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Some examples on lifting the commutant of a subnormal operator

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The commutant of an operator A is the algebra of operators which commute with A and will be denoted by $\mathscr{C}(A)$. If S is a subnormal operator on the Hilbert space $\mathscr{H}$, if N is the minimal normal extension of S, and if $\mathscr{C}(N, \mathscr{H})$ is the algebra of operators in $\mathscr{C}(N)$ which leave $\mathscr{H}$ invariant, then there is a contractive homomorphism

$$\Lambda\colon \mathscr{C}(N, \mathscr{H}) \to \mathscr{C}(S)$$

defined by setting $\Lambda(A)$ equal to the restriction of A to $\mathscr{H}$. If B is in the range of Λ, then B is said to lift to the commutant of N.

It is known that Λ is always one-to-one (lifts are unique) [8], p. 68, and that Λ may not be onto (lifts may not exist) [5], Corollary 7.2. In certain situations Λ is both isometric and onto, for example, if S is an isometry [5], Corollary 5.1, if S is cyclic [9], Theorem 3, or if S is a bundle shift [1], Theorem 4. It is not known whether S is isometric or onto when S is an analytic Toeplitz operator and this remains an important unsolved problem.

The purpose of this paper is to study a class of subnormal operators S, where the commutant of S and the map Λ can be computed explicitly and to exhibit ways that Λ can fail to be isometric and onto. In particular, there will be examples when Λ is (1) isometric and not onto, (2) onto and not isometric, and (3) not bounded below. Note that in the third example, by the open mapping theorem the range of Λ is not closed in the operator norm topology.

The class of subnormal operators to be studied consists of those operators of the form $S_1 \oplus S_2$, where each S_K is subnormal and similar to the

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unilateral shift. The analysis makes use of a model for subnormal operators similar to the shift developed by Clary [3] and it also uses a result due to Douglas on the intertwining maps between normal operators [5]. In terms of the Clary model for $S = S_1 \oplus S_2$, necessary and sufficient conditions are given for Λ to be onto.

Regarding the aforementioned result of Douglas that Λ need not be onto, two comments are in order. First, the example produced by Douglas is an example of an intertwining map between two subnormal operators T_1 and T_2 which does not lift to an intertwining map between their minimal normal extensions. It is then a simple matter to produce an operator in the commutant of $T_1 \oplus T_2$ which does not lift to the commutant of the minimal normal extension of $T_1 \oplus T_2$. (See Section 3 of this paper.) Second, there is an error in the development of this example which can be corrected in the following way. The conclusion of Lemma 7.1 in [5] should be changed to read: "Then a necessary condition that there exist B in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ so that $N_2B = BN_1$ and $A = B|\mathscr{H}_1$ is for N_1 to be unitarily equivalent to the restriction of N_2 to a reducing subspace." Then, in the proof of Corollary 7.2, one must add that in the example of Sarason [7], Problem 156, the operators T_1 and T_2 can be chosen so that the spectrum of N_1 is larger than the spectrum of N_2 and therefore N_1 is not unitarily equivalent to the restriction of N_2 to a reducing subspace.

NOTATIONS. In this paper, all Hilbert spaces are complex, all operators are bounded, and all subspaces are closed. If S is an operator on $\mathscr{H}$ and if T is an operator on $\mathscr{K}$, then the space $\mathscr{I}(S, T)$ is the space of operators B from $\mathscr{H}$ into $\mathscr{K}$ which intertwine S and T, that is, which satisfy $BS = TB$. If S and T are subnormal with minimal normal extensions M and N respectively, then the space $\mathscr{I}_L(S, T)$ is defined to be the set of operators in $\mathscr{I}(S, T)$ which are restrictions of operators in $\mathscr{I}(M, N)$. Thus, $\mathscr{I}_L(S, T)$ consists of those intertwining maps between S and T which lift to intertwining maps between M and N. Observe that $\mathscr{C}(S) = \mathscr{I}(S, S)$ and define $\mathscr{C}_L(S)$ to be the space $\mathscr{I}_L(S, S)$. Thus, $\mathscr{C}_L(S)$ is precisely the range of the map Λ discussed above.

