

On neighbourhoods of univalent starlike functions

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Abstract. Let A denote the class of analytic functions f in the unit disk with $f(0) = f'(0) - 1 = 0$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in A and $\delta \geq 0$ Ruscheweyh has defined the δ -neighbourhood of f as

$$N_{\delta}(f) = \{g \in A \mid g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k |a_k - b_k| \leq \delta\}.$$

In this paper we study various properties of $N_{\delta}(f)$, where f is assumed to be a univalent starlike function.

Introduction. Let A denote the class of analytic functions f in the unit disk $E = \{z \mid |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in A and $\delta \geq 0$ Ruscheweyh has defined the δ -neighbourhood $N_{\delta}(f)$ as follows:

$$N_{\delta}(f) = \{g \in A \mid g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k |a_k - b_k| \leq \delta\}.$$

He has shown in [3], among other results, that if $(f(z) + \varepsilon z)/(1 + \varepsilon) \in S^*$ for all complex numbers ε with $|\varepsilon| < \delta$, then

$$(1) \quad N_{\delta}(f) \subset S^*.$$

Let $B \subseteq A$. For $\delta \geq 0$ we define

$$B^{1,\delta} = \left\{ f \in B \mid \frac{f(z) + \varepsilon z}{1 + \varepsilon} \in B \text{ for all complex numbers } \varepsilon \text{ with } |\varepsilon| < \delta \right\}$$

and if $n \geq 2$ is an integer

$$B^{n,\delta} = \left\{ f \in B \mid f(z) + \frac{\varepsilon}{n} z^n \in B \text{ for all complex numbers } \varepsilon \text{ with } |\varepsilon| < \delta \right\}.$$

It follows easily from Ruscheweyh's result and the definition of neighbourhoods that for $\delta \geq 0$

$$(2) \quad f \in S^{*1,\delta} \Rightarrow N_{\delta}(f) \subset S^* \Rightarrow f \in S^{*n,\delta} \quad \text{for all } n \geq 2.$$

Ruscheweyh raised the following question: what can be said about the converse of the implications in (2)? For example is it true that, for any $\delta \geq 0$, $S^{**2,\delta} \subset S^{*1,\delta}$, or is it true that $N_\delta(f) \subset S^* \Rightarrow f \in S^{*1,\delta}$? It has been proved ([2]) by Rahman and Stankiewicz that, for any $\delta \geq 0$, $S^{**n,\delta} \subset S^{*1,\delta/n}$ if $n \geq 2$ and it follows, according to (1), that

$$f \in S^{**n,\delta} \Rightarrow N_{\delta/n}(f) \subset S^* \quad \text{if } n \geq 2 \text{ and } \delta \geq 0.$$

We prove:

THEOREM 1. *Let $\delta \geq 0$. Then $f \in S^{**2,\delta} \Rightarrow N_\delta(f) \subset S^*$.*

THEOREM 2. *Let $n > 2$ a positive integer. Then there is a function $f \in A$ and a real number δ , $0 < \delta < 1$, such that $f \in S^{**n,\delta}$ and $N_\delta(f) \not\subset S^*$.*

THEOREM 3. *There is a function $f \in A$ and a real number δ , $0 < \delta < 1$, such that $N_\delta(f) \subset S^*$ and $f \notin S^{*1,\delta}$.*

At this stage we may ask if the above theorems describe properties of starlike functions or more general structural properties of neighbourhoods. We introduce the following classes of univalent functions depending on a real parameter $t > \frac{1}{2}$:

$$(S^*)_t = \left\{ f \in A \mid \left| \frac{zf'(z)}{f(z)} - t \right| < t, z \in E \right\}.$$

It is well known that the classes $(S^*)_t$ contain univalent starlike functions and it can be shown ([1]) that Ruscheweyh's result (1) can be extended to $(S^*)_t$, only when $t \geq 1$, that Theorem 1 is valid for $(S^*)_t$ only when $t \geq 2$, and that Theorem 2 remains true for $(S^*)_t$ when $t \geq 1$. On the other hand the following results show that the behaviours of S^* and $(S^*)_t$ with respect to neighbourhoods may be quite different. We prove

THEOREM 4. *Let $\delta \geq 0$ and $f \in (S^*)_t$. Then $N_\delta(f) \subset (S^*)_t \Rightarrow f \in (S^*)_t^{1,\delta}$ if $\frac{1}{2} < t \leq 1$. Moreover, this result is false for each $t > 1$.*

THEOREM 5. *Let $n \geq 2$ a positive integer. There exist a positive real number δ and a function $f \in A$ such that $f \in (S^*)_{n/2}^{n,\delta}$ and, for any $\alpha > 0$, $N_\alpha(f) \not\subset (S^*)_{n/2}$.*

