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Some expansion theorems for the H-function

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Introduction. The H-function has been studied rather extensively in the recent years [3], [1], [7] and [4]. In the present paper an extension of the H-function is given which enables the author either to simplify or to generalize some already known results. Moreover, it is a counterpart of Meijer's results [6] for the G-function, while his earlier results have already been generalized by Ławrynowicz [4].

After extending the definition of the H-function in Section 1, the author quotes in Section 2 a theorem on analytic continuation for the H-function (due to Braaksma [1]) and adapts it to the case of the extended definition. In Section 3 some identities are established that are used in the continuation of this paper. Next, in Section 4 the author obtains some formulae for the successive derivatives of the H-function. Applying these formulae, he proves in Section 5 Theorems 1 and 2, which permit the expansion of $H_{p,q}^{m,n}(\eta w)$ into power series in η with the coefficients depending upon some $H_{p+1,q+1}^{m,n+1}(w)$ only. In Section 6 particular cases of Theorems 1 and 2 are discussed.

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1. An extension of the definition of the H-function.

DEFINITION 1 (see [1], pp. 239-240). Suppose that m, n, p, q are integers satisfying $0 \leqslant n \leqslant p, 1 \leqslant m \leqslant q$; α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$ are positive numbers and a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$ are complex numbers such that

$$(1.1) \qquad \alpha_j(b_h+\nu) \neq \beta_h(a_j-1-\lambda) \quad \text{for } \nu, \lambda = 0, 1, \ldots; \quad h = 1, \ldots, m;\ j = 1, \ldots, n.$$

Further, let C denote a contour which runs from $\infty - ik$ to $\infty + ik$ $(k > \operatorname{im}|b_j|/\beta_j,\ j = 1, \ldots, m)$, so that the points

$$(1.2) \qquad s = (b_j+\nu)/\beta_j \quad (j = 1, \ldots, m;\ \nu = 0, 1, \ldots)$$

and

(1.3) $$s = (a_j - 1 - \nu)/\alpha_j \quad (j = 1, \ldots, n;\ \nu = 0, 1, \ldots)$$

lie to the right and left of C, respectively. Then we assume

(1.4) $$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s) \prod_{j=1}^{m} \Gamma(b_j - \beta_j s)}{\prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)} z^s \, ds,$$

where

(1.5) $$z \neq 0 \quad \text{for} \quad \mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0,$$

and

(1.6) $$0 < |z| < \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{j=1}^{q} \beta_j^{\beta_j} = 1/\beta \quad \text{for} \quad \mu = 0.$$

Remark 1. The usual notation for

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right)$$

is

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right).$$

Remark 2 (see [1], pp. 278-279). The H-function is analytic in $\{z: z \neq 0\}$ for $\mu > 0$, and in $\{z: 0 < |z| < 1/\beta\}$ for $\mu = 0$. It does not depend on the choice of C and is in general multiple-valued but one-valued on the Riemann surface of $\log z$. Moreover, if the assumptions of Definition 1 are fulfilled, $H(z)$ may be written as the sum of residues of

(1.7) $$h(s) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s) \prod_{j=1}^{m} \Gamma(b_j - \beta_j s)}{\prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)} z^s$$

at the points (1.3).

If, in addition,

$$(b_j + \lambda)/\beta_j \neq (b_h + \nu)/\beta_h \quad \text{for } j \neq h;\ j, h = 1, \ldots, m;\ \nu, \lambda = 0, 1, \ldots,$$

then

$$(1.8)\quad H(z)$$

$$=\sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\big(b_j-\beta_j(b_h+\nu)/\beta_h\big)\prod_{j=1}^{n}\Gamma\big(1-a_j+\alpha_j(b_h+\nu)/\beta_h\big)}{\prod_{j=m+1}^{q}\Gamma\big(1-b_j+\beta_j(b_h+\nu)/\beta_h\big)\prod_{j=n+1}^{p}\Gamma\big(a_j-\alpha_j(b_h+\nu)/\beta_h\big)}\times$$

$$\times\frac{(-1)^{\nu}z^{(b_h+\nu)/\beta_h}}{\nu!\,\beta_h}.$$

In some cases in the definition of $H(z)$ restriction (1.1) may be omitted for some a_j, α_j, b_h, β_h. For instance, if r is a non-negative integer, δ is positive, and ϱ is a complex number, then the function

$$(1.9)\qquad \frac{\Gamma(1-\varrho+\delta s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\Gamma(1-\varrho-r+\delta s)}z^s$$

may have poles at points (1.2) and (1.3). Further, the factor $\Gamma(1-\varrho-$ $-r+\delta s)$ in the numerator does not yield the points

$$(1.10)\qquad (\varrho-1-\nu)/\delta\quad(\nu=0,1,\ldots)$$

as poles of expresion (1.9). Therefore, if

$$(1.11)\quad (\varrho-1-\nu)/\delta\neq(a_j-1-\lambda)/\alpha_j\quad(j=1,\ldots,n;\ \nu,\lambda=0,1,\ldots),$$

then in defining the function

$$H_{p+1,q+1}^{m,n+1}\left(z\,\middle|\,\begin{matrix}\varrho,\,\delta;\,a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots,\,b_q,\,\beta_q;\,\varrho+r,\,\delta\end{matrix}\right)\ (0\leqslant n\leqslant p,\,1\leqslant m\leqslant q,\,r=0,1,\ldots)$$

we can omit the restriction $(\varrho-1-\lambda)/\delta\neq(b_h+\nu)/\beta_h$ for $\nu,\lambda=0,1,\ldots$ $h=1,\ldots,m$, and so the hypotesis that points (1.10) lie to the left of the contour C is superfluous. Consequently, we arrive at the following extension of Definition 1.

