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A note on the coefficients of univalent functions

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Let $f \in S$, i.e., let f be regular and univalent in $|z| < 1$ and have the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n;$$

for integral t, $-\infty < t < \infty$, let

$$f(z)^t = z^t + \sum_{n=t+1}^{\infty} a_n^{(t)} z^n .$$

It is well known ([3]) that if f is extremal for the problem $\max\limits_{f\in S} |a_n|$, then

$$(n+1)a_{n+1} = 2a_2 a_n + (n-1)\bar{a}_{n-1} . \tag{1}$$

The following interesting generalization of (1) is easily obtained.

THEOREM. *If $f \in S$ is extremal for the problem $\max\limits_{f\in S} |a_n^{(t)}|$, then*

$$(n+1)a_{n+1}^{(t)} = t(2a_2 a_n^{(t)} + a_n^{(t-1)}) + (n-1)\bar{a}_{n-1}^{(t)} . \tag{2}$$

Proof. For $f \in S$ we make the variation $w^* = w + \varepsilon e^{i\theta}$, $\varepsilon > 0$, $0 \leqslant \theta \leqslant 2\pi$, of the image domain under f. Then for small ε

$$f^*(z) = f(z) + \varepsilon e^{i\theta}\left\{2a_2 f(z) + 1 - zf'(z)\left(\frac{1}{z} - \bar{z}\right)\right\} + O(\varepsilon^2) \tag{3}$$

belongs to S ([5]). We use (3) to obtain a variational formula for the tth power of $f \in S$. Let

$$k(z) = \left\{2a_2 f(z) + 1 - zf'(z)\left(\frac{1}{z} - \bar{z}\right)\right\},$$

then

$$\begin{aligned} f^*(z)^t &= [f(z) + \varepsilon e^{i\theta} k(z) + O(\varepsilon^2)]^t \\ &= f(z)^t + \varepsilon e^{i\theta} t f(z)^{t-1} k(z) + O(\varepsilon^2) \\ &= f(z)^t + \varepsilon e^{i\theta}\left\{2ta_2 f(z)^t + tf(z)^{t-1} - z\big(f(z)^t\big)'\left(\frac{1}{z} - \bar{z}\right)\right\} + O(\varepsilon^2). \end{aligned} \tag{4}$$

Instead of attacking the problem $\max_{f \in S} |a_n^{(t)}|$, $n = t+1, \ldots$, we may assume $a_n^{(t)}$ is real ([5]) and thus consider the problem

$$\max_{f \in S} \operatorname{Re}(a_n^{(t)}), \quad n = t+1, \ldots \tag{5}$$

Thus suppose $f \in S$ is extremal for (5); by comparing coefficients in (4) and taking real parts we see that

$$\begin{aligned} &\operatorname{Re}\{a_n^{(t)*} - a_n^{(t)}\} \\ &\quad = \operatorname{Re}\left\{\varepsilon e^{i\theta}\left(2ta_2 a_n^{(t)} + ta_n^{(t-1)} - (n+1)a_{n+1}^{(t)} + (n-1)\bar{a}_{n-1}^{(t)}\right) + O(\varepsilon^2)\right\} \leqslant 0 \end{aligned}$$

for all values of $\theta \in [0, 2\pi]$. This implies (2) and thus completes the proof of the theorem.

For $f \in S$ let $G = f\ (|z| < 1)$. Suppose $D = \{w \mid w = 1/\zeta,\ \zeta \in G\}$ and let E be the complement of D in the w-plane. Define the moments

$$s_n = \int w^n d\mu, \quad n = 1, 2, \ldots,$$

where μ is the natural mass distribution on E. Let

$$\tilde{f}(z) = 1/f(1/z) = z + \sum_{n=0}^{\infty} \tilde{a}_n z^{-n}$$

($\tilde{f}$ is regular, univalent and nonzero in $|z| > 1$, i.e., $f \in \Sigma$) and let

$$\varphi(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad \text{and} \quad \tilde{\varphi}(w) = w + \sum_{n=0}^{\infty} \tilde{b}_n w^{-n}$$

be the inverses of f and $\tilde{f}$ respectively. It is known ([1], [4]) that

1. $\tilde{a}_n = a_n^{(-1)}, \qquad n = 0, 1, \ldots,$
2. $b_n = \frac{1}{n} a_{-1}^{(-n)}, \qquad n = 1, 2, \ldots,$
3. $s_n = a_0^{(-n)}, \qquad n = 1, 2, \ldots,$
4. $\tilde{b}_n = -\frac{1}{n} a_1^{(-n)}, \qquad n = 1, 2, \ldots$

Therefore, making the appropriate substitution in (2), we have the following results.

I. *If $\tilde{f} \in \Sigma$ is extremal for the problem* $\max |\tilde{a}_n|$, *then*

$$(n+1)\tilde{a}_{n+1} = 2\tilde{a}_0 \tilde{a}_n - \sum_{\nu=-1}^{n+1} \tilde{a}_\nu \tilde{a}_{n-\nu} + (n-1)\bar{\tilde{a}}_{n-1}, \quad \tilde{a}_{-1} = 1 .$$

II. *If $\varphi \in S^{-1}$ is extremal for the problem* $\max|b_n|$, *then*

$$(n+1)b_{n-1} = 2nb_2b_n - \sum_{\nu=1}^{n-1} \bar{b}_\nu \bar{b}_{n-\nu}, \quad b_1 = 1.$$

III. *If $f \in S$ is extremal for the problem* $\max|s_n|$, *then*

$$a_1^{(-n)} = n(2s_1s_n - s_{n+1}) - a_{-1}^{(-n)}.$$

IV. *If $\tilde{\varphi} \in \Sigma^{-1}$ is extremal for the problem* $\max|\tilde{b}_n|$, *then*

$$(n+1)\tilde{b}_{n+1} = 2n\tilde{b}_0\tilde{b}_n - \sum_{\nu=-1}^{n+1} \tilde{b}_\nu \tilde{b}_{n-\nu}, \quad \tilde{b}_{-1} = 1.$$

In obtaining II and IV we have used this fact ([2]) that if φ is the inverse of f and $\varphi(w)^t = \sum_{\nu=t}^{\infty} b_\nu^{(t)} w^t$, then $b_\nu^{(t)} = \frac{t}{\nu} a_{-t}^{(-\nu)}$, $\nu \neq 0$.

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