It is especially surprising that, for any $\delta > 0$, $N_\delta(f) \subset S^* \Leftrightarrow f \in S^{**2,\delta}$ but $N_\delta(f) \subset (S^*)_1 \Leftrightarrow f \in (S^*)_1^{1,\delta}$. Nevertheless the classes $(S^*)_t$ and S^* also share some common behaviour with respect to neighbourhoods. For example it follows from Theorem 1 and some of the remarks above that for any $\delta > 0$

$$N_\delta(f) \subset S^* \Leftrightarrow f \in \bigcap_{n=2}^{\infty} S^{**n,\delta}$$

and

$$N_\delta(f) \subset (S^*)_t \Leftrightarrow f \in \bigcap_{n=2}^{\infty} (S^*)_t^{n,\delta} \quad \text{if } t \geq 2.$$

In fact this last statement is valid for any $t > \frac{1}{2}$ as will be shown by the proof of Theorem 4 (see Lemmas 4.1 and 4.2).

We would like to close this introduction with some remarks concerning neighbourhoods of univalent convex functions. It has been proved ([3]) by Ruscheweyh that $N_{1/4}(f) \subset S^*$ for any convex univalent function f , where the constant $1/4$ is the best possible. Since the function that will be used in the proof of Theorems 2 and 3 can be chosen to be convex no improvement of those theorems can be obtained by replacing S^* by $C = \{f \in A \mid f \text{ is convex univalent in } E\}$. But we can prove

THEOREM 6. *Let $\delta > 0$ and $f \in C$. Then $N_\delta(f) \not\subset C$. However, $N_\delta(f)$ contains only univalent functions if $\delta \leq \inf_{z \in E} |f'(z)|$.*

Finally we point out that in establishing most of the above mentioned theorems our main tool is the Hadamard product (or convolution) of analytic functions. If the two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to A their Hadamard product is the function $f * g$ in A defined as

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

It is not difficult to verify that many classes mentioned above can be defined in terms of convolution. For example

$$(3) \quad f \in S^* \Leftrightarrow \forall T \in \mathbf{R}, \quad \forall z \in E, \quad \frac{f * h_T(z)}{z} \neq 0,$$

where $h_T(z) = \frac{z/(1-z)^2 iTz/(1-z)}{1+iT}$, and

$$(4) \quad f \in (S^*)_t \Leftrightarrow \forall \theta \in [0, 2\pi], \quad \forall z \in E, \quad \frac{f * h_\theta(z)}{z} \neq 0,$$

where $h_\theta(z) = \frac{z/(1-z)^2 - t(1+e^{i\theta})z/(1-z)}{1-t(1+e^{i\theta})}$.

We shall also need, putting $S = \{f \in A \mid f \text{ is univalent}\}$,

$$(5) \quad f \in S \Leftrightarrow \forall x \in \mathbf{C} \quad (|x| \leq 1), \quad \forall z \in E, \quad \frac{f * h_x(z)}{z} \neq 0,$$

where $h_x(z) = \frac{z}{(1-z)(1-xz)}$.

Proof of Theorem 1. We need the following lemma. It is an extension of the result given by (3).

LEMMA 1.1. *Let $\delta \geq 0$ and n a positive integer ≥ 2 . Then*

$$f \in S^{*n,\delta} \Leftrightarrow \left| \frac{f * h_T(z)}{zc_n(T)} \right| > \frac{\delta}{n}, \quad T \in \mathbf{R}, z \in E,$$

where $c_n(T) = \frac{n+iT}{1+iT} = \frac{h_T^{(n)}(0)}{n!}$.

The proof of Lemma 1.1 depends essentially on (3). We have

$$\begin{aligned} f \in S^{*n,\delta} &\Leftrightarrow \frac{\left(f(z) + \frac{\varepsilon}{n} z^n \right) * h_T(z)}{z} \neq 0, \quad T \in \mathbf{R}, z \in E, |\varepsilon| \leq \delta \\ &\Leftrightarrow \left| \frac{f(z) * h_T(z)}{z} \right| \geq \frac{\delta}{n} |c_n(T)| |z|^{n-1}, \quad T \in \mathbf{R}, z \in E. \end{aligned}$$

A straightforward application of the maximum principle to the non-vanishing function $f * h_T(z)/z$ will then show that the last condition is equivalent to

$$\left| \frac{f * h_T(z)}{z} \right| > \frac{\delta}{n} |c_n(T)|, \quad T \in \mathbf{R}, z \in E.$$

This completes the proof of Lemma 1.1.