DEFINITION 2. Suppose that the numbers m, n, p, q, α_j $(j=1,\ldots,p)$, β_j $(j=1,\ldots,q)$, a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$ and the contour C fulfil the same conditions as in Definition 1. Further, let r be a non-negative integer, δ – a positive number, and ϱ – a complex number which satisfies conditions (1.11). Then we assume

$$(1.12)\qquad H^{m,n+1}_{p+1,q+1}\left(z\left|\begin{matrix}\varrho,\delta;a_1,\alpha_1;\ldots;a_p,\alpha_p\\ b_1,\beta_1;\ldots;b_q,\beta_q;\varrho+r,\delta\end{matrix}\right.\right)$$

$$=\frac{1}{2\pi i}\int_C\frac{\Gamma(1-\varrho+\delta s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\Gamma(1-\varrho-r+\delta s)}z^s\,ds$$

provided that one of cases (1.5) and (1.6) holds.

2. Analytic continuation. Before quoting the theorem on analytic continuation we need a proposition and some notation.

PROPOSITION (see [1], p. 263). Suppose that the numbers m, n, p, q, α_j $(j=1,\ldots,p)$, β_j $(j=1,\ldots,q)$, a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$ fulfil the same conditions as in Definition 1. Further, let

$$l=1+\max\left\{\left|\operatorname{im}\frac{a_j}{\alpha_j}\right|\ (j=1,\ldots,p),\ \left|\operatorname{im}\frac{b_j}{\beta_j}\right|\ (j=1,\ldots,q),\ \left|\operatorname{im}\frac{a}{\mu}\right|\right\}\quad\text{for } \mu>0,$$

$$l=1+\max\left\{\left|\operatorname{im}\frac{a_j}{\alpha_j}\right|\ (j=1,\ldots,p),\ \left|\operatorname{im}\frac{b_j}{\beta_j}\right|\ (j=1,\ldots,q)\right\}\quad\text{for } \mu=0,$$

where

$$a=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j+\tfrac{1}{2}(q-p+1),$$

$$(2.1)\qquad h_0(s)=\prod_{j=1}^{p}\Gamma(1-a_j+\alpha_j s)\Big/\prod_{j=1}^{q}\Gamma(1-b_j+\beta_j s)$$

for

$$s\neq(a_j-1-\nu)/\alpha_j\qquad(j=1,\ldots,p;\ \nu=0,1,\ldots),$$

$$(2.2)\qquad h_1(s)=\pi^{m+n-p}\prod_{j=n+1}^{p}\sin\pi(a_j-\alpha_j s)\Big/\prod_{j=1}^{m}\sin\pi(b_j-\beta_j s)$$

for

$$s=(b_j+\nu)/\beta_j\qquad(j=1,\ldots,m;\ \nu=0,\pm1,\pm2,\ldots),$$

and

$$\delta_0=\Big(\sum_{j=1}^{m}\beta_j-\sum_{j=n+1}^{p}\alpha_j\Big)\pi.$$

Then

$$h_1(s)=c_0e^{i\gamma_0 s}\prod_{j=n+1}^{p}\left(1-e^{-2\pi i(\alpha_j s-a_j)}\right)\prod_{j=1}^{m}\sum_{\nu=0}^{\infty}e^{2\pi i(\beta_j s-b_j)}\qquad\text{for im } s\geqslant l$$

and

$$h_1(s) = d_0 e^{-i\gamma_0 s} \prod_{j=n+1}^{p} (1 - e^{-2\pi i(a_j s - a_j)}) \prod_{j=1}^{m} \sum_{\nu=0}^{\infty} e^{-2\nu\pi i(\beta_j s - b_j)} \quad \text{for } \operatorname{im} s \leqslant -l,$$

where

$$\gamma_0 = \Big(\sum_{j=1}^{m} \beta_j - \sum_{j=n+1}^{p} a_j\Big)\pi = \delta_0, \tag{2.3}$$

$$c_0 = (2\pi i)^{m+n-p} \exp\Big(\sum_{j=n+1}^{p} a_j - \sum_{j=1}^{m} b_j\Big)\pi i, \tag{2.4}$$

$$d_0 = (-2\pi i)^{m+n-p} \exp\Big(\sum_{j=1}^{m} b_j - \sum_{j=n+1}^{p} a_j\Big)\pi i. \tag{2.5}$$

These formulae may be written as

$$h_1(s) = \sum_{j=0}^{\infty} c_j e^{i\gamma_j s} \quad \text{for } \operatorname{im} s \geqslant l$$

and

$$h_1(s) = \sum_{j=0}^{\infty} d_j e^{-i\gamma_j s} \quad \text{for } \operatorname{im} s \leqslant -l,$$

where the corresponding series are absolutely convergent. Here $\{\gamma_j\}$ $(j = 0, 1, \ldots)$ is an increasing sequence of real numbers independent of s; c_j and d_j $(j = 1, 2, \ldots)$ are complex numbers independent of s; γ_0, c_0 and d_0 are given by (2.3), (2.4) and (2.5); the numbers $(\gamma_j - \gamma_0)/2\pi$ $(j = 1, 2, \ldots)$ are linear combinations of $a_{n+1}, \ldots, a_p, \beta_1, \ldots, \beta_m$ with coefficients which are non-negative integers.

