ANNALES
POLONICI MATHEMATICI
XXXVIII (1980)

Differential-functional inequalities of parabolic type in unbounded regions

by P. Besala and G. Paszek (Gdańsk)

Abstract. The paper deals with a diagonal system of parabolic differential-functional inequalities of the form

$$(0.1)\quad u_t^k(t,x) < \sum_{i,j=1}^{n} \big(a_{ij}^k(t,x)u_{x_i}^k(t,x)\big)_{x_j} + \sum_{i=1}^{n} b_i^k(t,x)u_{x_i}^k(t,x) + f^k(t,x,u(t,x),u(t,\cdot)) \qquad (k = 1,\ldots,N)$$

in an arbitrary open set D (bounded or not) in the half-space: $t > 0$, $x = (x_1,\ldots,x_n) \in R^n$. The function $u(t,x) = (u^1(t,x),\ldots,u^N(t,x))$ is continuous in the closure $\bar{D}$ of D and $u(t,\cdot) = (u^1(t,\cdot),\ldots,u^N(t,\cdot))$ denotes the function from Δ_t to R^N such that $u(t,\cdot)(x) = u(t,x)$, $\Delta_{\tilde{t}}$ being the intersection of $\bar{D}$ with the plane $t = \tilde{t}$.

Under adequate assumptions we prove theorems on differential-functional inequalities, from which the maximum principle is derived, as are also some theorems on estimates and uniqueness for solutions of the first Fourier problem for differential-functional equations.

Parabolic differential-functional inequalities in regions bounded with respect to x-variables were treated by J. Szarski [4]–[6], A. Sobolewska [3] and in an unboundeds trip by T. Stanisz in his thesis. Our improvement of their results consists in that the solution $u(t,x)$ is allowed to belong to some wider function classes and that the region is arbitrary. Thus, in particular, we extend the corresponding result of [1] (concerning parabolic differential inequalities) from the strip to an arbitrary unbounded region and from L^1 to L^p-solutions. The method used here is patterned on that employed in papers [1], [2].

1. Preliminaries. We denote by t points of the interval $\langle 0,T\rangle$, $T > 0$, and by $x = (x_1,\ldots,x_n)$ points of the real n-space R^n $(n \geqslant 1)$ with $|x| = \big(\sum_{i=1}^{n} x_i^2\big)^{1/2}$. Let $S = \langle 0,T) \times R^n$, $\bar{S} = \langle 0,T\rangle \times R^n$ and let G be an open set contained in S. We denote $\Gamma = S \cap \partial G$, ∂G being the boundary of G. The following lemma is an extension of Lemma 1 of [1].

Lemma. *Let $w(t,x)$ be a function continuous in S such that $w > 0$ in G and $w = 0$ in $S \setminus G$. Assume w has derivatives w_t, w_{x_i}, $w_{x_ix_j}$ at each point of G and the first derivatives are bounded in any bounded set $G_r = G \cap (|x| < r)$, $r > 0$ (by a constant $M_r > 0$). Then the function*

$$(1.1)\qquad z = (w^3+\varepsilon)^{p/3}, \quad \varepsilon > 0,\ p \geqslant 1,$$

possesses the derivatives $z_t, z_{x_i}, z_{x_ix_j}$ $(i, j = 1, \ldots, n)$ *at each point of* S *and they are given by the formulae*

$$(1.2)\qquad \begin{aligned} z_{x_i} &= pz^{(p-3)/p}w^2 w_{x_i}, \qquad z_t = pz^{(p-3)/p}w^2 w_t, \\ z_{x_ix_j} &= p[(p-1)w^3+2\varepsilon]z^{(p-6)/p}w w_{x_i}w_{x_j} + pz^{(p-3)/p}w^2 w_{x_ix_j} \end{aligned}$$

for $(t, x) \in G$ *and* $z_t = z_{x_i} = z_{x_ix_j} = 0$ *for* $(t, x) \in S\setminus G$.

Proof. It is sufficient to show that the function $\tilde{z} = w^3$ has the derivatives $\tilde{z}_t, \tilde{z}_{x_i}, \tilde{z}_{x_ix_j}$ in S and that

$$(1.3)\qquad \tilde{z}_{x_i} = 3w^2 w_{x_i}, \quad \tilde{z}_t = 3w^2 w_t, \quad \tilde{z}_{x_ix_j} = 6ww_{x_i}w_{x_j} + 3w^2 w_{x_ix_j}$$

for $(t, x) \in G$ and $\tilde{z}_t = \tilde{z}_{x_i} = \tilde{z}_{x_ix_j} = 0$ for $(t, x) \in S\setminus G$, because then the derivatives of z exist and can be calculated in the usual way, whence we obtain (1.2). We prove that $\tilde{z}_t = \tilde{z}_{x_i} = \tilde{z}_{x_ix_j} = 0$ for $(t, x) \in \Gamma$; at other points the formulae are obvious.