We are now ready to prove Theorem 1. Let $f \in S^{*2,\delta}$ and $g \in N_\delta(f)$. In view of (3) it will be enough to show that

$$\frac{g * h_T(z)}{z} \neq 0, \quad T \in \mathbf{R}, z \in E.$$

We have, according to Lemma 1.1,

$$\left| \frac{g * h_T(z)}{zc_2(T)} \right| \geq \left| \frac{f * h_T(z)}{zc_2(T)} \right| - \left| \frac{(g-f) * h_T(z)}{zc_2(T)} \right| > \frac{\delta}{2} - \left| \frac{(g-f) * h_T(z)}{zc_2(T)} \right| \geq 0$$

because if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$,

$$(6) \quad \left| \frac{(g-f) * h_T(z)}{zc_2(T)} \right| = \left| \sum_{k=2}^{\infty} \frac{c_k(T)}{c_2(T)} (b_k - a_k) z^{k-1} \right|$$

$$\begin{aligned} (7) \quad &\leq \sum_{k=2}^{\infty} \frac{k}{2} |b_k - a_k| \\ &\leq \frac{1}{2} \delta \quad \text{because } g \in N_\delta(f). \end{aligned}$$

The passage from (6) to (7) is justified by

$$\left| \frac{c_k(T)}{c_2(T)} \right| = \left| \frac{k+iT}{2+iT} \right| \leq \frac{k}{2} \quad \text{if } k \geq 2 \text{ and } T \in \mathbb{R}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We need the following lemma. Its proof is essentially computational and will be omitted here.

LEMMA 2.1. Let $f(z) = z(1+pz)^i$, where $p > 0$ and the branch of $(1+pz)^i$ is chosen in such a way that $f'(0) = 1$. If p is close enough to zero we have that $f(z)$ and $zf'(z)/f(z)$ are bounded analytic functions in the closed unit disk and

- (8) there is $K_1 > 0$ such that $\operatorname{Re}(zf'(z)/f(z)) > K_1$, $|z| \leq 1$,
- (9) there is $K_2 > 0$ such that $|f(z)/z| > K_2$, $|z| \leq 1$,
- (10) if $|u| = 1$ and $0 < |f'(u)| = \inf_{z \in E} |f'(z)|$, then $uf'(u)/f(u) \notin \mathbb{R}$.

Let $n > 2$ and f be defined as in Lemma 2.1. We first show that

$$(11) \quad \inf_{\substack{T \in \mathbb{R} \\ z \in E}} \left| \frac{f * h_T(z)}{zc_n(T)} \right| < \inf_{z \in E} \frac{|f'(z)|}{n}.$$

In fact if $|u| = 1$ and $|f'(u)| = \inf_{z \in E} |f'(z)|$ we obtain, in view of (10),

$$\inf_{T \in \mathbb{R}} \left| \frac{\frac{uf'(u)}{f(u)} + iT}{n+iT} \right| = \frac{\left| \frac{uf'(u)}{f(u)} + n \right| - \left| \frac{uf'(u)}{f(u)} - n \right|}{2n} < \frac{1}{n} \left| \frac{uf'(u)}{f(u)} \right|$$

and there must exist a real number T such that

$$(12) \quad \left| \frac{f * h_T(u)}{uc_n(T)} \right| = \left| \frac{f'(u) + iTf(u)/u}{n+iT} \right| < \frac{|f'(u)|}{n} = \inf_{z \in E} \frac{|f'(z)|}{n}.$$

It is clear from (12) that (11) is valid.

Then we define δ by

$$\frac{\delta}{n} = \inf_{\substack{T \in \mathbb{R} \\ z \in E}} \left| \frac{f * h_T(z)}{zc_n(T)} \right|.$$

It is clear from (11) that $\delta < 1$; since

$$\left| \frac{f * h_T(z)}{zc_n(T)} \right| \geq \left| \frac{f(z)}{z} \right| \left(\frac{\left| \frac{zf'(z)}{f(z)} + n \right| - \left| \frac{zf'(z)}{f(z)} - n \right|}{2n} \right)$$

it follows easily from (8), (9) and the fact that $|zf'(z)/f(z)|$ is bounded that

$\delta > 0$. It is a consequence of Lemma 1.1 that

$$(13) \quad f \in S^{*n, \delta}.$$

There must exist $T_0 \in \mathbf{R} \cup \{\infty\}$ and z_0 with $|z_0| \leq 1$ such that

$$\frac{\delta}{n} = \left| \frac{f * h_{T_0}(z_0)}{z_0 c_n(T_0)} \right|$$

and it should be clear from (11) that $T_0 \neq 0$. On the other hand

$$(14) \quad \left| \frac{f * h_{T_0}(z_0)}{z_0 c_2(T_0)} \right| = \left| \frac{f * h_{T_0}(z_0)}{z_0 c_n(T_0)} \right| \left| \frac{c_n(T_0)}{c_2(T_0)} \right| < \frac{\delta}{n} \left(\frac{n}{2} \right) = \frac{\delta}{2}$$

because $\left| \frac{c_n(T_0)}{c_2(T_0)} \right| = \left| \frac{n+iT_0}{2+iT_0} \right| < \frac{n}{2}$ if $n > 2$ and $T_0 \neq 0$.