Now we introduce some notation (see [1], pp. 265 and 270). Let $\{\delta_j\}$ $(j = 0, \pm 1, \pm 2, \ldots)$ denote the monotonic increasing sequence which arises if we write down the set of numbers γ_g and $-\gamma_h$ $(g, h = 0, 1, \ldots)$ in the order of increasing magnitude, so that if there happen to be two equal numbers in this set, we only write down this number once, while further $\delta_0 = \gamma_0$. Thus $\delta_j < \delta_h$ for $j < h$.

If r is an arbitrary integer, we may distinguish three different cases for δ_r:

(a) there exists a non-negative integer g such that $\delta_r = \gamma_g$ while $\delta_r \neq -\gamma_j$ for $j = 0, 1, \ldots$;

(b) there exists a non-negative integer h such that $\delta_r = -\gamma_h$ while $\delta_r \neq \gamma_j$ for $j = 0, 1, \ldots$;

(c) there exist two non-negative integers g and h such that $\delta_r = \gamma_g = -\gamma_h$.

With the preceding notation we define the integer $\varkappa$ by $\delta_\varkappa = -\gamma_0 = -\delta_0$. Further, if r is an arbitrary integer, we define: $C_r = c_g$, $D_r = 0$ in case (a); $C_r = 0$, $D_r = -d_h$ in case (b), and $C_r = c_g$, $D_r = -d_h$ in case (c).

By $Q(z)$ and $P(z)$ we denote the formal series of residues of $h(s)$ at points (1.3), and of $h_0(s)z^s$ at the points

$$s = (a_j - 1 - \nu)/a_j \quad (j = 1, \ldots, p;\ \nu = 0, 1, \ldots), \tag{2.6}$$

respectively. Here $h_0(s)$ and $h(s)$ are defined by (2.1) and (1.7).

Now we can quote the mentioned above theorem on analytic continuation of $H(z)$ (in the sense of Definition 1).

THEOREM ON ANALYTIC CONTINUATION OF $H(z)$ (see [1], p. 280). *Suppose $\mu = 0$, and r is an integer. Let D be a contour in the complex s-plane which runs from $s = -\infty i + \sigma$ to $s = \infty i + \sigma'$ (σ being an arbitrary real number), so that points* (2.6) *lie to the left of D and those points* (1.2) *which do not occur among points* (2.6) *lie to the right of D. Further, let $V(z)$ be the sum of the residues of $h(s)z^s$ at the points, where* (1.2) *and* (2.6) *hold simultaneously, $h_0(s)$ and $h_1(s)$ being defined in* (2.1) *and* (2.2), *respectively.*

Then $H(z)$ can be continued analytically by

$$H(z) = \frac{1}{2\pi i} \int_D h_0(s)\Big\{h_1(s) + \sum_{j=r}^{\varkappa} D_j e^{i\delta_j s} - \sum_{j=0}^{r-1} C_j e^{i\delta_j s}\Big\} z^s ds - V(z)$$

into the sector

$$-\delta_r < \arg z < -\delta_{r-1}. \tag{2.7}$$

Here $V(z) = 0$ if

$$a_h(b_j + \nu) \neq \beta_j(a_h - 1 - \lambda) \quad \textit{for } j = 1, \ldots, m;\ h = n+1, \ldots, p;$$
$$\nu, \lambda = 0, 1, \ldots,$$

Furthermore, the function $H(z)$ can be continued analytically from sector (2.7) *into the domain $|z| > 1/\beta$ by*

$$H(z) = Q(z) + \sum_{j=r}^{\varkappa} D_j P(ze^{i\delta_j}) - \sum_{j=0}^{r-1} C_j P(ze^{i\delta_j}).$$

Here the formal series for $Q(z)$ and $P(z)$ converge for $|z| > 1/\beta$.

3. Some identities. Now we establish some identities needed in the continuation of this paper.

LEMMA 1. *Suppose that the numbers m, n, p, q, a_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$, a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$, r, δ and ϱ fulfil the* *same conditions as in Definitions* 1 *and* 2. *Further, let* τ *be an arbitrary complex number, and* γ — *a positive number. We assume also that one of cases* (1.5) *and* (1.6) *holds. Then*

$$(3.1)\qquad H^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}\varrho,\,\delta;\,a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q;\,\varrho,\,\delta\end{matrix}\right) = H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right),$$

$$(3.2)\qquad H^{m+1,n}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p;\,\varrho,\,\delta\\ \varrho,\,\delta;\,b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right) = H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right),$$

$$(3.3)\qquad z^{\tau}H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right) = H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1+\alpha_1\tau,\,\alpha_1;\,\ldots;\,a_p+\alpha_p\tau,\alpha_p\\ b_1+\beta_1\tau,\,\beta_1;\,\ldots;\,b_q+\beta_q\tau,\,\beta_q\end{matrix}\right),$$

$$(3.4)\qquad (1/\gamma)H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right) = H^{m,n}_{p,q}\left(z^{\gamma}\,\middle|\,\begin{matrix}a_1,\,\alpha_1\gamma;\,\ldots;\,a_p,\,\alpha_p\gamma\\ b_1,\,\beta_1\gamma;\,\ldots;\,b_q,\,\beta_q\gamma\end{matrix}\right),$$

$$(3.5)\qquad H^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}0,\,\delta;\,a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q;\,r,\,\delta\end{matrix}\right)$$
$$= (-1)^rH^{m+1,n}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p;\,0,\,\delta\\ r,\,\delta;\,b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right)\qquad (p\leqslant q),$$

$$(3.6)\qquad H^{m+1,n}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p;\,1-r,\,\delta\\ 1,\,\delta;\,b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right)$$
$$= (-1)^rH^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}1-r,\,\delta;\,a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q;\,1,\,\delta\end{matrix}\right)\qquad (p\leqslant q),$$

where the branches of the functions are suitably chosen.