1. Intertwining maps between cyclic normal operators. In this paper the term *measure* will refer to a finite positive compactly supported Borel measure on the complex plane. Thus, given a measure μ there is a normal operator W_μ on $L^2(\mu)$ defined by the equation $W_\mu(f)(z) = zf(z)\,d\mu$-a.e. and it is well known that any normal operator with a cyclic vector is unitarily equivalent to W_μ for some measure μ [4], Theorem 4.58. It is well known that an operator commutes with W_μ if and only if it is multiplication by ψ for some ψ in $L^\infty(\mu)$ (see for example [7], Problem 115). It is a result of Douglas that for two measures μ and ν, the space of intertwining maps $\mathscr{I}(W_\mu, W_\nu)$ is zero if and only if μ and ν are relatively singular [5], Theorem 3. In Theorem 1 below, these two results are combined to give a description of the space of intertwining maps $\mathscr{I}(W_\mu, W_\nu)$ for arbitrary measures μ and ν.

To state Theorem 1, one needs the notion of an absolutely continuous support for a pair of measures which is defined as follows. For a Borel subset E of the complex plane, let μ_E be the measure defined by the equation $\mu_E(F) = \mu(E \cap F)$. An absolutely continuous support of a pair of measures μ and ν is a Borel subset E of the complex plane such that μ_E and ν_E are mutually absolutely continuous and μ_F and ν_F are relatively singular, where $F = C \setminus E$. Lemma 1.1 establishes the existence and uniqueness (modulo sets of measure zero) of absolutely continuous supports. For two sets E and E', the set $E \triangle E'$ is the symmetric difference of E and E', that is, $E \triangle E' = (E \setminus E') \cup (E' \setminus E)$.

LEMMA 1.1. *If μ and ν are measures, then there is an absolutely continuous support for μ and ν. If E and E' are two such supports, then $\mu(E \triangle E') = 0 = \nu(E \triangle E')$.*

Proof. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to ν, where μ_a is absolutely continuous with respect to ν and μ_s is singular relative to ν. Let A be a Borel set with $\mu_s(A) = 0 = \mu_a(C \setminus A)$, let h be a Borel measurable function such that $d\mu_a = h d\nu$, and let $E = \{x \text{ in } A: h(x) > 0\}$. For any Borel set S

$$\mu_E(S) = \mu(E \cap S) = \mu_a(S) = \int_{E \cap S} h\,d\nu = \int_S h\,d\nu_E$$

and therefore μ_E and ν_E are mutually absolutely continuous. If $F = C \setminus E$, then $\mu_F = \mu_s$ and therefore μ_F is singular relative to ν_F. This proves that E is an absolutely continuous support for μ and ν. If E' is a second absolutely continuous support for μ and ν and if $F' = C \setminus E'$, then $\mu = \mu_E + \mu_F$ and $\mu = \mu_{E'} + \mu_{F'}$ are both Lebesgue decompositions of μ with respect to ν. Thus, $\mu_E = \mu_{E'}$ and $\mu_F = \mu_{F'}$ and it follows that $\mu(E \triangle E') = 0$. Similarly, $\nu = \nu_E + \nu_F$ and $\nu = \nu_{E'} + \nu_{F'}$ are both Lebesgue decompositions of ν with respect to μ, hence $\nu_E = \nu_{E'}$ and $\mu_E = \mu_{E'}$ and therefore $\nu(E \triangle E') = 0$. This proves Lemma 1.1.

For Theorem 1, let E be an absolutely continuous support for the measures μ and ν, let F be the set $C \setminus E$, let h be the function $\sqrt{d\mu_E / d\nu_E}$ and let $I(\mu, \nu)$ be the set of functions ψ on the complex plane such that (i) $\psi | F \equiv 0$ and (ii) there is a constant C such that $|\psi| \leqslant Ch$ $d\mu$-a.e. It is easily verified that for ψ in $I(\mu, \nu)$ there is an operator $A(\psi, \mu, \nu)$ from $L^2(\mu)$ into $L^2(\nu)$ defined by the equation $A(\psi, \mu, \nu)(f) = \psi f$.