It follows from Lemma 1.1 and (14) that

$$(15) \quad f \notin S^{*2, \delta} \quad \text{and} \quad N_\delta(f) \notin S^*.$$

In view of (13) and (15) the proof of Theorem 2 is completed.

Proof of Theorem 3. The proof of Theorem 3 is similar to the proof of Theorem 2 and only the main steps will be supplied. We choose the function f as in Lemma 2.1. We first establish that

$$(16) \quad \inf_{\substack{T \in \mathbf{R} \\ z \in E}} \left| \frac{f * h_T(z)}{z c_2(T)} \right| < \inf_{z \in E} \frac{|f'(z)|}{2},$$

and we define δ by $\frac{1}{2}\delta = \inf_{\substack{T \in \mathbf{R} \\ z \in E}} |f * h_T(z)/z c_2(T)|$. It follows from (16) and

Lemma 2.1 that $0 < \delta < 1$; it follows from Lemma 1.1 that $f \in S^{*2, \delta}$ and by Theorem 1 we have

$$(17) \quad N_\delta(f) \subset S^*.$$

On the other hand if $\frac{1}{2}\delta = |f * h_{T_0}(z_0)/z_0 c_2(T_0)|$ with $|z_0| \leq 1$ and $T_0 \in \mathbf{R} \cup \{\infty\}$ it follows from (16) that $T_0 \neq 0$ and

$$(18) \quad \left| \frac{f * h_{T_0}(z_0)}{z_0} \right| = \frac{\delta}{2} |c_2(T_0)| < \delta$$

because $|c_2(T)| = \left| \frac{2+iT}{1+iT} \right| < 2$ if $T \neq 0$. It is then obvious from (18) that

$\inf_{\substack{T \in \mathbf{R} \\ z \in E}} |f * h_T(z)/z| < \delta$ and $f \notin S^{*1, \delta}$. This, together with (17), completes the proof of Theorem 3.

Remark about Theorems 2–3. Theorems 2 and 3 answer Ruscheweyh’s question in the negative. But it is possible to give sufficient conditions under which the question may be answered positively. We can prove (the proofs are omitted):

THEOREM 2’. Let $f \in A$ such that

$$\inf_{\substack{T \in \mathbb{R} \\ z \in E}} \left| \frac{f * h_T(z)}{zc_2(T)} \right| = \inf_{z \in E} \frac{|f'(z)|}{2}.$$

Then $\forall \delta > 0 \forall n \geq 2 (f \in S^{*n,\delta} \Rightarrow N_\delta(f) \subset S^*)$.

THEOREM 3’. Let $f \in A$ such that

$$\inf_{\substack{T \in \mathbb{R} \\ z \in E}} \left| \frac{f * h_T(z)}{z} \right| = \inf_{z \in E} |f'(z)|.$$

Then $\forall \delta > 0 \forall n \geq 2 (f \in S^{*n,\delta} \Rightarrow f \in S^{*1,\delta})$.

In general the result of Rahman and Stankiewicz quoted in the introduction shows that $f \in S^{*n,\delta} \Rightarrow N_{\delta/n}(f) \subset S^*$ if $n \geq 2$. We think that this result is best possible when $n > 2$ but we are unable to prove it.

Proof of Theorem 4. For the proof of the first part of Theorem 4 we need the following lemma which is an extension of (4). The proof is very similar to the proof of Lemma 1.1 and will be omitted. We recall that $h_\theta(z) = \sum_{n=1}^\infty c_n(\theta)z^n$, where

$$c_n(\theta) = \frac{n-t(1+e^{i\theta})}{1-t(1+e^{i\theta})}.$$

LEMMA 4.1. Let $\delta \geq 0$ and $t > \frac{1}{2}$. Let also $n \geq 1$ be a positive integer. Then

$$f \in (S^*)_t^{n,\delta} \Leftrightarrow \left| \frac{f * h_\theta(z)}{zc_n(\theta)} \right| > \frac{\delta}{n} \Leftrightarrow \left| \frac{f'(z) - t(1+e^{i\theta})f(z)/z}{n-t(1+e^{i\theta})} \right| > \frac{\delta}{n},$$