Let $(t, x) \in \Gamma$ and let

$$x^{ih} = (x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n), \qquad h \neq 0.$$

Choose $r > \max(|x|, |x^{ih}|)$ and suppose first that $(t, x^{ih}) \in G$. Let $(t, x^{ih'})$, $|h'| < |h|$, be the point (on the segment joining the points (t, x), (t, x^{ih})) such that $(t, x^{ih'}) \in \Gamma$ and all the points of the segment between $(t, x^{ih'})$ and (t, x^{ih}) belong to G. In particular, h' may be equal to zero, i.e., $(t, x^{ih'})$ may coincide with (t, x). Applying the mean value theorem we get

$$h^{-1}w(t, x^{ih}) = h^{-1}[w(t, x^{ih}) - w(t, x^{ih'})] = w_{x_i}(t, x^{i\theta(h-h')})$$

for a certain $0 < \theta < 1$. The boundedness of the first derivatives of w in G_r implies

$$(1.4)\qquad |h^{-1}w(t, x^{ih})| \leqslant M_r,$$

where M_r can be chosen to be independent of h.

If $(t, x^{ih}) \in S\setminus G$, (1.4) is evidently satisfied, too. Inequality (1.4) yields

$$|h^{-1}[\tilde{z}(t, x^{ih}) - \tilde{z}(t, x)]| = |h^{-1}w^3(t, x^{ih})| \leqslant M_r w^2(t, x^{ih}) \to 0 \quad \text{as } h \to 0.$$

Hence $\tilde{z}_{x_i}(t, x) = 0$ $(i = 1, \ldots, n)$. Similarly we find $\tilde{z}_t(t, x) = 0$ for $(t, x) \in \Gamma$.

Now we show that $\tilde{z}_{x_ix_j}(t, x) = 0$ for $(t, x) \in \Gamma$. Let $(t, x^{jh}) \in G$. Then

$$\begin{aligned} |h^{-1}[\tilde{z}_{x_i}(t, x^{jh}) - \tilde{z}_{x_i}(t, x)]| &= |h^{-1}3w^2(t, x^{jh})w_{x_i}(t, x^{jh})| \\ &\leqslant 3M_{r'}|h^{-1}w^2(t, x^{jh})|, \end{aligned}$$

whence, according to (1.4), we obtain

$$(1.5) \qquad |h^{-1}[\tilde{z}_{x_i}(t, x^{jh}) - \tilde{z}_{x_i}(t, x)]| \leqslant 3M_{r'}M_{r''}w(t, x^{jh})$$

for suitable $r', r'' > 0$. Note that inequality (1.5) is also satisfied if $(t, x^{jh}) \in S \setminus G$. Since the right-hand side of (1.5) tends to zero as $h \to 0$, we get $\tilde{z}_{x_i x_j}(t, x) = 0$ for $(t, x) \in \Gamma$ and the lemma is proved.

Notice that, assuming additionally that for each $r > 0$ the second derivatives of w with respect to x are bounded in G_r, the derivatives z_t, z_{x_i}, $z_{x_i x_j}$ are continuous on $S \setminus G$, in particular on Γ.

2. Notations and main assumptions. Throughout this paper D will denote an arbitrary open set (bounded or not) contained in the zone S and Σ will stand for the portion of the boundary of D that is not situated on the plane $t = T$, i.e., $\Sigma = S \cap \partial D$. We assume that the projection $\Delta_{\tilde{t}}$ of the intersection of $\bar{D}$ (the closure of D) with the plane $t = \tilde{t}$ onto the space R^n is non-void for $0 \leqslant \tilde{t} \leqslant T$.

Let $h(x) = (h^1(x), \ldots, h^N(x))$ be a continuous function from Δ_t into R^N. A matrix $\varrho = (\varrho_l^k(t, x, y))$ $(k, l = 1, \ldots, N)$ of continuous and non-negative functions in $\langle 0, T\rangle \times R^n \times R^n$ being given, we say that $h(x)$ belongs to the space $L_\varrho^p(\Delta_t)$, $1 \leqslant p < \infty$, if for any fixed $(t, x) \in \bar{D}$ and $1 \leqslant k \leqslant N$ the norm

$$(2.1) \qquad \|h\|_{p,k}(t, x) = \Big(\int_{\Delta_t} \sum_{l=1}^{N} |h^l(y)|^p \varrho_l^k(t, x, y)\, dy\Big)^{1/p}$$

is finite. In $L_\varrho^p(\Delta_t)$ the following partial order is introduced: for $h = (h^1(x), \ldots, h^N(x)) \in L_\varrho^p(\Delta_t)$, $\tilde{h} = (\tilde{h}^1(x), \ldots, \tilde{h}^N(x)) \in L_\varrho^p(\Delta_t)$ the inequality $h \leqslant \tilde{h}$ means that $h^k(x) \leqslant \tilde{h}^k(x)$, $x \in \Delta_t$ $(k = 1, \ldots, N)$.

Considering inequalities of type (0.1) in D, we always assume the following:

I. The coefficients a_{ij}^k $(a_{ij}^k = a_{ji}^k)$, b_j^k $(i, j = 1, \ldots, n;\ k = 1, \ldots, N)$ given in D can be extended to the whole strip S so that the extended coefficients (denoted also by a_{ij}^k, b_j^k) have the derivatives $(a_{ij}^k)_{x_j}$, $(b_j^k)_{x_j}$ at each point of S, the extended coefficients and the derivatives being measurable and bounded in any finite cylinder $S \cap (|x| < r)$.