THEOREM 1. *An operator A from $L^2(\mu)$ into $L^2(\nu)$ satisfies $AW_\mu = W_\nu A$ if and only if $A = A(\psi, \mu, \nu)$ for some ψ in $I(\mu, \nu)$. In this case, the norm of the operator A is the norm of the function ψ / h in $L^\infty(\mu)$.*

The proof of Theorem 1 follows from the known special cases when $\mu = \nu$ and when $\mu \perp \nu$ and from the following lemma which establishes an intimate relationship between the intertwining maps between two direct sums $M_1 \oplus M_2$ and $N_1 \oplus N_2$ and the intertwining maps between M_i and N_j. This idea is well known and seems to go back to Berberian [2].

LEMMA 1.2. *For $i = 1$ and $i = 2$, let M_i be an operator on $\mathscr{H}_i$ and let N_i be an operator on $\mathscr{K}_i$. Let $M = M_1 \oplus M_2$, let $N = N_1 \oplus N_2$, let A be an operator from $\mathscr{H}_1 \oplus \mathscr{H}_2$ into $\mathscr{K}_1 \oplus \mathscr{K}_2$, and let*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be the matrix of A with $A_{ij}\colon \mathscr{H}_j \to \mathscr{K}_i$. Then A is in $\mathscr{I}(M, N)$ if and only if each A_{ij} is in $\mathscr{I}(M_j, N_i)$.

Proof. The result is established by carrying out the matrix multiplication in the expressions

$$AM = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

and

$$NA = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Proof of Theorem 1. Let

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be the matrix for A with respect to the decompositions $L^2(\mu) = L^2(\mu_E) \oplus \oplus L^2(\mu_F)$ and $L^2(\nu) = L^2(\nu_E) \oplus L^2(\nu_F)$. With respect to these decompositions, $W_\nu = W_{\nu_E} \oplus W_{\nu_F}$. Thus, by Lemma 1.2, the operator A is in $\mathscr{I}(W_\mu, W_\nu)$ if and only if A_{11} is in $\mathscr{I}(W_{\mu_E}, W_{\nu_E})$, A_{12} is in $\mathscr{I}(W_{\mu_F}, W_{\nu_E})$, A_{21} is in $\mathscr{I}(W_{\mu_E}, W_{\nu_F})$, and A_{22} is in $\mathscr{I}(W_{\mu_F}, W_{\nu_F})$. Since each pair of measures (μ_F, ν_E), (μ_E, ν_F), and (μ_F, ν_F) is relatively singular, the latter three spaces contain only the zero operator by the aforementioned result of Douglas [5], Theorem 3. Thus, the operator A is in $\mathscr{I}(W_\mu, W_\nu)$ if and only if A_{11} is in $\mathscr{I}(W_{\mu_E}, W_{\nu_E})$ and $A_{12} = A_{21} = A_{22} = 0$. To characterize A_{11}, observe that there is a unitary operator U in $\mathscr{I}(W_{\nu_E}, W_{\mu_E})$ defined by the equation $U(f) = \sqrt{d\nu_E/d\mu_E} f$. Thus, the operator A is in $\mathscr{I}(W_{\mu_E}, W_{\nu_E})$ if and only if UA is in $\mathscr{I}(W_{\mu_E}, W_{\mu_E})$. This occurs if and only if there is a bounded Borel measurable function η on the complex plane such that $\eta | F = 0$ and $(UA)(f) = \eta f$ for all f in $L^2(\mu_E)$, [7], Problem 115. Moreover, in this case the norm of UA is the norm of η in $L^\infty(\mu_E)$. The theorem now follows by setting $\psi = h\eta$.

2. Intertwining maps between two subnormal operators similar to the shift. For a measure μ, let $H^2(\mu)$ be the closure in $L^2(\mu)$ of the polynomials in z and let U_μ be the restriction of W_μ to $H^2(\mu)$. It is an immediate consequence of the Stone–Weierstrass Theorem that W_μ is the minimal normal extension of the subnormal operator U_μ. Furthermore, if D is the open unit disk, if T denotes the boundary of D, and if m is normalized linear Lebesgue measure on T, then U_m is a model for the unilateral shift.