$\theta \in [0, 2\pi], z \in E.$

Let $\frac{1}{2} < t \leq 1$ and $N_\delta(f) \subset (S^*)_t$. This means that $f \in \bigcap_{n=2}^\infty (S^*)_t^{n,\delta}$ and letting $n \rightarrow \infty$ in Lemma 4.1 we obtain easily

$$(19) \quad N_\delta(f) \subset (S^*)_t \Rightarrow |f(z)/z| \frac{1 - \left| \frac{1}{t} \frac{zf'(z)}{f(z)} - 1 \right|}{(1/t - 1) + 1} > \delta, \quad z \in E.$$

On the other hand it also follows from Lemma 4.1 that

$$(20) \quad f \in (S^*)_t^{1,\delta}$$

$$\Leftrightarrow |f(z)/z| \frac{\left| 1 - \left(\frac{1}{t} - 1\right) \left(\frac{1zf'(z)}{t f(z)} - 1\right) \right| - \left| \left(\frac{1zf'(z)}{t f(z)} - 1\right) - \left(\frac{1}{t} - 1\right) \right|}{1 - (1/t - 1)^2} > \delta,$$

$z \in E.$

The definition of the class $(S^*)_t$, and a comparison of (19) and (20) show that the proof of the first part of Theorem 4 will be completed if

$$(21) \quad \frac{|1 - w(0)w(z)| - |w(z) - w(0)|}{1 - w(0)^2} \geq \frac{1 - |w(z)|}{1 + w(0)}, \quad z \in E,$$

for any analytic function $w(z)$ in the unit disk E with $|w(z)| < 1$ and $0 \leq w(0) < 1$. But we have

$$\begin{aligned} |1 - w(0)w(z)| - |w(z) - w(0)| &\geq \frac{|1 - w(0)w(z)|^2 - |w(z) - w(0)|^2}{1 + |w(0)||w(z)| + |w(z)| + |w(0)|} \\ &= \frac{(1 - |w(z)|^2)(1 - w(0)^2)}{(1 + |w(z)|)(1 + w(0))} \end{aligned}$$

from which follows the truth of (21).

We now proceed to show Theorem 4 when $t \geq 2$. Since Theorem 1 is valid for $(S^*)_t$ when $t \geq 2$ ([1]) it will be enough to show the existence of a function f and of a positive number δ such that

$$f \in (S^*)_t^{2,\delta} \quad \text{and} \quad f \notin (S^*)_t^{1,\delta}.$$

We choose f as the function in Lemma 2.1 and show that

$$0 < \frac{\delta}{2} =: \inf_{\substack{\theta \in [0, 2\pi] \\ z \in E}} \left| \frac{f * h_\theta(z)}{z c_2(\theta)} \right| < \inf_{z \in E} \frac{|f'(z)|}{2} < 1.$$

Just as in the case of Theorem 3

$$\frac{\delta}{2} = \left| \frac{f * h_{\theta_0}(z_0)}{z_0 c_2(\theta_0)} \right|, \quad \text{where } |z_0| \leq 1 \quad \text{and} \quad \theta_0 \in [0, 2\pi], \theta_0 \neq \pi.$$

It will follow that

$$\left| \frac{f * h_{\theta_0}(z_0)}{z_0} \right| < \delta$$

and by Lemma 4.1, $f \in (S^*)_t^{2,\delta}$ and $f \notin (S^*)_t^{1,\delta}$. This completes the proof of Theorem 4 when $t \geq 2$.

For the remaining case we need:

LEMMA 4.2. Let $\frac{1}{2} < t \leq 2$ and $\delta > 0$. Then

$$N_\delta(f) \subset (S^*)_t \Leftrightarrow \left| t \frac{f(z)}{z} \right| - \left| f'(z) - t \frac{f(z)}{z} \right| > \delta, \quad z \in E.$$

Proof of Lemma 4.2. The sufficiency is just the statement (19). Assume that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_\delta(f)$ and

$$\left| t \frac{f(z)}{z} \right| - \left| f'(z) - t \frac{f(z)}{z} \right| > \delta, \quad z \in E.$$

Then

$$\left| t \frac{g(z)}{z} \right| - \left| g'(z) - t \frac{g(z)}{z} \right| > \delta - \left(t \left| \frac{g(z)}{z} - \frac{f(z)}{z} \right| + \left| (g'(z) - f'(z)) - t \left(\frac{g(z)}{z} - \frac{f(z)}{z} \right) \right| \right) \geq 0$$

because

$$\left| t \left(\frac{g(z)}{z} - \frac{f(z)}{z} \right) \right| + \left| (g'(z) - f'(z)) - t \left(\frac{g(z)}{z} - \frac{f(z)}{z} \right) \right| \leq \sum_{k=2}^{\infty} (t + |k - t|) |b_k - a_k| \leq 0,$$

since $g \in N_\delta(f)$ and $\frac{1}{2} < t \leq 2$.

