ANNALES
POLONICI MATHEMATICI
XXIII (1970)

On the partial sums of certain analytic functions in the unit disc

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The problem of determining to what extent a given property of a power series is carried over to its partial sums has interested several authors. G. Szegö [5] has shown that all the partial sums of a function regular and univalent in the unit disc are themselves univalent in $|z| < 1/4$ and the partial sums of a function regular and starlike (convex) in $|z| < 1$ are starlike (convex) for $|z| < 1/4$. The object of this paper is to investigate the partial sums of 2-valently starlike and 2-valently close-to-convex analytic functions in the unit disc. Goodman [2] calls a function $F(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$; $1 \leqslant q \leqslant p$; p, q being positive integers, p-valently starlike in $|z| < 1$, if $F(z)$ is regular in the unit disc, satisfying for all r in a certain interval $\varrho < r < 1$ the conditions

$$H(r, \theta) = \mathrm{Re}\{re^{i\theta}F'(re^{i\theta})/F(re^{i\theta})\} > 0, \qquad 0 \leqslant \theta \leqslant 2\pi,$$

$$\int_0^{2\pi} H(r, \theta)\, d\theta = 2\pi p.$$

Goodman [2] has also introduced a p-valently convex function. A function $\varphi(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$; $1 \leqslant q \leqslant p$; p, q being positive integers, is called *p-valently convex* for $|z| < 1$ if $\varphi(z)$ is regular for $|z| < 1$ and if there exists a ϱ, $0 < \varrho < 1$, such that for $\varrho < r < 1$, $G(r, \theta) = 1 + \mathrm{Re}\{re^{i\theta}\varphi''(re^{i\theta})/\varphi'(re^{i\theta})\} > 0$, $0 \leqslant \theta \leqslant 2\pi$ and $\int_0^{2\pi} G(r, \theta)\, d\theta = 2\pi p$. Goodman has further proved that if $\varphi(z)$ is p-valently convex in $|z| < 1$, $F(z) = cz\varphi'(z)$, where $c \neq 0$ is any constant, is p-valently starlike for $|z| < 1$ and conversely. Umezawa [6] introduced the concept of p-valently close to convex function as follows. A function $f(z) = z^q + \sum b_n z^n$, $1 \leqslant q \leqslant p$, is called *p-valently close-to-convex* in $|z| < 1$ if there exists a p-valently convex function $\varphi(z)$ in the unit disc such that $\mathrm{Re}\{f'(z)/\varphi'(z)\} > 0$ in $|z| < 1$ or alternately, if there exists a p-valently starlike function $F(z)$ in the unit disc such that $\mathrm{Re}\{zf'(z)/F(z)\} > 0$ in $|z| < 1$. In view of Goodman's result mentioned earlier, the two definitions are equivalent.

In this paper we confine ourselves to the 2-valent case and prove that the partial sums of a 2-valently starlike function of the form $z^2+\sum a_n z^n$ are also 2-valently starlike for $|z|<1/6$ and use this to prove that a similar result is true of the partial sums of a two-valently close to convex function of the same form. Our results are best possible.

We prove the following

THEOREM 1. *Let* $F(z)=z^2+\sum^{\infty} A_n z^n$ *be a 2-valently starlike function for* $|z|<1$. *Then any partial sum* $S_n(z)=z^2+A_3z^3+\ldots+A_nz^n$ *is 2-valently starlike for* $|z|<1/6$ *and the result is sharp.*

Proof. Writing $R_n(z)=A_{n+1}z^{n+1}+A_{n+2}z^{n+2}+\ldots$ we have $F(z)=S_n(z)+R_n(z)$,

$$(1)\qquad \mathrm{Re}\{zS_n'(z)/S_n(z)\}=\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}-\frac{zR_n'(z)-zR_n(z)F'(z)/F(z)}{F(z)-R_n(z)}\right\}$$

$$\geqslant \mathrm{Re}\left\{\frac{zF'(z)}{F(z)}\right\}-\frac{|z|\big(|R_n'(z)|+|R_n(z)|\,|F'(z)/F(z)|\big)}{|F(z)|-|R_n(z)|}.$$

If we set $zF'(z)/F(z)=H(z)$, we observe that $H(z)$ is regular in $|z|<1$, $H(0)=2$ and $\mathrm{Re}H(z)>0$ for $|z|<1$. Hence we have

$$(2)\qquad 2(1-|z|)/(1+|z|)\leqslant \mathrm{Re}\{zF'(z)/F(z)\}\leqslant 2(1+|z|)/(1-|z|).$$

We have also the following estimate

$$(3)\qquad |F(z)|\leqslant r^2/(1+r)^4,\qquad r=|z|<1.$$

By a lemma of Robertson ([4], Lemma 2) we further have

$$(4)\qquad |A_n|\leqslant (n+1)!/\big((n-2)!\,3!\big),\qquad n\geqslant 2.$$

Therefore

$$(5)\qquad |R_n(z)|\leqslant |A_{n+1}|r^{n+1}+|A_{n+2}|r^{n+2}+\ldots$$
$$\leqslant \binom{n+2}{3}r^{n+1}+\binom{n+3}{3}r^{n+2}+\ldots$$
$$=E_1(r),$$

where $E_1(r)=r^2\left\{(1-r)^{-4}-1-4r-\ldots-\binom{n+1}{3}r^{n-2}\right\}$.