If μ is a measure such that $\mu_T = m$, then there is a contractive and densely ranged operator X_μ from $H^2(\mu)$ into $H^2(m)$ which maps the equivalence class of a function in $H^2(\mu)$ to the equivalence class of the same function in $H^2(m)$. It is apparent that $X_\mu U_\mu = U_m X_\mu$ and it is a consequence of Carleson's Theorem [6], Theorem 9.3, that X_μ is invertible when $\mu(\{z\colon |z| > 1\}) = 0$ and μ_D is a Carleson measure. Thus, if μ is a measure such that (1) $\mu_T = m$, (2) $\mu(\{z\colon |z| > 1\}) = 0$, and (3) μ_D is a Carleson measure, then X_μ implements a similarity between U_μ and the unilateral shift U_m. Conversely, if S is any subnormal operator similar to the unilaterial shift, then there is a unique measure μ satisfying (1), (2), and (3) such that S is unitarily equivalent to U_μ. This result is due to Clary [3], Corollary 6.4, and it provides a model for all subnormal operators similar to the unilaterial shift. Examples of Carleson measures are measures with closed support in D, planar Lebesgue measure on D, and linear Lebesgue measure on $(-1, 1)$.

Let μ and ν be two measures satisfying (1), (2), and (3) above. It is the purpose of this section to describe the spaces $\mathscr{I}(U_\mu, U_\nu)$ and $\mathscr{I}_L(U_\mu, U_\nu)$. To this end, let $H^\infty(D)$ be the space of bounded Borel measurable functions φ on the complex plane such that $\varphi | D$ is analytic and

$$\lim_{r\uparrow 1} \varphi(rz) = \varphi(z)$$

for dm-almost-every z in T and, for φ in $H^\infty(D)$, let $\|\Phi\|$ be the norm of the equivalence class of φ in $L^\infty(m)$. Furthermore, for φ in $H^\infty(D)$, let T_φ be the analytic Toeplitz operator on $H^2(m)$ with symbol φ, that is, $T_\varphi(f) = \varphi f$ for f in $H^2(m)$ and let $B(\varphi, \mu, \nu)$ be the operator $X_\nu^{-1} T_\varphi X_\mu$ from $H^2(\mu)$ into $H^2(\nu)$. Thus $B(\varphi, \mu, \nu)(f) = \varphi f$ for all f in $H^2(\mu)$.

LEMMA 2.1. *If* $K = \|X_\mu\| \|X_\nu^{-1}\|$, *then for all* φ *in* $H^\infty(D)$, $\|\varphi\| \leqslant \|B(\varphi, \mu, \nu)\| \leqslant K \|\Phi\|$.

Proof. It is known that $\|T_\varphi\| = \|\Phi\|$, [7], p. 139, and thus $\|B(\varphi, \mu, \nu)\| = \|X_\nu^{-1} T_\varphi X_\mu\| \leqslant K \|\Phi\|$. Furthermore, for $\varepsilon > 0$, there is a polynomial p such that $\int |p|^2 dm = 1$ and $\int |\varphi|^2 |p|^2 dm \geqslant \|\Phi\|^2 - \varepsilon$. Thus, by the bounded convergence theorem, as $n \to \infty$

$$\|B(\varphi, \mu, \nu)(z^n p)\|^2 = \int |\varphi|^2 |z|^{2n} |p|^2 d\nu \to \int |\varphi|^2 |p|^{2n} dm \geqslant \|\Phi\|^2 - \varepsilon$$

and
$$\int |z|^{2n}|p|^2 d\mu \to \int |p|^2 dm = 1.$$

It follows that $\|B(\varphi, \mu, \nu)\| \geqslant \|\Phi\|$.

LEMMA 2.2. *For φ and ψ in $H^\infty(D)$, the following four assertions are equivalent*: (i) $B(\varphi, \mu, \nu) = B(\psi, \mu, \nu)$; (ii) $\varphi = \psi$ *dm-a.e.*; (iii) $\varphi = \psi$ *$d\nu$-a.e.*; (iv) $\varphi = \psi$ *$d\mu$-a.e.*

Proof. The equivalence of (i) and (ii) is immediate from Lemma 2.1. If $\varphi = \psi$ dm-a.e., then $\varphi(z) = \psi(z)$ for all z in D and the equivalence of (ii), (iii), and (iv) follows.