This completes the proof of Lemma 4.2.

We shall also need (stated here without proof)

LEMMA 4.3. Let $-1 < \xi < 0$. There exists $\varepsilon > 0$ such that

$$z \in E \quad \text{and} \quad |\arg(z)| < \varepsilon \Rightarrow \frac{|1 - \xi z| - |z - \xi|}{1 - \xi^2} \leq \frac{1 - |z|}{1 + \xi}.$$

We are now ready to prove Theorem 4 when $1 < t \leq 2$. Let w be a Möbius transform with $w(0) = 1/t - 1$ and such that the image of the unit disk under w is the interior of a circle C with the following properties:

- (22) the interior of C is included in E ,
- (23) the circle C is tangent to the unit circle at 1.

Let also $0 < p < 1$ and define $f \in (S^*)_t$ by the equation

$$\frac{1}{t} \frac{zf'(z)}{f(z)} - 1 = w(pz).$$

It is clear that

$$(24) \quad 0 < \inf_{z \in E} \left| \frac{f(z)}{z} \right| \frac{1 - |w(pz)|}{1 + w(0)} := \delta < 1$$

and, by Lemma 4.2, $N_\delta(f) \subset (S^*)_t$. It is also true that for p close enough to 1 it follows from (23) that the inf in (24) is attained when $|\arg(w(pz))| < \varepsilon$ and

by Lemma 4.3

$$(25) \quad 0 < \inf_{z \in E} \left| \frac{f(z)}{z} \right| \frac{|1 - w(0)w(pz)| - |w(pz) - w(0)|}{1 - w(0)^2} < \inf_{z \in E} \left| \frac{f(z)}{z} \right| \frac{1 - |w(pz)|}{1 + w(0)} = \delta.$$

The comparison of (25) and (20) gives $f \notin (S^*)_t^{1,\delta}$. This completes the proof of Theorem 4.

Proof of Theorem 5. We shall need the following lemma. It was first proved by Ruscheweyh ([3]) for the class S^* .

LEMMA 5.1. *Let $t > \frac{1}{2}$ and $f \in (S^*)_t$ such that $\inf_{\substack{\theta \in [0, 2\pi] \\ z \in E}} |f * h_\theta(z)/z| = 0$. Then for any $\alpha > 0$, $N_\alpha(f) \notin (S^*)_t$.*

In order to prove Lemma 5.1 we may assume that $t \geq 1$, since it follows from Theorem 4 and Lemma 4.1 that

$$N_\alpha(f) \subset (S^*)_t \Rightarrow f \in (S^*)_t^{1,\alpha} \Rightarrow \inf_{\substack{\theta \in [0, 2\pi] \\ z \in E}} \left| \frac{f * h_\theta(z)}{z} \right| \geq \alpha \quad \text{if } \frac{1}{2} < t \leq 1.$$

Let $t \geq 1$ and n be an integer ≥ 2 , $n \neq 2t$. Let $\alpha > 0$. There must exist $z_0 \in E$ and $\theta_0 \in [0, 2\pi]$ such that

$$(26) \quad \left| \frac{f * h_{\theta_0}(z_0)}{z_0^n} \right| < \frac{\alpha |n - 2t|}{n |2t - 1|}.$$

Otherwise

$$\left| \frac{f * h_\theta(z)}{z} \right| \geq |z|^{n-1} \frac{\alpha |n - 2t|}{n |2t - 1|}, \quad \theta \in [0, 2\pi], \quad z \in E,$$

and an application of the maximum principle to the non-vanishing function $f * h_\theta(z)/z$ would then contradict the assumption of Lemma 5.1. We put $\mu = f * h_{\theta_0}(z_0)/z_0$ and

$$g(z) = f(z) - \frac{\mu}{c_n(\theta_0)} z^n.$$

Since

$$|c_n(\theta_0)| = \frac{|n - t(1 + e^{i\theta_0})|}{|1 - t(1 + e^{i\theta_0})|} \geq \frac{|n - 2t|}{2t - 1} \quad \text{when } t \geq 1,$$

we obtain from (26)

$$n \left| \frac{\mu}{c_n(\theta_0)} \right| < \alpha \quad \text{and} \quad g \in N_\alpha(f).$$

On the other hand

$$\frac{g * h_{\theta_0}(z_0)}{z_0} = \frac{f * h_{\theta_0}(z_0)}{z_0} - \mu z_0^{n-1} = 0$$

which, according to (4), means that $g \notin (S^*)_r$. Therefore $N_\alpha(f) \notin (S^*)$, and this completes the proof of Lemma 5.1.