Lemma 1 follows directly from Definitions 1 and 2.

Remark 3. By the theorem on analytic continuation the restriction that $|z| < 1/\beta$ for $\mu = 0$ in formulae (3.1)-(3.4) can be removed, since the functions standing on the right-hand side of these formulae can be continued analytically outside the circle $|z| = 1/\beta$; thereby the functions standing on the left-hand side of formulae (3.1)-(3.4) can be continued analytically outside the circle $|z| = 1/\beta$ as well. In an analogous way we deduce that formulae (3.5) and (3.6) are also valid for $|z| \geqslant 1/\beta$, but we have to apply Theorem 1 for the function $H^{m',n'}_{p',q'}(z)$, where $m' = m$, $n' = n+1$, $p' = p+1$ and $q' = q+1$.

4. Formulae for successive derivatives of the H-function.

LEMMA 2. *Suppose that the numbers* m, n, p, q, α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$, a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$, r *and* δ *fulfil the same conditions as in Definitions* 1 *and* 2. *Then*

$$(4.1)\qquad w^r\frac{d^r}{dw^r}\left\{H^{m,n}_{p,q}\left(w^{\delta}\,\middle|\,\begin{matrix}a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q\end{matrix}\right)\right\}$$
$$= H^{m,n+1}_{p+1,q+1}\left(w^{\delta}\,\middle|\,\begin{matrix}0,\,\delta;\,a_1,\,\alpha_1;\,\ldots;\,a_p,\,\alpha_p\\ b_1,\,\beta_1;\,\ldots;\,b_q,\,\beta_q;\,r,\,\delta\end{matrix}\right).$$

Proof. We have

$$(4.2)\qquad w^r\frac{d^r(w^{\delta s})}{dw^r}=\delta s(\delta s-1)\cdot\ldots\cdot(\delta s-r+1)w^{\delta s}=\frac{\Gamma(1+\delta s)w^{\delta s}}{\Gamma(1-r+\delta s)}.$$

Hence, according to Definition 1, we get

$$w^r\frac{d^r}{dw^r}H_{p,q}^{m,n}\left(w^\delta\left|\begin{matrix}a_1,\alpha_1;\ldots;a_p,\alpha_p\\ b_1,\beta_1;\ldots;b_q,\beta_q\end{matrix}\right.\right)$$
$$=\frac{1}{2\pi i}\int_C\frac{\Gamma(1+\delta s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)w^{\delta s}}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\Gamma(1-r+\delta s)}ds,$$

since differentiation under the sign of integration is allowed (see e.g. [2], p. 110). Consequently, by (1.12), we obtain (4.1).

Remark 4. The condition $|w^\delta|<1/\beta$ for $\mu=0$ can be removed (cf. Remark 3).

Lemma 3. *Suppose that the numbers* m,n,p,q,α_j $(j=1,\ldots,p)$, β_j $(j=1,\ldots,q)$, a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$, r *and* δ *fulfil the same conditions as in Definitions* 1 *and* 2. *Moreover, suppose that* $w\neq 0,\infty$ *for* $\mu>0$ *and* $0<1/|w^\delta|<1/\beta$ *for* $\mu=0$. *Then*

$$w^r\frac{d^r}{dw^r}H_{p,q}^{m,n}\left(1/w^\delta\left|\begin{matrix}a_1,\alpha_1;\ldots;a_p,\alpha_p\\ b_1,\beta_1;\ldots;b_q,\beta_q\end{matrix}\right.\right)$$
$$=(-1)^rH_{p+1,q+1}^{m,n+1}\left(1/w^\delta\left|\begin{matrix}1-r,\delta;a_1,\alpha_1;\ldots;a_p,\alpha_p\\ b_1,\beta_1;\ldots;b_q,\beta_q;1,\delta\end{matrix}\right.\right).$$

Proof. This proof resembles the proof of Lemma 2, but instead of (4.2) we apply the identity

$$w^r\frac{d^r(w^{-\delta s})}{dw^r}=\frac{(-1)^r\Gamma(r+\delta s)w^{-\delta s}}{\Gamma(\delta s)}.$$

Remark 5. The condition $0<1/|w^\delta|<1/\beta$ for $\mu=0$ can be removed (cf. Remark 3).