Let E be an absolutely continuous support for the measures μ and ν, let h be the function $\sqrt{d\mu_E/d\nu_E}$, and let $I_L(\mu, \nu)$ be the set of functions φ in $H^\infty(D)$ such that (i) $\varphi|F = 0$ $d\nu$-a.e. and (ii) there is a constant C such that $|\varphi| \leqslant Ch$ $d\mu_E$-a.e. Observe that $I(\mu, \nu)$, $H^\infty(D)$, and $I_L(\mu, \nu)$ are all spaces of functions (not equivalence classes) on the complex plane. The relationships between these spaces and the operators $A(\psi, \mu, \nu)$ and $B(\varphi, \mu, \nu)$ are expressed in the following two lemmas. Let χ_E denote the characteristic function of the set E.

LEMMA 2.3. *Let φ be in $I(\mu, \nu)$ and ψ be in $I_L(\mu, \nu)$. The operator $A(\varphi, \mu, \nu)$ is an extension of $B(\psi, \mu, \nu)$ if and only if $\varphi = \psi$ $d\nu$-a.e.*

Proof. If $A(\varphi, \mu, \nu)$ extends $B(\psi, \mu, \nu)$, then $\varphi = A(\varphi, \mu, \nu)(1) = B(\psi, \mu, \nu)(1) = \psi$ $d\nu$-a.e. and the converse implication is immediate.

LEMMA 2.4. *Let φ be a function in $H^\infty(D)$. Then φ is in $I_L(\mu, \nu)$ if and only if there is a function ψ in $I(\mu, \nu)$ with $\varphi = \psi$ $d\nu$-a.e. In this case, the function ψ can be chosen to be $\chi_E\varphi$.*

Proof. Assume that φ is in $I_L(\mu, \nu)$ and set $\psi = \chi_E\varphi$. Then ψ is in $I(\mu, \nu)$ and $\varphi = (\chi_F + \chi_E)\varphi = \chi_E\varphi$ $d\nu$-a.e. because $\varphi|F \equiv 0$ $d\nu$-a.e. The converse assertion is immediate.

THEOREM 2. *An operator B is in $\mathscr{I}(U_\mu, U_\nu)$ if and only if $B = B(\varphi, \mu, \nu)$ for some φ in $H^\infty(D)$. For φ in $H^\infty(D)$, the operator $B(\varphi, \mu, \nu)$ is in $\mathscr{I}_L(U_\mu, U_\nu)$ if and only if φ is in $I_L(\mu, \nu)$. If φ is in $I_L(\mu, \nu)$, then $B(\varphi, \mu, \nu)$ extends to the operator $A(\psi, \mu, \nu)$ in $\mathscr{I}(W_\mu, W_\nu)$, where $\psi = \chi_E\varphi$.*

Proof. An operator B from $H^2(\mu)$ into $H^2(\nu)$ is in $\mathscr{I}(U_\mu, U_\nu)$ if and only if $X_\nu B X_\mu^{-1}$ is in $\mathscr{C}(U_m)$ and this occurs if and only if $X_\nu B X_\mu^{-1} = T_\varphi$ for some φ in $H^\infty(D)$, [7], Problem 116. The latter equation says that $B = X_\nu^{-1} T_\varphi X_\mu = B(\varphi, \mu, \nu)$ which proves the first assertion of the theorem. To prove the second assertion, let φ be in $H^\infty(D)$. By Theorem 1, the operator $B(\varphi, \mu, \nu)$ is in $\mathscr{I}_L(\mu, \nu)$ if and only if there is a function ψ in $I(\mu, \nu)$ such that $A(\psi, \mu, \nu)$ extends $B(\varphi, \mu, \nu)$. By Lemmas 2.3 and 2.4 this is equivalent to saying that φ is in $I_L(\mu, \nu)$. Also, by Lemma 2.4, the function ψ can be chosen to be $\chi_E\varphi$. This completes the proof of Theorem 2.

COROLLARY 2.5. *The following three conditions are equivalent*: (i) $\mathscr{I}_L(U_\mu, U_\nu) = \mathscr{I}(U_\mu, U_\nu)$; (ii) *the operator* $B(l, \mu, \nu)$ *is in* $\mathscr{I}_L(U_\mu, U_\nu)$; (iii) *the measure* ν *is absolutely continuous with respect to* μ *and the Radon–Nikodym derivative* $d\nu/d\mu$ *is in* $L^\infty(d\mu)$.