We also remark that for any $\delta > 0$ and $n \geq 2$

$$(27) \quad f \in (S^*)_{n/2}^{\delta} \Leftrightarrow n \left| \frac{f(z)}{z} \right| \frac{1 - \left| \frac{2zf'(z)}{nf(z)} - 1 \right|^2}{2 \left| 1 - \left(\frac{2zf'(z)}{nf(z)} - 1 \right) \right|} > \delta, \quad z \in E.$$

In fact by Lemma 4.1

$$\begin{aligned} f \in (S^*)_{n/2}^{\delta} &\Leftrightarrow \left| \frac{f'(z) - \frac{1}{2}n(1 + e^{i\theta}) \frac{f(z)}{z}}{n - \frac{1}{2}n(1 + e^{i\theta})} \right| > \frac{\delta}{n}, \quad z \in E, \theta \in [0, 2\pi] \\ &\Leftrightarrow n \left| \frac{f(z)}{z} \right| \left| \frac{\left(\frac{2zf'(z)}{nf(z)} - 1 \right) - e^{i\theta}}{1 - e^{i\theta}} \right| > \delta, \quad z \in E, \theta \in [0, 2\pi] \end{aligned}$$

and (27) is then valid because

$$\inf_{\theta \in [0, 2\pi]} \left| \frac{\xi - e^{i\theta}}{1 - e^{i\theta}} \right| = \frac{1 - |\xi|^2}{2|1 - \xi|} \quad \text{if } |\xi| < 1.$$

We are now ready to prove Theorem 5. Define a function f in A by

$$\frac{2zf'(z)}{nf(z)} - 1 = W(z),$$

where the function $W(z)$ maps E onto the domain E' illustrated by Fig. 1.

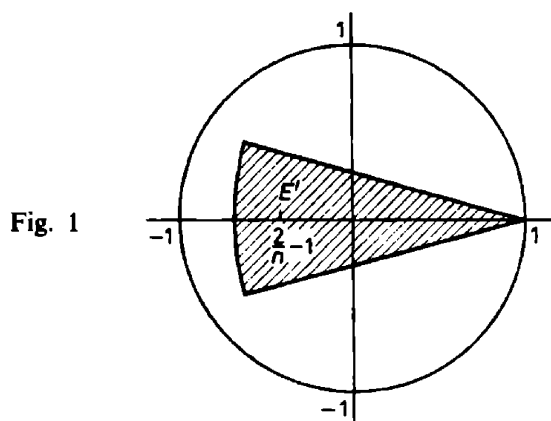


Fig. 1

Since $E' \subset E$ it is clear that $|W(z)| < 1$ and $f \in (S^*)_{n/2}$. There exist positive constants K_1 and K_2 such that

$$(28) \quad \left| \frac{f(z)}{z} \right| > K_1, \quad z \in E,$$

$$(29) \quad \frac{1 - |W(z)|^2}{|1 - W(z)|} > K_2, \quad z \in E.$$

The statement (28) is valid because $f \in (S^*)_{n/2} \subset S$ and the statement (29) depends on a well-known property of Stoltz angles. We define δ as

$$\delta = \inf_{z \in E} n \left| \frac{f(z)}{z} \right| \frac{1 - \left| \frac{2zf'(z)}{nf(z)} - 1 \right|^2}{2 \left| 1 - \left(\frac{2zf'(z)}{nf(z)} - 1 \right) \right|}.$$

It follows from (28) and (29) that $0 < \delta < 1$ and in view of (27)

$$f \in (S^*)_{n/2}^{\delta}.$$

On the other hand the boundary of E' meets the unit circle and

$$\inf_{\substack{\theta \in [0, 2\pi] \\ z \in E}} \left| \frac{f * h_\theta(z)}{z} \right| = \inf_{\substack{\theta \in [0, 2\pi] \\ z \in E}} \frac{n}{2} \left| \frac{f(z)}{z} \right| \frac{\left| \left(\frac{2zf'(z)}{nf(z)} - 1 \right) - e^{i\theta} \right|}{\left| 1 - \frac{1}{2}n(1 + e^{i\theta}) \right|} = 0.$$

This means, in view of Lemma 5.1, that $N_\alpha(f) \notin (S^*)_{n/2}$ for any $\alpha > 0$. The proof of Theorem 5 is completed.