5. Expansion theorems.

Theorem 1. *Suppose that the numbers* m,n,p,q,α_j $(j=1,\ldots,p)$, β_j $(j=1,\ldots,q)$, a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$ *and* r *fulfil the same conditions as in Definitions* 1 *and* 2. *Then one of the following cases arises*:

(i) *If* $\mu>0$ *and*

$$(5.1)\qquad w\neq 0,\quad |\eta-1|<1,$$

then

$$(5.2)\qquad H_{p,q}^{m,n}\left(\eta w\middle| \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{(\eta-1)^r}{r!} H_{p+1,q+1}^{m,n+1}\left(w\middle| \begin{matrix} 0, 1; a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q; r, 1 \end{matrix}\right).$$

In case (i) *the values of the many-valued functions* $H_{p,q}^{m,n}(\eta w)$ *and* $H_{p+1,q+1}^{m,n+1}(w)$ *are connected in the following way: if the value of* $\arg w$ *is chosen, then the value of* $\arg \eta w$ *is determined by*

$$(5.3)\qquad \arg\eta w = \arg\eta + \arg w, \qquad -\tfrac{1}{2}\pi < \arg\eta < \tfrac{1}{2}\pi.$$

(ii) *If* $\mu = 0$ *and*

$$(5.4)\qquad w \neq 0, \qquad -\delta_r < \arg w < -\delta_{r-1}, \qquad |\eta - 1| < k,$$

where δ_{r-1}, δ_r *are the same as in Section* 2, *and*

$k = \sin(\arg w + \delta_r)$ *for* $-\delta_r < \arg w < -\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$ *and* $|w| > \beta^{-1}/\cos(\arg w + \delta_r)$;

$k = |1 - \beta^{-1} e^{-i\delta_r}/w|$ *for* $-\delta_r < \arg w < -\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$ *and* $\frac{1}{2}\beta^{-1}/\cos(\arg w + \delta_r) \leqslant |w| \leqslant \beta^{-1}/\cos(\arg w + \delta_r)$;

$k = 1$:

1° *for* $-\delta_r < \arg w < -\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$ *and* $0 < |w| < \frac{1}{2}\beta^{-1}/\cos(\arg w + \delta_r)$,

2° *for* $-\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) \leqslant \arg w \leqslant -\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$,

3° *for* $-\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) < \arg w < -\delta_{r-1}$ *and* $0 < |w| < \frac{1}{2}\beta^{-1}/\cos(\arg w + \delta_{r-1})$;

$k = |1 - \beta^{-1} e^{-i\delta_{r-1}}/w|$ *for* $-\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) < \arg w < -\delta_{r-1}$ *and* $\frac{1}{2}\beta^{-1}/\cos(\arg w + \delta_{r-1}) \leqslant |w| \leqslant \beta^{-1}/\cos(\arg w + \delta_{r-1})$;

$k = \sin(\arg w + \delta_{r-1})$ *for* $-\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) < \arg w < -\delta_{r-1}$ *and* $|w| > \beta^{-1}/\cos(\arg w + \delta_{r-1})$,

then formula (5.2) *is also valid.*

In case (ii) *the values of the the many-valued functions* $H_{p,q}^{m,n}(\eta w)$ *and* $H_{p+1,q+1}^{m,n+1}(w)$ *are connected in the following way: we consider the sector* (2.7) *in the z-plane; now, if we take an arbitrary w from this sector, then the values of* $H_{p,q}^{m,n}(\frac{1}{2}\beta^{-1}w/|w|)$ *and* $H_{p+1,q+1}^{m,n+1}(\frac{1}{2}\beta^{-1}w/|w|)$ *are determined uniquely by means of* (1.4). *The values of the functions* $H_{p,q}^{m,n}(w)$ *and* $H_{p+1,q+1}^{m,n+1}(w)$ *are derived from those of* $H_{p,q}^{m,n}(\frac{1}{2}\beta^{-1}w/|w|)$ *and* $H_{p+1,q+1}^{m,n+1}(\frac{1}{2}\beta^{-1}w/|w|)$ *by analytic continuation into the sector* (5.4) *along the straight line which joins* $\frac{1}{2}\beta^{-1}w/|w|$ *to w.*

(iii) *If* $\mu > 0$, $m = 1$, *and* b_1, β_1 *are real numbers such that* $(b_1+\nu)/\beta_1$ $(\nu = 0, 1, \ldots)$ *are non-negative integers, then formula* (5.2) *is valid for all values of* w *and* η.

(iv) *If* $\mu = 0$, $m = 1$, *and* b_1, β_1 *satisfy the same conditions as in* (iii), *then formula* (5.2) *is also valid provided that*

$$|\eta - 1| < k_1,$$

where

$k_1 = \sin(\arg w + \delta_r)$ *for* $-\delta_r < \arg w < -\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$ *and* $|w| > \beta^{-1}/\cos(\arg w + \delta_r)$;

$k_1 = |1 - \beta^{-1} e^{-i\delta_r}/w|$:

1° *for* $-\delta_r < \arg w < -\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$ *and* $0 < |w| \leqslant \beta^{-1}/\cos(\arg w + \delta_r)$,

2° *for* $-\delta_r + \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) \leqslant \arg w \leqslant \frac{1}{2}(\delta_r - \delta_{r-1})$;

$k_1 = |1 - \beta^{-1} e^{-i\delta_{r-1}}/w|$:

1° *for* $\frac{1}{2}(\delta_r - \delta_{r-1}) < \arg w \leqslant -\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1})$,

2° *for* $-\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) < \arg w < -\delta_{r-1}$ *and* $0 < |w| \leqslant \beta^{-1}/\cos(\arg w + \delta_{r-1})$;

$k_1 = \sin(\arg w + \delta_{r-1})$ *for* $-\delta_{r-1} - \frac{1}{2}\min(\pi, \delta_r - \delta_{r-1}) < \arg w < -\delta_{r-1}$ *and* $|w| > \beta^{-1}/\cos(\arg w + \delta_{r-1})$.