Proof. That (i) implies (ii) is immediate. Assume (ii), let E be an absolutely continuous support for μ and ν, and let $F = CE$. By Theorem 2, the function l is in $\mathscr{I}_L(\mu, \nu)$ which implies that $l|F = 0$ $d\nu$-a.e. Thus, $\nu(F) = 0$ and so ν is absolutely continuous with respect to μ. Furthermore, since l is in $I_L(\mu, \nu)$, there is a constant C such that $1 \leqslant Ch$ $d\mu_E$-a.e., where $h = \sqrt{d\mu_E/d\nu_E}$. Thus, $d\nu/d\mu = d\nu_E/d\mu_E = h^{-2} \leqslant C^{-2}$ $d\mu$-a.e. and this establishes (iii). Finally, if (iii) is satisfied, then it follows easily that $I_L(\mu, \nu) = H^\infty(D)$ and therefore (i) is satisfied by Theorem 2. This completes the proof of the corollary.

3. The commutant of the direct sum of two subnormal operators similar to the shift. Let S and T be subnormal operators with minimal normal extensions M and N respectively. By Lemma 1.1, the commutant of $S \oplus T$ is the matrix algebra

$$\begin{bmatrix} \mathscr{C}(S) & \mathscr{I}(T, S) \\ \mathscr{I}(S, T) & \mathscr{C}(T) \end{bmatrix}.$$

It is easily verified that the minimal normal extension of $S \oplus T$ is $M \oplus N$ and that the space $\mathscr{C}_L(S \oplus T)$ of operators in $\mathscr{C}(S \oplus T)$ which lift to $\mathscr{C}(M \oplus N)$ is the space

$$\begin{bmatrix} \mathscr{C}_L(S) & \mathscr{I}_L(T, S) \\ \mathscr{I}_L(S, T) & \mathscr{C}_L(T) \end{bmatrix}.$$

The following theorem is an immediate consequence of these remarks and Theorem 2. In this theorem, μ and ν are measures satisfying conditions (1), (2), and (3) of Section 2. Furthermore, the spaces of operators $\{B(\varphi, \mu, \mu): \varphi \text{ in } H^\infty(D)\}$, $\{B(\varphi, \mu, \nu): \varphi \text{ in } H^\infty(D)\}$, $\{B(\varphi, \nu, \mu): \varphi \text{ in } H^\infty(D)\}$, and $\{B(\varphi, \nu, \nu): \varphi \text{ in } H^\infty(D)\}$ shall all be denoted $H^\infty(D)$, the space $\{B(\varphi, \mu, \nu): \varphi \text{ in } I_L(\mu, \nu)\}$ shall be denoted $I_L(\mu, \nu)$, and the space $\{B(\varphi, \nu, \mu): \varphi \text{ in } I_L(\nu, \mu)\}$ shall be denoted $I_L(\nu, \mu)$.

THEOREM 3. *The commutant of* $U_\mu \oplus U_\nu$ *is the matrix algebra*

$$\begin{bmatrix} H^\infty(D) & H^\infty(D) \\ H^\infty(D) & H^\infty(D) \end{bmatrix}$$

and the space $\mathscr{C}_L(U_\mu \oplus U_\nu)$ *is*

$$\begin{bmatrix} H^\infty(D) & I_L(\nu, \mu) \\ I_L(\mu, \nu) & H^\infty(D) \end{bmatrix}.$$

Furthermore, if

$$B = \begin{bmatrix} B(\varphi_{11}, \mu, \mu) & B(\varphi_{12}, \nu, \mu) \\ B(\varphi_{21}, \mu, \nu) & B(\varphi_{22}, \nu, \nu) \end{bmatrix}$$

is in $\mathscr{C}_L(U_\mu \oplus U_\nu)$, *then* $\Lambda(A) = B$, *where*

$$A = \begin{bmatrix} A(\varphi_{11}, \mu, \mu) & A(\chi_E \varphi_{12}, \nu, \mu) \\ A(\chi_E \varphi_{21}, \mu, \nu) & A(\varphi_{22}, \nu, \nu) \end{bmatrix}$$

and E *is an absolutely continuous support for* μ *and* ν.

Two measures μ and ν are mutually boundedly absolutely continuous and if $d\mu/d\nu$ is an invertible function in $L^\infty(\nu)$. The following corollary is an immediate consequence of Theorem 3 and Corollary 2.5.