II. $\sum_{i,j=1}^{n} a_{ij}^k(t, x)\xi_i\xi_j \geqslant 0$ for $(t, x) \in D$, $(\xi_1, \ldots, \xi_n) \in R^n$ $(k = 1, \ldots, N)$.

III. There exists a matrix $\varrho = (\varrho_l^k(t, x, y))$ $(k, l = 1, \ldots, N)$ of continuous and non-negative functions in $\langle 0, T\rangle \times R^n \times R^n$ and a number $p \geqslant 1$ such that the functionals $f^k(t, x, u, h)$ $(k = 1, \ldots, N)$ are defined for $(t, x) \in D$, $u = (u^1, \ldots, u^N)$ arbitrary and for $h \in L_\varrho^p(\Delta_t)$. Moreover, each f^k is, for almost all $(t, x) \in D$, non-decreasing in $u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^N, h$.

IV. There exist functions $c_l^k(t,x)$, $c_l^k \geqslant 0$ for $l \neq k$, continuous in S and such that for $\tilde{u} \leqslant u$, $\tilde{h} \leqslant h$ the inequalities

$$f^k(t,x,u,h) - f^k(t,x,\tilde{u},\tilde{h}) \leqslant \sum_{l=1}^{N} c_l^k(t,x)(u^l - \tilde{u}^l) + \|h-\tilde{h}\|_{p,k}(t,x)$$

($k = 1, \ldots, N$) hold almost everywhere in D.

For $k = 1, \ldots, N$ and $\Phi = \big(\Phi^1(t,x), \ldots, \Phi^N(t,x)\big) \geqslant 0$ we define

$$L^k(u^k) = \sum_{i,j=1}^{n} \big(a_{ij}^k(t,x)u_{x_i}^k\big)_{x_j} + \sum_{j=1}^{n} b_j^k(t,x)u_{x_j}^k, \tag{2.2}$$

$$\overset{*}{L}{}^k(\Phi^k) = \sum_{i,j=1}^{n} \big(a_{ij}^k(t,x)\Phi_{x_i}^k\big)_{x_j} - \sum_{j=1}^{n} \big(b_j^k(t,x)\Phi^k\big)_{x_j}, \tag{2.3}$$

$$I^k(\Phi) = \int_{\Delta_t} \sum_{l=1}^{N} \Phi^l(t,y)\varrho_k^l(t,y,x)\,dy, \tag{2.4}$$

$$\Lambda^k(\Phi) = \overset{*}{L}{}^k(\Phi^k) + \sum_{l=1}^{N} c_k^l\Phi^l + (p-1)\Big(\sum_{l=1}^{N} c_l^k + 1\Big)\Phi^k + \Phi_t^k + I^k(\Phi), \tag{2.5}$$

$$\begin{aligned}(2.6)\qquad & \Psi^k(t,x) \\ &= \max_i \sum_{j=1}^{n} |a_{ij}^k\Phi_{x_j}^k| + \Phi^k\Big(\max_{i,j}|a_{ij}^k| + \max_i\Big|\sum_{j=1}^{n}(a_{ij}^k)_{x_j} - b_i^k\Big|\Big) + I^k(\Phi).\end{aligned}$$

We also assume

V. There exists a function $\Phi(t,x) = \big(\Phi^1(t,x), \ldots, \Phi^N(t,x)\big)$ of class $C^2(\bar{S})$ such that for $k = 1, \ldots, N$ we have $\Phi^k(t,x) > 0$ in D,

$$I^k(\Phi) < \infty \quad \text{for } (t,x) \in \bar{D}, \tag{2.7}$$

and the integro-differential inequalities

$$\Lambda^k(\Phi) \leqslant 0 \tag{2.8}$$

hold true almost everywhere in D.

For vectors $u = (u^1, \ldots, u^N)$, $v = (v^1, \ldots, v^N)$ we write

$$u \leqslant v \quad \text{iff} \quad u^k \leqslant v^k \quad (k = 1, \ldots, N).$$

3. Differential-functional inequalities.

THEOREM 1. *Suppose $u(t,x) = \big(u^1(t,x), \ldots, u^N(t,x)\big)$, $v(t,x) = \big(v^1(t,x), \ldots, v^N(t,x)\big)$ are continuous functions in the closure $\bar{D}$ of D and $u(t,x) \leqslant v(t,x)$ on Σ.*

Let

$$G^l = \{(t,x) \in D\colon\ u^l(t,x) > v^l(t,x)\}.$$

Assume that the derivatives $u_t^k, u_{x_i}^k, u_{x_ix_j}^k, v_t^k, v_{x_i}^k, v_{x_ix_j}^k$ $(i, j = 1, \ldots, n)$ *exist in* G^k, *are measurable and bounded in every set* $G^k \cap (|x| < r)$, $r > 0$, *and that the differential-functional inequalities*

$$u_t^k(t, x) \leqslant L^k(u^k)(t, x) + f^k\big(t, x, u(t, x), u(t, \cdot)\big), \tag{3.1}$$

$$v_t^k(t, x) \geqslant L^k(v^k)(t, x) + f^k\big(t, x, v(t, x), v(t, \cdot)\big) \tag{3.2}$$

are satisfied whenever $(t, x) \in G^k$.