Proof of (i). It is well known that the H-function has, in general, branch-points at 0 and ∞. In our case these points are the only singularities of H. We now take an arbitrary number $w \neq 0$ and make a choice for $\arg w$ (not necessarily the principial value). Then the function $H_{p,q}^{m,n}(z)$ is analytic in the open disc D with centre at w and radius $|w|$. Moreover, the function is one-valued within D provided the value of $\arg z$ is determined uniquely by an appropriate agreement, e.g. by

$$-\tfrac{1}{2}\pi < \arg z - \arg w < \tfrac{1}{2}\pi. \tag{5.5}$$

Now, the function $H_{p,q}^{m,n}(z)$ can be expanded in the Taylor series in D:

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{(z-w)^r}{r!} \frac{d^r}{dw^r} H_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right). \tag{5.6}$$

Substituting ηw for z in (5.6) and applying (4.1), we obtain (5.2), and the restriction (5.5) takes form (5.1).

Proof of (ii). As has been mentioned above, the only singularities of the H-function are 0 and ∞, and so we take an arbitrary number w

such that

$$w \neq 0, \qquad -\delta_r < \arg w < -\delta_{r-1}$$

(cf. the theorem on analytic continuation) and choose one of the values of $H_{p,q}^{m,n}(w)$. Next, we consider the open disc D with centre at w and radius R in the z-plane, where $R = k|w|$. The function $H_{p,q}^{m,n}(w)$ is one-valued in D provided that the value of $H_{p,q}^{m,n}(z)$ is derived from the value (chosen earlier) of $H_{p,q}^{m,n}(w)$ by analytic continuation along a path which starts at w and lies entirely within D. Thus for $\mu = 0$ and $|z-w| < R$ we also get the same expansion (5.6). Substituting ηw for z in (5.6) and applying (4.1) we obtain (5.2) again. Here $|\eta-1| < k$, where k satisfies the hypotheses of (ii).

Proof of (iii). In this case, by (1.8), the function $H_{p,q}^{1,n}(z)$ is an integral function of z, and so formula (5.6) holds for all values of z and w. Substituting ηw for z in (5.6) and applying (4.1), we obtain (5.2) again.

Proof of (iv). In this case the proof resembles that of (ii).

THEOREM 2. *Suppose that the numbers* m, n, p, q, α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$, a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$ *and* r *fulfil the same conditions as in Definitions* 1 *and* 2. *Then one of the following cases arises*:

(v) *If* $\mu > 0$ *and* $w \neq 0$, $\operatorname{re}\eta > \frac{1}{2}$, *then*

$$(5.7) \qquad H_{p,q}^{m,n}\left(\eta w \,\middle|\, \begin{matrix} a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{1}{r!}\left(1-\frac{1}{\eta}\right)^r H_{p+1,q+1}^{m,n+1}\left(w \,\middle|\, \begin{matrix} 1-r, 1; a_1, \alpha_1; \ldots; a_p, \alpha_p \\ b_1, \beta_1; \ldots; b_q, \beta_q; 1, 1 \end{matrix}\right).$$

(vi) *If* $w \neq 0$ *and* $-\delta_r < \arg w < -\delta_{r-1}$, $|1/\eta - 1| < k$, *where* k *is the same as in* (ii), *then formula* (5.7) *is also valid.*

In the cases (v) and (vi) the values of the many-valued functions $H_{p,q}^{m,n}(\eta w)$ and $H_{p+1,q+1}^{m,n+1}(w)$ are connected in the same way as in the corresponding cases of Theorem 1.

Proof of (v). If $\mu > 0$, the point $z = 0$ is the only finite singularity of $H(z)$. We now take an arbitrary number $\omega \neq 0$. Then the function $H_{p,q}^{m,n}(1/\zeta)$ is analytic in the open disc D with centre at ω and radius $|\omega|$. Moreover, this function is one-valued in D provided that we make an appropriate choice for $\arg(1/\zeta)$; we may do it in the following way: first we choose an arbitrary value of $\arg(1/\omega)$, and then we determine uniquely, for every ζ within D, the value of $\arg(1/\zeta)$ by

$$(5.8) \qquad -\tfrac{1}{2}\pi < \arg(1/\zeta) - \arg(1/\omega) < \tfrac{1}{2}\pi.$$

Now, the function $H_{p,q}^{m,n}(1/\zeta)$ can be expanded in the Taylor series in D:

$$H_{p,q}^{m,n}\left(\frac{1}{\zeta}\middle|\begin{matrix}a_1, \alpha_1; \ldots, a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_q, \beta_q\end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{(\zeta-\omega)^r}{r!} \frac{d^r}{d\omega^r}\left\{H_{p,q}^{m,n}\left(\frac{1}{\omega}\middle|\begin{matrix}a_1, \alpha_1; \ldots; a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_q, \beta_q\end{matrix}\right)\right\}.$$

Applying (4.3) we obtain

$$(5.9)\quad H_{p,q}^{m,n}\left(\frac{1}{\zeta}\middle|\begin{matrix}a_1, \alpha_1; \ldots; a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_q, \beta_q\end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!}\left(1-\frac{\zeta}{\omega}\right)^r H_{p+1,q+1}^{m,n+1}\left(\frac{1}{\omega}\middle|\begin{matrix}1-r, 1; a_1, \alpha_1; \ldots; a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_q, \beta_q; 1, 1\end{matrix}\right).$$

Substituting $1/w$ for ω and $1/\eta w$ for ζ in (5.9) we obtain (5.7), and the condition $|\zeta-\omega| < |\omega|$ takes the form $|1/\eta-1| < 1$, that is $\mathrm{re}\,\eta > \frac{1}{2}$, while the restriction (5.8) is to be replaced by (5.3).