COROLLARY 3.1. *The spaces* $\mathscr{C}_L(U_\mu \oplus U_\nu)$ *and* $\mathscr{C}(U_\mu \oplus U_\nu)$ *are equal if and only if the measures* μ *and* ν *are mutually boundedly absolutely continuous.*

Theorem 3 and Corollary 3.1 are now applied to several specific situations.

EXAMPLE A. This example is an elaboration of the example of Douglas [5]. Let δ be a unit point mass at the origin, let $\mu = m$, let $\nu = m + \delta$, and let $H_0^\infty(D)$ be the space of functions φ in $H^\infty(D)$ such that $\varphi(0) = 0$. An absolutely continuous support for μ and ν is any Borel subset E of the plane which does not contain 0 and thus $I_L(\mu, \nu) = H_0^\infty(D)$ and $I_L(\nu, \mu) = H^\infty(D)$. Thus,

$$\mathscr{C}_L(U_\mu \oplus U_\nu) = \begin{bmatrix} H^\infty(D) & H^\infty(D) \\ H_0^\infty(D) & H^\infty(D) \end{bmatrix}$$

and therefore $\mathscr{C}_L(U_\mu \oplus U_\nu) \neq \mathscr{C}(U_\mu \oplus U_\nu)$.

EXAMPLE B. This example shows that the map Λ can be isometric and not onto. Let σ be linear Lebesgue measure on $(-1, 1)$, let τ be linear Lebesgue measure on $(-i, i)$, let $\mu = m + \sigma$, and let $\nu = m + \tau$. In this case, $I_L(\mu, \nu) = \{0\} = I_L(\nu, \mu)$ and therefore

$$\mathscr{C}_L(U_\mu \oplus U_\nu) = \begin{bmatrix} H^\infty(D) & 0 \\ 0 & H^\infty(D) \end{bmatrix}.$$

For any μ and any φ in $H^\infty(D)$, it is easily verified that the restriction of $A(\varphi, \mu, \mu)$ to $H^2(\mu)$ is $B(\varphi, \mu, \mu)$ and that $\|A(\varphi, \mu, \mu)\| = \|\Phi\|$. Thus, by Lemma 2.1, $\|\Phi\| \leqslant \|B(\varphi, \mu, \mu)\| A(\varphi, \mu, \mu)\| = \|\Phi\|$ and therefore $\|B(\varphi, \mu, \mu)\| = \|A(\varphi, \mu, \mu)\|$. It follows that in this particular example Λ is isometric and it clearly is not onto.

EXAMPLE C. In this example, Λ is onto and not isometric. Let δ be a unit point mass at the origin, let $\mu = m + \delta$, and let $\nu = m + 2\delta$.

Since μ and ν are mutually boundedly absolutely continuous, it follows from Corollary 3.1 that Λ is onto. Since $h = \chi_T + \frac{1}{\sqrt{2}}\chi_{\{0\}}$, it follows from Theorem 1 that $\|\Lambda(1, \mu, \nu)\| = \sqrt{2}$. For $n = 1, 2, \ldots$, let e_n be the function $e_n(z) = z^n$. The space $H^2(\mu)$ has an orthonormal basis $\{1/\sqrt{2}, e_1, e_2, \ldots\}$, the space $H^2(\nu)$ has an orthonormal basis $\{1/\sqrt{3}, e_1, e_2, \ldots\}$, and with respect to these bases, the operator $B(1, \mu, \nu)$ is diagonal with entries along the diagonal $(\sqrt{3/2}, 1, 1, \ldots)$. Thus, $\|B(1, \mu, \nu)\| = \sqrt{3/2}$. Therefore, if A is the operator

$$A = \begin{bmatrix} 0 & 0 \\ A(1, \mu, \nu) & 0 \end{bmatrix},$$

then $\|A\| = \sqrt{2}$ and $\|\Lambda(A)\| = \sqrt{3/2}$.