Let assumptions I–V *concerning the operators* L^k *and the functionals* f^k *be fulfilled and let* $u(t, \cdot), v(t, \cdot) \in L_\varrho^p(\Delta_t)$.

Moreover, assume that

$$\iint_D \big[\big(u^k(t, x) - v^k(t, x)\big)^+\big]^p \Psi^k(t, x)\,dt\,dx < \infty. \tag{3.3}$$

Under these assumptions we have

$$u(t, x) \leqslant v(t, x) \quad \textit{in } D.$$

Proof. Let $\tilde{u} = (\tilde{u}^1, \ldots, \tilde{u}^N)$, $\tilde{v} = (\tilde{v}^1, \ldots, \tilde{v}^N)$ be continuous extensions of u, v, respectively, to the whole strip $\bar{S}$ such that $\tilde{u} \leqslant \tilde{v}$ in $S \setminus D$. We define

$$z^k = \big((w^k)^3 + \varepsilon\big)^{p/3}, \tag{3.4}$$

where $\varepsilon > 0$, $p \geqslant 1$, $w^k = (u^k - v^k)^+ = \max(0, \tilde{u}^k - \tilde{v}^k)$ for $(t, x) \in S$. At points of the set G^k we have, by the lemma,

$$L^k(z^k) - z_t^k = p(z^k)^{(p-3)/p}(w^k)^2\big(L^k(w^k) - w_t^k\big) + \\ + p\,[(p-1)(w^k)^3 + 2\varepsilon](z^k)^{(p-6)/p} w^k \sum_{i,j=1}^n a_{ij}^k w_{x_i}^k w_{x_j}^k.$$

Hence, by inequalities (3.1), (3.2) and their parabolicity, we obtain

$$L^k(z^k) - z_t^k \geqslant -p(z^k)^{(p-3)/p}(w^k)^2\big[f^k\big(t, x, u(t, x), u(t, \cdot)\big) - \\ - f^k\big(t, x, v(t, x), v(t, \cdot)\big)\big]. \tag{3.5}$$

In the set G^k we have $u^l(t, x) \leqslant v^l(t, x) + w^l(t, x)$ $(l = 1, \ldots, N)$, $u^k(t, x) = v^k(t, x) + w^k(t, x)$ and in the space $L_\varrho^p(\Delta_t)$ we have $u(t, \cdot) \leqslant v(t, \cdot) + w(t, \cdot)$, where $w = (w^1, \ldots, w^N)$. Applying successively assumptions III, IV we get

$$f^k\big(t, x, u(t, x), u(t, \cdot)\big) - f^k\big(t, x, v(t, x), v(t, \cdot)\big) \\ \leqslant f^k\big(t, x, v(t, x) + w(t, x), v(t, \cdot) + w(t, \cdot)\big) - f^k\big(t, x, v(t, x), v(t, \cdot)\big) \\ \leqslant \sum_{l=1}^N \sigma_l^k w^l + \|w\|_{p,k}$$

which together with (3.5) gives

$$(3.6)\qquad L^k(z^k)-z_t^k \geqslant -p(z^k)^{(p-3)/p}(w^k)^2\Big(\sum_l c_l^k w^l+\|w\|_{p,k}\Big).$$

Inequality (3.6) is derived for (almost all) points $(t, x)\in G^k$; however, it is clear that it holds true also in $S\setminus G^k$.

Now we make use of the identity

$$(3.7)\qquad \sum_k (z^k\varphi^k)_t = \sum_k \big[z^k\big(\overset{*}{L}{}^k(\varphi^k)+\varphi_t^k\big)-\varphi^k\big(L^k(z^k)-z_t^k\big)\big]+ \\ +\sum_k\sum_i\Big[\varphi^k\sum_j a_{ij}^k z_{x_j}^k - z^k\sum_j a_{ij}^k\varphi_{x_j}^k + b_i^k z^k\varphi^k\Big]_{x_i},$$

in which we set for z^k the functions defined by (3.4) and for $\varphi = (\varphi^1, \ldots, \varphi^N)$ the product $\gamma^r\Phi$, where Φ appears in the assumptions of the theorem, $\gamma^r(x)$ $(r>0)$ being a function of class $C^2(R^n)$ such that $\gamma^r = 1$ for $|x|\leqslant r$, $\gamma^r = 0$ for $|x|\geqslant r+1$, $0\leqslant\gamma^r\leqslant 1$ in R^n and the first and second derivatives of γ^r are bounded in R^n by a constant independent of r.