Proof of Theorem 6. We first prove that for any $f \in C$ and $\delta > 0$, $N_\delta(f) \notin C$. In fact, if $N_\delta(f) \subset C$ then, for any $n \geq 2$, $f \in C^{n, \delta}$ and $zf'(z) \in S^{*n, n\delta}$. In view of Lemma 1.1 this means that

$$(30) \quad \left| \frac{zf'(z) * h_0(z)}{zc_n(0)} \right| = \left| \frac{zf''(z) + f'(z)}{n} \right| > \delta, \quad z \in E, \quad n \geq 2$$

and putting $z = 0$ in (30) we obtain $\delta < 1/n$ for any $n \geq 2$ and therefore $\delta = 0$. This completes the proof of the first statement of Theorem 6. We remark that this statement means that Ruscheweyh's result (1) admits no extension to the class C . On the other hand, by defining $N'_\delta(f)$ for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ as

$$N'_\delta(f) = \left\{ g \in A \mid g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k^2 |a_k - b_k| \leq \delta \right\},$$

we obtain (the proof is omitted), for any $\delta > 0$,

$$f \in C^{1, \delta} \Rightarrow N'_\delta(f) \subset C.$$

In order to prove the second statement of Theorem 6 it will be enough, in view of (5) and Ruscheweyh's work (see [3]), to show that for $f \in C$,

$$(31) \quad \inf_{\substack{z \in E \\ |x| \leq 1}} \left| \frac{f(z) - f(xz)}{(1-x)z} \right| = \inf_{z \in E} |f'(z)|.$$

It will be clearly enough to show that

$$(32) \quad \inf_{\substack{z \in E \\ |x| \leq 1}} \left| \frac{f(z) - f(xz)}{(1-x)z} \right| \geq \inf_{z \in E} |f'(z)|,$$

and the truth of (32) follows from

$$(33) \quad \inf_{\substack{z \in E \\ |x| \leq 1}} \left| \frac{f(z) - f(xz)}{(1-x)z} \right| = \inf_{a, b \in f(E)} \left| \frac{a - b}{f^{-1}(a) - f^{-1}(b)} \right| \\ = \left(\sup_{a, b \in f(E)} \left| \frac{f^{-1}(a) - f^{-1}(b)}{a - b} \right| \right)^{-1}$$

$$(34) \quad \geq \left(\sup_{c \in f(E)} |(f^{-1})'(c)| \right)^{-1} \\ = \inf_{z \in E} |f'(z)|.$$

The passage from (33) to (34) is justified by the fact that, since $f(E)$ is convex, we have for $a, b \in f(E)$

$$\left| \frac{f^{-1}(a) - f^{-1}(b)}{a - b} \right| = \left| \int_0^1 (f^{-1})'(b + \rho(a - b)) d\rho \right| \leq \sup_{c \in f(E)} |(f^{-1})'(c)|.$$

This completes the proof of the second statement of Theorem 6. The example $f(z) = (e^{nz} - 1)/\pi$ shows that (31) is not valid for all univalent functions; in fact

$$\inf_{z \in E} |f'(z)| = e^{-\pi} > 0 \quad \text{and} \quad \inf_{\substack{z \in E \\ |x| \leq 1}} \left| \frac{f(z) - f(xz)}{(1-x)z} \right| = 0 \quad \text{because } f(i) = f(-i).$$

We also remark that (31) can be used to show (the proof is omitted) that for $f \in C$, $\delta > 0$ and $n \geq 2$,

$$f \in S^{n, \delta} \Leftrightarrow N_\delta(f) \subset S \Leftrightarrow f \in S^{1, \delta}.$$

Conclusion. We would like to mention that some of the results of this paper can be extended to certain classes of non-starlike univalent functions. For example if

$$H = \{f \in A \mid \operatorname{Re}(f'(z)) > 0, z \in E\}$$

we can prove that for any $n \geq 2$ and $\delta \geq 0$,

$$f \in H^{n,\delta} \Leftrightarrow N_\delta(f) \subset H \Leftrightarrow \operatorname{Re}(f'(z)) > \delta, \quad z \in E,$$

and

$$f \in H^{1,\delta} \Leftrightarrow |f'(z)+1| - |f'(z)-1| > 2\delta, \quad z \in E.$$

Since $\operatorname{Re}(f'(z)) > 0 \Leftrightarrow \frac{f'(z)+i^\alpha}{1+i^\alpha} \neq 0, \alpha \in \mathbf{R}$, the class H can be defined in terms of convolution. The function $f_\delta \in H$ defined by

$$f'_\delta(z) = \delta + (1-\delta) \frac{1+z}{1-z}, \quad \delta \geq 0,$$

has the interesting properties that $N_\delta(f_\delta) \subset H$ and $f_\delta \notin H^{1,\beta}$ for any $\beta > 0$.

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