Proof of (vi). In this case the proof resembles that of (ii). The function $H_{p,q}^{m,n}(1/\zeta)$ is analytic in the domain $-\delta_r < \arg(1/\zeta) < -\delta_{r-1}$. Hence, if $\omega \neq 0$ and $-\delta_r < \arg(1/\omega) < -\delta_{r-1}$, then the open disc D with centre at ω and radius $\hat{k}$, where $\hat{k}$ is a number satisfying the conditions assumed for k in (ii) with w replaced by ω, $-\delta_r$ by δ_{r-1}, $-\delta_{r-1}$ by δ_r and β^{-1} by β, lies entirely within the above-mentioned sector, and the function $H_{p,q}^{m,n}(1/\zeta)$ is analytic in D. The rest of the proof is similar to that of (v).

6. Particular cases. 1. Suppose that $m < q$ and put $b_q = 0$, $\beta_q = 1$ in Theorem 1. Then, by the symmetry of the function $H_{p+1,q+1}^{m,n+1}(z)$ with respect to the parameters $b_{m+1}, \beta_{m+1}; \ldots; b_{q+1}, \beta_{q+1}$, and by (3.1), formula (5.2) implies

$$H_{p,q}^{m,n}\left(\eta w\middle|\begin{matrix}a_1, \alpha_1; \ldots; a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_{q-1}, \beta_{q-1}; 0, 1\end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{(\eta-1)^r}{r!} H_{p,q}^{m,n}\left(w\middle|\begin{matrix}a_1, \alpha_1; \ldots; a_p, \alpha_p\\ b_1, \beta_1; \ldots; b_{q-1}, \beta_{q-1}; r, 1\end{matrix}\right).$$

Multiplying through by w^{b_q}/β_q we reduce this expansion by means of (3.3) and (3.4) to

$$\eta^{-b_q} H_{p,q}^{m,n}\left(\eta^{\beta_q} w^{\beta_q}\middle|\begin{matrix}a_1+\alpha_1 b_q, \alpha_1\beta_q; \ldots; a_p+\alpha_p b_q, \alpha_p\beta_q\\ b_1+\beta_1 b_q, \beta_1\beta_q; \ldots; b_{q-1}+\beta_{q-1}b_q, \beta_{q-1}\beta_q; b_q, \beta_q\end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{(\eta-1)^r}{r!} H_{p,q}^{m,n}\left(w^{\beta_q}\middle|\begin{matrix}a_1+\alpha_1 b_q, \alpha_1\beta_q; \ldots; a_p+\alpha_p b_q, \alpha_p\beta_q\\ b_1+\beta_1 b_q, \beta_1\beta_q; \ldots; b_{q-1}+\beta_{q-1}b_q, \beta_{q-1}\beta_q; r+b_q, \beta_q\end{matrix}\right).$$

Now, putting $w^{\beta_q} = z$, $\eta^{\beta_q} = \lambda$, $a_j + \alpha_j b_q = c_j$ $(j = 1, \dots, p)$, $\alpha_j \beta_q = \gamma^j$ $(j = 1, \dots, p)$, $b_j + \beta_j b_q = d_j$ $(j = 1, \dots, q-1)$, $\beta_j b_q = \delta_j$ $(j = 1, \dots, q-1)$, $b_q = d_q$, $\beta_q = \delta_q$, we obtain

$$H_{p,q}^{m,n}\left(\lambda z\,\middle|\,\begin{matrix} c_1, \gamma_1; \dots; c_p, \gamma_p \\ d_1, \delta_1; \dots; d_q, \delta_q \end{matrix}\right)$$
$$= \lambda^{d_q/\delta_q} \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda^{1/\delta_q} - 1)^r H_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} c_1, \gamma_1; \dots; c_p, \gamma_p \\ d_1, \delta_1; \dots; d_{q-1}, \delta_{q-1}; r + d_q, \delta_q \end{matrix}\right).$$

2. By (3.5), formula (5.2), under the hypotheses of Theorem 1, also implies

$$H_{p,q}^{m,n}\left(\eta w\,\middle|\,\begin{matrix} a_1, \alpha_1; \dots; a_p, \alpha_p \\ b_1, \beta_1; \dots; b_q, \beta_q \end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{(1-\eta)^r}{r!} H_{p+1,q+1}^{m+1,n}\left(w\,\middle|\,\begin{matrix} a_1, \alpha_1; \dots; a_p, \alpha_p; 0, 1 \\ r, 1; b_1, \beta_1; \dots; b_q, \beta_q \end{matrix}\right).$$

Putting $b_1 = 0$, $\beta_1 = 1$, and using (3.2), we obtain

$$H_{p,q}^{m,n}\left(\eta w\,\middle|\,\begin{matrix} a_1, \alpha_1; \dots; a_p, \alpha_p \\ 0, 1; b_1, \beta_1; \dots; b_q, \beta_q \end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{(1-\eta)^r}{r!} H_{p,q}^{m,n}\left(w\,\middle|\,\begin{matrix} a_1, \alpha_1; \dots; a_p, \alpha_p \\ r, 1; b_2, \beta_2; \dots; b_q, \beta_q \end{matrix}\right).$$