By assumption I and by the lemma, we may integrate (3.7) over the strip $(0, t_0)\times R^n$, $t_0\in(0, T)$, obtaining

$$(3.8)\qquad \int_{R_n}\sum_k z^k\varphi^k|_{t=t_0}dx = \int_{R_n}\sum_k z^k\varphi^k|_{t=0}dx+ \\ +\int_0^{t_0}dt\int_{R_n}\sum_k\big[z^k\big(\overset{*}{L}{}^k(\varphi^k)+\varphi_t^k\big)-\varphi^k\big(L^k(z^k)-z_t^k\big)\big]dx.$$

Hence, keeping (3.6) in mind, we get

$$(3.9)\qquad \int_{R_n}\sum_k z^k\varphi^k|_{t=t_0}dx \leqslant \int_{R_n}\sum_k z^k\varphi^k|_{t=0}dx+ \\ +\int_0^{t_0}dt\int_{\Delta_t}\sum_k \varphi^k p(z^k)^{(p-3)/p}(w^k)^2\Big(\sum_l c_l^k w^l+\|w\|_{p,k}\Big)dx+ \\ +\int_0^{t_0}dt\int_{R_n}\sum_k z^k\big(\overset{*}{L}{}^k(\varphi^k)+\varphi_t^k\big)dx.$$

Now, $c_l^k\geqslant 0$ for $l\neq k$ implies

$$(3.10)\qquad (z^k)^{(p-3)/p}(w^k)^2\sum_l c_l^k w^l \leqslant (z^k)^{(p-1)/p}\sum_l c_l^k(z^l)^{1/p}-\varepsilon c_k^k(z^k)^{(p-3)/p}.$$

By Young's inequality we have

$$(3.11)\qquad p(z^k)^{(p-1)/p}(z^l)^{1/p}\leqslant(p-1)z^k+z^l,$$

which for $l = k$ turns into equality. We also have

(3.12) $$p(z^k)^{(p-3)/p}(w^k)^2\|w\|_{p,k} \leqslant p(z^k)^{(p-1)/p}\|w\|_{p,k} \leqslant (p-1)z^k + \|w\|_{p,k}^p.$$

By (3.10), (3.11), (3.12) we obtain from (3.9), by letting $\varepsilon \to 0$,

$$\int_{R_n}\sum_k (w^k)^p \varphi^k|_{t=t_0} dx \leqslant \int_0^{t_0} dt \int_{\Delta_t}\sum_k \Big\{(w^k)^p\big(\overset{*}{L}{}^k(\varphi^k)+\varphi_t^k\big)+$$
$$+\varphi^k\Big[\sum_l c_l^k (w^l)^p + (p-1)\Big(\sum_l c_l^k + 1\Big)(w^k)^p + \|w\|_{p,k}^p\Big]\Big\}dx,$$

which can be written as follows:

(3.13) $$\int_{R_n}\sum_k [w^k(t_0, x)]^p \varphi^k(t_0, x)\,dx \leqslant \iint_D \sum_k [w^k(t, x)]^p \Lambda^k(\varphi)(t, x)\,dt\,dx,$$

where $\Lambda^k(\varphi)$ is defined by (2.5). Setting $\varphi = \gamma^r \Phi$ yields

(3.14) $$\Lambda^k(\varphi) = \gamma^r \Lambda^k(\Phi) + 2\sum_{i,j} a_{ij}^k \gamma_{x_i}^r \Phi_{x_j}^k + \Phi^k\Big\{\sum_{i,j} a_{ij}^k \gamma_{x_i x_j}^r +$$
$$+\sum_i\Big[\sum_j (a_{ij}^k)_{x_j} - b_i^k\Big]\gamma_{x_i}^r\Big\} + I^k(\gamma^r\Phi) - \gamma^r(x) I^k(\Phi).$$

By assumptions V, (3.3) and by (3.14) and the properties of γ^r, it follows that the upper limit (as $r \to \infty$) of the integral on the right-hand side of (3.13) is less than or equal to zero. Thus

$$\int_{R_n}\sum_k [w^k(t_0, x)]^p \Phi^k(t_0, x)\,dx \leqslant 0 \qquad \text{for } t_0 \in (0, T),$$

whence $w^k(t, x) = 0$ in S $(k = 1, \ldots, N)$ and the proof is complete.

4. Some corollaries.

Remark 1. If the functionals $f^k(t, x, u, h)$ are independent of the functional argument h, Theorem 1 (and, similarly, subsequent theorems) is concerned with differential inequalities of parabolic type. The result is obtained by setting $\varrho_l^k(t, x, y) \equiv 0$ in Theorem 1 and constitutes a generalization of a result of [1].

DEFINITION. By a solution of a system of inequalities (or equations) of type (0.1) in D we mean a function $u(t, x) = \big(u^1(t, x), \ldots, u^N(t, x)\big)$ continuous in the closure $\bar{D}$, having the derivatives $u_t^k, u_{x_i}^k, u_{x_i x_j}^k$ $(i, j = 1, \ldots, n;\ k = 1, \ldots, N)$ in D, which are measurable and bounded in any bounded set $D \cap (|x| < r)$, and satisfying the system in D.

