. Pisa (2) 12 (1943), 135–145.

[2] P. Brandi, *Sul problema di Cauchy per una equazione differenziale non lineare alle derivate parziali del primo ordine con argomento funzionale*, Atti Sem. Mat. Fis. Univ. Modena 28 (1979), 86–101.

[3] –, R. Ceppitelli, *On the existence of the solution of non-linear functional partial differential equations of the first order*, ibidem 29 (1980), 166–186.

[4] M. Cinquini Cibrario, S. Cinquini, *Equazioni a derivative parziali di tipo iperbolico*, Edizioni Cremonese, Roma 1964.

[5] J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York and London 1960.

[6] L. E. El'sgol'ts, S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*, Acad. Press, London 1973.

[7] J. K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York 1977.

[8] —, M. A. Cruz, *Existence, uniqueness and continuous dependence for hereditary systems*, Ann. Mat. Pura Appl. (4) 85 (1970), 63–82.

[9] G. S. Jones, *Hereditary structures in differential equations*, Math. Sys. Theory, Vol. I, No. 3 (1967), 263–278.

[10] Z. Kamont, *On the Cauchy problem for linear partial differential-functional equations of the first order*, Ann. Polon. Math. 35 (1977), 27–48.

[11] —, *On the existence and uniqueness of solutions of the Cauchy problem for linear partial differential-functional equations of the first order*, Math. Nachr. 80 (1977), 183–200.

[12] —, *On the Cauchy problem for non-linear partial differential-functional equations of the first order*, ibidem 88 (1979), 13–29.

[13] —, *On the estimation of the existence domain for solutions of a non-linear partial differential-functional equation of the first order*, Glasnik Mat. (33) 13 (1979), 277–291.

[14] —, *On differential-integral equations with first order partial derivatives*, University of Gdańsk, Institute of Mathematics, Preprint No. 23, May 1979.

[15] —, M. Kwapisz, *On non-linear Volterra integral-functional equations in several variables*, Ann. Polon. Math. 40 (1981), 1–29.

[16] A. Pliś, *On the estimation of the existence domain for solutions of a nonlinear partial differential equations of the first order*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 3 (1956), 125–129.

[17] J. Szarski, *Differential inequalities*, Warszawa 1965.

[18] K. Zima, *Sur les équations aux dérivées partielles du premier ordre à argument fonctionnel*, Ann. Polon. Math. 13 (1969), 49–59.

*Reçu par la Rédaction le 1984.07.20*