Multiplying through by w^{b_1}/β_1, we reduce this expansion by means of (3.3) and (3.4) to

$$H_{p,q}^{m,n}\left(\lambda z\,\middle|\,\begin{matrix} c_1, \gamma_1; \dots; c_p, \gamma_p \\ d_1, \delta_1; \dots; d_q, \delta_q \end{matrix}\right)$$
$$= \lambda^{d_1/\delta_1} \sum_{r=0}^{\infty} \frac{1}{r!} (1 - \lambda^{1/\delta_1})^r H_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} c_1, \gamma_1; \dots; c_p, \gamma_p \\ r + d_1, \delta_1; d_2, \delta_2; \dots; d_q, \delta_q \end{matrix}\right),$$

where $z = w^{\beta_1}$, $\lambda = \eta^{\beta_1}$, $c_j = a_j + \alpha_j b_1$ $(j = 1, \dots, p)$, $\gamma_j = \alpha_j \beta_1$ $(j = 1, \dots, p)$, $d_1 = b_1$, $\delta_1 = \beta_1$, $d_j = b_j + \beta_j b_1$ $(j = 2, \dots, q)$, $\delta_j = \beta_j \beta_1$ $(j = 2, \dots, q)$.

3. Now, we put $a_1 = 1$ and $\alpha_1 = 1$ in Theorem 2. Then, by (3.1), formula (5.7) implies

$$H_{p,q}^{m,n}\left(\eta w\,\middle|\,\begin{matrix} 1, 1; a_2, \alpha_2; \dots; a_p, \alpha_p \\ b_1, \beta_1; \dots; b_q, \beta_q \end{matrix}\right)$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!} \left(1 - \frac{1}{\eta}\right)^r H_{p,q}^{m,n}\left(w\,\middle|\,\begin{matrix} 1-r, 1; a_2, \alpha_2; \dots; a_p, \alpha_p \\ b_1, \beta_1; \dots; b_q, \beta_q \end{matrix}\right) \quad \text{for } \eta > 0.$$

Multiplying through by w^{a_1-1}/a_1, we reduce this expansion by means of (3.4) and (3.5) to

$$H^{m,n}_{p,q}\left(\lambda z\left|\begin{matrix}c_1,\gamma_1;\ldots;c_p,\gamma_p\\ d_1,\delta_1;\ldots;d_q,\delta_q\end{matrix}\right.\right)$$

$$=\lambda^{(c_1-1)/\gamma_1}\sum_{r=0}^{\infty}\frac{1}{r!}(1-\lambda^{-1/\gamma_1})^r H^{m,n}_{p,q}\left(z\left|\begin{matrix}-r+c_1,\gamma_1;c_2,\gamma_2;\ldots;c_p,\gamma_p\\ d_1,\delta_1;\ldots;d_q,\delta_q\end{matrix}\right.\right),$$

where $\lambda=\eta^{a_1}$, $z=w^{a_1}$, $c_1=a_1$, $\gamma_1=a_1$, $c_j=a_j+a_j(a_1-1)$ $(j=2,\ldots,p)$, $\gamma_j=a_ja_1$ $(j=2,\ldots,p)$, $d_j=b_j+\beta_j(a_1-1)$ $(j=1,\ldots,q)$, $\delta_j=\beta_ja_1$ $(j=1,\ldots,q)$.

4. By (3.2), (3.4) and (3.6) formula (5.7), under the hypotheses of Theorem 2, for $n<p$ also implies

$$H^{m,n}_{p,q}\left(\lambda z\left|\begin{matrix}c_1,\gamma_1;\ldots;c_p,\gamma_p\\ d_1,\delta_1;\ldots;d_q,\delta_q\end{matrix}\right.\right)$$

$$=\lambda^{(c_p-1)/a_p}\sum_{r=0}^{\infty}\frac{1}{r!}(\lambda^{-1/\gamma_p}-1)^r H^{m,n}_{p,q}\left(z\left|\begin{matrix}c_1,\gamma_1;\ldots;c_{p-1},\gamma_{p-1};-r+c_p,\gamma_p\\ d_1,\delta_1;\ldots;d_q,\delta_q\end{matrix}\right.\right),$$

and the reasoning is analogous to that given in the previous cases.

Summing up, we obtain the theorem proved by Ławrynowicz in [4].

On the other hand, Theorems 1 and 2 of the present paper with $a_j=1$ $(j=1,\ldots,p)$, $\beta_j=1$ $(j=1,\ldots,q)$ reduce to Theorems 1 and 2 proved by Meijer in [6]; particularly we obtain Assertion 3 in Theorem 1 of Meijer and Assertion 2 in Theorem 2 of Meijer as a consequence of (ii) and (vi) respectively (cf. also [1], p. 320).

References

[1] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes-integrals*, Compositio Math. 15 (1964), pp. 239-341.

[2] E. T. Copson, *Theory of functions of a complex variable*, Oxford 1946.

[3] C. Fox, *The G and H-functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. 98 (1961), pp. 395-429.

[4] J. Ławrynowicz, *Remarks on the preceding paper of P. Anandani*, Ann. Polon. Math. 21 (1969), pp. 119-123.

[5] C. S. Meijer, *Multiplikationstheoreme für die Funktion* $G^{m,n}_{p,q}(z)$, Proc. Kon. Akad. v. Wetensch. 44 (1941), pp. 1062-1070.

[6] — *Expansion theorems for the G-function I-II*, Proc. Kon. Ned. Akad. v. Wetensch., Series A, 55 (1952), pp. 369-379 and pp. 483-487.

[7] R. K. Saxena, *A formal solution of certain dual integral equations involving H-functions*, Proc. Cambridge Philos. Soc. 63 (1967), pp. 171-178.

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