An immediate consequence of Theorem 1 is the following

MAXIMUM PRINCIPLE. *Let* $u(t, x) = \big(u^1(t, x), \ldots, u^N(t, x)\big)$ *be a solution in* D *of system* (3.1) *and let assumptions* I–V *be fulfilled with* ϱ

satisfying the condition

$$\int_{\Delta_t} \varrho_l^k(t, x, y)\,dy < \infty \quad \textit{for } (t, x) \in D \ (k, l = 1, \ldots, N).$$

Suppose that $u(t, \cdot) \in L_\varrho^p(\Delta_t)$. *If, moreover, for some constant vector* $M = (M^1, \ldots, M^N) \geqslant 0$ *we have*

$$u(t, x) \leqslant M \quad \textit{on} \quad \Sigma, \quad f^k(t, x, M, M) \leqslant 0, \quad (t, x) \in D \quad (k = 1, \ldots, N)$$

and

$$\iint_D \left[(u^k(t, x) - M^k)^+\right]^p \Psi(t, x)\,dt\,dx < \infty \quad (k = 1, \ldots, N),$$

then

$$u(t, x) \leqslant M \quad \textit{in } D.$$

For the proof one can easily verify that the functions u and $v = M$ satisfy all the assumptions of Theorem 1.

A particular case of Theorem 1 is the following

THEOREM 2. *Assume that* $u(t, x), v(t, x)$ *are solutions, in* D, *of systems* (3.1), (3.2) *respectively, such that*

$$u(t, x) \leqslant v(t, x) \quad \textit{on } \Sigma$$

and that

$$\int_{\Delta_t} \left(|u^k(t, x)|^p + |v^k(t, x)|^p\right) \exp\{-K(|x|^2+1)^{\lambda/2}\}\,dx \leqslant \bar{K}, \tag{4.1}$$

for $0 \leqslant t \leqslant T$, $k = 1, \ldots, N$ *and for some constants* $K \geqslant 0$, $\bar{K} > 0$, $\lambda > 0$, $p \geqslant 1$. *Let assumptions* I–IV *be satisfied with*

$$\varrho_l^k(t, x, y) \leqslant M \exp\{K(|x|^2+1)^{\lambda/2} - K_1(|y|^2+1)^{\lambda/2}\}, \tag{4.2}$$

where $M \geqslant 0$, $K_1 > K$ *are constants and let* $u(t, \cdot), v(t, \cdot) \in L_\varrho^p(\Delta_t)$. *Moreover, assume there is a constant* $C > 0$ *such that the inequalities*

$$|a_{ij}^k| \leqslant C(|x|^2+1)^{(2-\lambda)/2}, \quad \Big|\sum_j (a_{ij}^k)_{x_j} - b_i^k\Big| \leqslant C(|x|^2+1)^{1/2}, \tag{4.3}$$

$$c_l^k, \ -(b_i^k)_{x_i} \leqslant C(|x|^2+1)^{\lambda/2}$$

hold true almost everywhere in S.

Then $u(t, x) \leqslant v(t, x)$ *in* D.

Proof. We have only to check assumptions V and (3.3). To this effect choose

$$\Phi^k(t, x) = \exp -\left\{\frac{K_0}{1-\mu t}(|x|^2+1)^{\lambda/2}\right\}, \tag{4.4}$$

where $K < K_0 < K_1$,

(4.5) $\mu = C[nK_0\lambda^2 + n\lambda|\lambda-2| + \sqrt{n}\lambda + (n+pN)/K_0] + (p-1+\bar{M})/K_0$

and

$$\bar{M} = MN\int_{R_n} \exp\{(K-K_0)(|x|^2+1)^{\lambda/2}\}\,dx.$$

We first show that Theorem 2 is true for the domain

$$D^1 = D\cap[(0, T_1)\times R^n], \quad \text{where} \quad T_1 = \min\left(T, \frac{K_1-K_0}{K_1\mu}\right).$$

By (4.2), (4.4) we obtain

$$I^k(\Phi) \leqslant \bar{M}\exp\{-K_1(|x|^2+1)^{\lambda/2}\}.$$

One can verify that assumption (4.3) together with (4.4), (4.5) imply $\Lambda^k(\Phi) \leqslant 0$ in D^1 (we omit the details). Furthermore, it can easily be shown that (4.1) implies (3.3).

Now the proof can be extended similarly to the domains $D^{i+1} = D\cap \cap\big((T_i, 2T_i)\times R^n\big)$, $i = 1, 2, \ldots$

5. Estimates. Let $\sigma^k(t, x, u, s)$ $(k = 1, \ldots, N)$ be functions subject to the following conditions:

VI. Each σ^k is defined for $(t, x)\in D$, arbitrary $u = (u^1, \ldots, u^N)$, $s = (s^1, \ldots, s^N)$ and is non-decreasing in $u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^N, s^1, \ldots, s^N$.

VII. For $\tilde{u} \leqslant u$, $\tilde{s} \leqslant s$ we have

$$\sigma^k(t, x, u, s) - \sigma^k(t, x, \tilde{u}, \tilde{s}) \leqslant \sum_{l=1}^{N}[c_l^k(t, x)(u^l-\tilde{u}^l) + (s^l-\tilde{s}^l)] \quad (k = 1, \ldots, N)$$

for almost all $(t, x)\in D$, where c_l^k are some continuous functions in $\bar{S}$ such that $c_l^k \geqslant 0$ for $l \neq k$.