EXAMPLE D. In this example, Λ is not bounded below. To produce such an example, it is sufficient to produce μ and ν with absolutely continuous support E and functions φ_n in $I_L(\mu, \nu)$ such that $\|\Phi_n\| \leqslant 2$ and $\|A(\chi_E\varphi_n, \mu, \nu)\|$ diverges to infinity. For given such μ, ν, and φ_n, set

$$A_n = \begin{bmatrix} 0 & 0 \\ A(\chi_E\varphi_n, \mu, \nu) & 0 \end{bmatrix}.$$

Then the sequence $\{\|A_n\|\}$ diverges to infinity and by Theorem 3 and Lemma 2.1 $\|\Lambda(A_n)\| = \|B(\varphi_n, \mu, \nu)\| \leqslant 2\|X_\mu\|\|X_\nu^{-1}\|$. These facts imply that Λ is not bounded below. To produce μ, ν, and φ_n, let τ be linear Lebesgue measure on the interval $(\frac{1}{2}, 1)$, let g be the function on $(\frac{1}{2}, 1)$ defined by the equation $g(x) = [-\log(1-x)]^{-2}$, let σ be the measure $d\sigma(x) = g(x)d\tau(x)$, let $\mu = m+\sigma$, let $\nu = m+\tau$, and let $\varphi_n(z) = (1-z)^{1/n}$ defined with branch cut $[1, \infty)$. Then clearly $\|\Phi_n\| \leqslant 2$. Since the measures μ and ν are mutually absolutely continuous, the entire plane is an absolutely continuous support for μ and ν. Thus, the function φ_n is in $I_L(\mu, \nu)$ if and only if there is a constant C such that $|\varphi_n(x)| \leqslant C\sqrt{g(x)}$ $d\tau$-a.e. Setting $y = 1/(1-x)$ and observing that both φ_n and g are continuous on $(\frac{1}{2}, 1)$, this inequality becomes $h^{-1/n}\log y \leqslant C$ for $y \geqslant 2$. Since $y^{-1/n}\log y$ converges to zero as y converges to ∞, the existence of the constant C is established and thus φ_n is in $\mathscr{I}_L(\mu, \nu)$. Applying Theorem 1 and using again the continuity of φ_n and g on $(\frac{1}{2}, 1)$, one obtains $\|A(\varphi_n, \mu, \nu)\| \geqslant \sup\{|\varphi_n(x)|g(x)^{-1/2}: \frac{1}{2} < x < 1\} = \sup\{y^{-1/n}\log y: y \geqslant 2\} \geqslant (e^n)^{-1/n}\log(e^n) = n/e$ and thus the sequence $\{\|A(\varphi_n, \mu, \nu)\|\}$ diverges to infinity.

4. Comments and problems. Let S be a subnormal operator on $\mathscr{H}$ with minimal normal extension N and let $\Lambda: \mathscr{C}(N, \mathscr{H}) \to \mathscr{C}(S)$ be defined as in the introductory paragraph of this paper. The examples in Section 3 of Λ failing to be onto or failing to be isometric occur when S is the direct sum of two non-zero subnormal operators. This suggests the following problem.

PROBLEM 1. If S is an irreducible subnormal operator, must Λ be isometric and onto?

For an operator A, the second commutant of A is the algebra of operators which commute with every operator in the commutant of A. Problem 2 inquires into the possibility of a lifting theorem for operators in the second commutant of a subnormal operator. This problem arose in a conversation between the author and C. Berger.

PROBLEM 2. If B is in the second commutant of S, must there be an operator A in $\mathscr{C}(N, \mathscr{K})$ such that $\Lambda(A) = B$ and $\|A\| = \|B\|$.

If $S = U_\mu \oplus U_\nu$ as in Section 3, then by using Theorem 3 the second commutant of S can be shown to be the space $\{B(\varphi, \mu, \mu) \oplus B(\varphi, \nu, \nu): \varphi$ in $H^\infty(D)\}$ and an affirmative answer to Problem 2 is obtained for this case.

Finally, the examples in Section 3 of the failure of commutant lifting are all examples involving pure subnormal operators (no normal part). However, if μ and ν are chosen as in Example A and if B is the operator $B(1, \mu, \nu)$ followed by the injection of $H^2(\nu)$ in $L^2(\nu)$, then B is an operator in $\mathscr{I}(U_\mu, W_\nu)$ which does not lift to an intertwining map in $\mathscr{I}(W_\mu, W_\nu)$. Thus, even though $\mathscr{C}_L(U_\mu) = \mathscr{C}(U_\mu)$ and W_ν is normal, the commutant of $U_\mu \oplus W_\nu$ does not lift to the commutant of $W_\mu \oplus W_\nu$.

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