Remark 2. Notice that if σ^k satisfy VI, VII, then the functionals

$$f^k(t, x, u, h) := \sigma^k\Big(t, x, u, \int_{\Delta_t} h^1(y)\,\varrho_1^k(t, x, y)\,dy, \ldots$$
$$\ldots, \int_{\Delta_t} h^N(y)\,\varrho_N^k(t, x, y)\,dy\Big),$$

$(k = 1, \ldots, N)$, $h = (h^1, \ldots, h^N) \in L_\varrho^1(\Delta_t)$, satisfy conditions III and IV stated in Section 2. It follows that Theorem 1 involves some integro-differential inequalities.

THEOREM 3. *Assume that* $u(t, x) = \big(u^1(t, x), \ldots, u^N(t, x)\big)$, $v(t, x) = \big(v^1(t, x), \ldots, v^N(t, x)\big)$ *are solutions, in* D, *of the systems of equations*

$$\begin{aligned} &(5.1)\quad u_t^k(t, x) = L^k(u^k)(t, x) + f^k\big(t, x, u(t, x), u(t, \cdot)\big) \\ &(5.2)\quad v_t^k(t, x) = L^k(v^k)(t, x) + g^k\big(t, x, v(t, x), v(t, \cdot)\big) \end{aligned} \qquad (k = 1, \ldots, N),$$

respectively, $u(t,\cdot)$, $v(t,\cdot)$ *being elements of* $L^p_\varrho(\Delta_t)$ *for some* $\varrho^k_l(t,x,y) \geqslant 0$, *where* $f^k(t,x,u,h)$, $g^k(t,x,u,h)$ $(k=1,\ldots,N)$ *are functionals defined for* $(t,x)\in D$, $u=(u^1,\ldots,u^N)$ *arbitrary and* $h=(h^1,\ldots,h^N)\in L^p_\varrho(\Delta_t)$, *such that*

$$(5.3)\qquad [f^k(t,x,u,h)-g^k(t,x,\tilde u,\tilde h)]\operatorname{sgn}(u^k-\tilde u^k) \leqslant \sigma^k(t,x,|u^1-\tilde u^1|,\ldots,|u^N-\tilde u^N|,\|h^1-\tilde h^1\|_{p,k},\ldots,\|h^N-\tilde h^N\|_{p,k})$$

$(k=1,\ldots,N)$; *here*

$$\|h^l-\tilde h^l\|_{p,k} = N^{-1}\Big(\int\limits_{\Delta_t}|h^l(y)-\tilde h^l(y)|^p\varrho^k_l(t,x,y)\,dy\Big)^{1/p},$$

$\operatorname{sgn} z=1$ *for* $z\geqslant 0$, $\operatorname{sgn} z=-1$ *for* $z<0$. $\sigma^k(t,x,u,s)$ $(k=1,\ldots,N)$ *are functions satisfying conditions* VI *and* VII.

Let assumptions I, II *and* V *of Section* 2 *be satisfied and let* $\omega(t,x) = (\omega^1(t,x),\ldots,\omega^N(t,x))$ *be a non-negative solution, in* D, *of the system of integro-differential inequalities*

$$(5.4)\qquad \omega^k_t(t,x)\geqslant L^k(\omega^k)(t,x)+\sigma^k\big(t,x,\omega(t,x),\|\omega^1\|_{p,k}(t,x),\ldots \ldots,\|\omega^N\|_{p,k}(t,x)\big)$$

$(k=1,\ldots,N)$ *such that* $\omega(t,\cdot)\in L^p_\varrho(\Delta_t)$ *and*

$$|u^k(t,x)-v^k(t,x)|\leqslant\omega^k(t,x)\qquad(k=1,\ldots,N)$$

for $(t,x)\in\Sigma$. *If, moreover,*

$$(5.5)\qquad \iint\limits_D |u^k(t,x)-v^k(t,x)|^p\Psi^k(t,x)\,dt\,dx<\infty,$$

then

$$(5.6)\qquad |u^k(t,x)-v^k(t,x)|\leqslant\omega^k(t,x)\qquad(k=1,\ldots,N)\ \textit{in}\ D.$$

Proof. Denote $w^k=|u^k-v^k|$, $w=(w^1,\ldots,w^N)$ and

$$G^l=\{(t,x)\in D\colon\ w^l(t,x)>\omega^l(t,x)\}.$$

Suppose that for some k the set G^k is not empty. Since $w^k>0$ in G^k, the derivatives w^k_t, $w^k_{x_i}$, $w^k_{x_ix_j}$ exist in G^k and are measurable and bounded in any bounded set $G^k\cap(|x|<r)$. Consequently, by (5.1), (5.2) we get

$$w^k_t=L^k(w^k)+\big[f^k\big(t,x,u(t,x),u(t,\cdot)\big)-g^k\big(t,x,v(t,x),v(t,\cdot)\big)\big]\operatorname{sgn}w^k$$

whence, by (5.3), we obtain

$$w^k_t\leqslant L^k(w^k)+\sigma^k\big(t,x,w(t,x),\|w^1\|_{p,k}(t,x),\ldots,\|w^N\|_{p,k}(t,x)\big)$$

for $(t,x)\in G^k$. It follows that all the assumptions of Theorem 1 with u, v replaced by w, ω respectively, are satisfied. Thus we conclude that $w\leqslant\omega$ in D, which was to be proved.

The estimate of the solution itself stated in the next theorem can be deduced from Theorem 1 in a similar way.

THEOREM 4. *Suppose that* $u(t,x) = (u^1(t,x), \ldots, u^N(t,x))$ *is a solution of system* (5.1) *in* D, $u(t,\cdot) \in L^p_\varrho(\Delta_t)$, *where* $f^k(t,x,u,h)$ *are functionals defined for* $(t,x) \in D$, $u = (u^1, \ldots, u^N)$ *arbitrary,* $h = (h^1, \ldots, h^N) \in L^p_\varrho(\Delta_t)$ *and satisfy the inequalities*

$$f^k(t,x,u,h)\operatorname{sgn} u^k \leqslant \sigma^k(t,x,|u^1|,\ldots,|u^N|,\|h^1\|_{p,k},\ldots,\|h^N\|_{p,k})$$

$(k = 1,\ldots,N)$, $\sigma^k(t,x,u,s)$ *being functions satisfying conditions* VI *and* VII. *Let assumptions* I, II *and* V *of Section* 2 *be fulfilled and let* $\omega(t,x) = (\omega^1(t,x),\ldots,\omega^N(t,x))$ *be a non-negative solution, in* D, *of system* (5.4) *such that*

$$|u^k(t,x)| \leqslant \omega^k(t,x) \quad (k = 1,\ldots,N) \quad \textit{for } (t,x) \in \Sigma.$$

Moreover, assume that

$$\iint_D |u^k(t,x)|^p \Psi(t,x)\,dt\,dx < \infty \quad (k = 1,\ldots,N).$$

Under these assumptions we have

$$|u^k(t,x)| \leqslant \omega^k(t,x) \quad (k = 1,\ldots,N) \quad \textit{for } (t,x) \in D.$$

The uniqueness of a solution of the first Fourier problem for parabolic differential-functional equations is the objective of the following

THEOREM 5. *Let* $u(t,x)$, $v(t,x)$ *be solutions of system* (5.1) *in* D *such that* $u = v$ *on* Σ *and* $u(t,\cdot), v(t,\cdot) \in L^p_\varrho(\Delta_t)$, *where* $p \geqslant 1$, $\varrho = (\varrho^k_l(t,x,y))$ $(k,l = 1,\ldots,N)$, ϱ^k_l *being non-negative functions continuous in* $\langle 0,T\rangle \times R^n \times R^n$. *Suppose that* $f^k(t,x,u,h)$ *are defined for* $(t,x) \in D$, $u = (u^1, \ldots, u^N)$ *arbitrary,* $h = (h^1,\ldots,h^N) \in L^p_\varrho(\Delta_t)$ *and satisfy the inequalities*

$$[f^k(t,x,u,h) - f^k(t,x,\tilde{u},\tilde{h})]\operatorname{sgn}(u^k - \tilde{u}^k) \leqslant \sum_{l=1}^N c^k_l(t,x)|u^k - \tilde{u}^k| + \|h - \tilde{h}\|_{p,k}(t,x) \quad (k = 1,\ldots,N),$$

where c^k_l *are some continuous functions in* $\bar{S}$ *and* $c^k_l \geqslant 0$ *for* $l \neq k$. *Assume that conditions* I, II, V *of Section* 2 *and condition* (5.5) *hold true.*

Then $u(t,x) \equiv v(t,x)$ *in* D.

Proof. It is sufficient to note that the function $\omega(t,x) \equiv 0$ satisfies in D the system

$$\omega^k_t(t,x) \geqslant L^k(\omega^k)(t,x) + \sum_{l=1}^N c^k_l(t,x)\omega^l(t,x) + \|\omega\|_{p,k}(t,x)$$

$(k = 1,\ldots,N)$ and to apply Theorem 3.

In particular, Theorems 3–5 contain some results on parabolic differential equations.

References

[1] P. Besala, *Function classes pertaining to differential inequalities of parabolic type in unbounded regions*, Ann. Polon. Math. 25 (1972), p. 281–291.

[2] — *On L^p-estimates for solutions of the Cauchy problem for parabolic differential equations*, ibidem 25 (1971), p. 179–185.

[3] A. Sobolewska, *Sur un système d'inégalités différentielles partielles du second ordre à argument fonctionnel*, ibidem 25 (1971), p. 103–108.

[4] J. Szarski, *Sur un système non linéaire d'inégalités différentielles paraboliques contenant des fonctionnelles*, Colloq. Math. 16 (1967), p. 141–145.

[5] — *Uniqueness of solutions of a mixed problem for parabolic differential-functional equations*, Ann. Polon. Math. 28 (1973), p. 57–65.

[6] — *Strong maximum principle for non-linear parabolic differential-functional inequalities*, ibidem 29 (1974), p. 207–214.

Reçu par la Rédaction le 30. 10. 1973