Again we have

$$(6)\qquad |R_n'(z)|\leqslant (n+1)|A_{n+1}|r^n+(n+2)|A_{n+2}|r^{n+1}+\ldots$$
$$\leqslant (n+1)\binom{n+2}{3}r^n+(n+2)\binom{n+3}{3}r^{n+1}+\ldots$$
$$\leqslant E_2(r),$$

where

$$E_2(r) = 4r^2\left\{(1-r)^{-5}-1-5r-\ldots-\binom{n+1}{4}r^{n-3}\right\}+$$
$$+2r\left\{(1-r)^{-4}-1-4r-\ldots-\binom{n+1}{3}r^{n-2}\right\}.$$

Also

(7) $$|F(z)|-|R_n(z)| \geqslant r^2/(1+r)^4-E_1(r) = E_3(r), \quad \text{say}.$$

It is easily verified that $E_3(r) > 0$ for $r = 1/6, n \geqslant 3$ by actual computation. Hence it follows from Rouche's theorem that $S_n(z), n > 3$, has the same number of zeros in $|z| < 1/6$ as $F(z)$. Thus $S_n(z)$ does not vanish in $|z| \leqslant 1/6$ except at $z = 0$, where it has a double zero.

Using the above estimates in (1) we obtain

(8) $$\operatorname{Re}\{zS_n'(z)/S_n(z)\} \geqslant \frac{2(1-r)}{(1+r)} - \frac{rE_2(r)+2E_1(r)(1+r)/(1-r)}{E_3(r)} > 0,$$

for $r = 1/6$ and $n \geqslant 5$ by actual verification.

Since $\operatorname{Re}\{zS_n'(z)/S_n(z)\}$ is a harmonic function for $n \geqslant 5$ in $|z| < 1/6$, it assumes its minimum value on the boundary and hence $\operatorname{Re}\{zS_n'(z)/S_n(z)\} > 0$ for $|z| < 1/6$. Thus $S_n(z)$ is starlike for $n \geqslant 5$ in $|z| < 1/6$. To show that $S_n(z)$ is 2-valent in $|z| < 1/6$, we proceed as follows. As already shown, $S_n(z)$ has just two zeros in $|z| < 1/6$, none on $|z| = 1/6$ and $\operatorname{Re}\{zS_n'(z)/S_n(z)\} > 0$ on $|z| = 1/6$. Hence according to a theorem of Ozaki [3], $S_n(z)$ is 2-valent in the disc $|z| < 1/6$. The theorem is, therefore, proved for $n \geqslant 5$. We now proceed to consider the cases $n = 3, 4$. For $n = 3$ we have

$$\begin{aligned}\operatorname{Re}\{zS_3'(z)/S_3(z)\} &= \operatorname{Re}\{(2+3A_3z)/(1+A_3z)\}\\ &\geqslant 2-\{|A_3z|/|(1+A_3z)|\}\\ &\geqslant 2-\{4|z|/(1-4|z|)\}.\end{aligned}$$

Thus $\operatorname{Re}\{zS_3'(z)/S_3(z)\} > 0$ for $|z| < 1/6$. If we choose any $r < 1/6$, we have $\operatorname{Re}\{zS_3'(z)/S_3(z)\} > 0$ for $|z| = r$. Further for $|z| < r$, $|1+A_3z| \geqslant 1-|A_3z| \geqslant 1-4|z| > 1/3$, since $r < 1/6$. Thus $(1+A_3z)$ does not vanish in $|z| < r$ and so $S_3(z) = z^2(1+A_3z)$ has exactly two zeros for $|z| < r$. Hence, from Ozaki's theorem mentioned earlier, we conclude that $S_3(z)$ is 2-valent and starlike in $|z| < r$ for any $r < 1/6$. Next we take up the case $n = 4$. $S_4(z) = z^2(1+A_3z+A_4z^2)$ and $S_4(z)$ vanishes only at $z = 0$ in $|z| < 1/6$, where it has a double zero. Indeed, $|1+A_3z+A_4z^2| \geqslant 1-|A_3|/6-|A|_4/36$ for $|z| < 1/6$. Since $|A_3| \leqslant 4$ and $|A_4| \leqslant 10$, it follows that $1+A_3z+A_4z^2$ does not vanish for $|z| < 1/6$. Hence $zS_4'(z)/S_4(z)$ is regular in $|z| < 1/6$ and so

(9) $$\operatorname{Re}\{zS_4'(z)/S_4(z)\} = \operatorname{Re}\{(2+3A_3z+4A_4z^2)/(1+A_3z+A_4z^2)\}$$

is harmonic in $|z| < 1/6$. We shall now proceed to show that the above expression is positive for $|z| = 1/6$ from which would follow that it is positive for $|z| < 1/6$ also, by the minimum principle for harmonic functions. Again it is sufficient to prove that expression (9) is positive for $z = 1/6$, for we can consider $\bar{\varepsilon}^2 F(\varepsilon z)$ instead of $F(z)$, with a suitable ε such that $|\varepsilon| = 1$. Thus, we need only show that

$$(10) \qquad \operatorname{Re}\{(2+A_3/2+A_4/9)/(1+A_3/6+A_4/36)\} > 0.$$

We have

$$(11) \qquad zF'(z)/F(z) = (2+3A_3z+4A_4z^2+\ldots)/(1+A_3z+A_4z^2+\ldots)$$
$$= 2+B_1z+B_2z^2+\ldots, \quad \text{say}.$$

Since $F(z)$ is starlike for $|z| < 1$, we have

$$\operatorname{Re}\{1+(B_1/2)z+B_2z^2/2+\ldots\} > 0 \quad \text{for } |z| < 1.$$

Hence, by Carathéodory-Toeplitz's theorem, we conclude (see, for example, [1])

$$(12) \qquad |B_2-B_1^2/4| \leqslant 4-|B_1|^2/4.$$

From (11) we obtain

$$B_1+2A_3 = 3A_3,$$
$$B_2+A_3B_1+2A_4 = 4A_4.$$

Hence $B_1 = A_3$ and $B_2 = 2A_4-A_3^2$. So (12) takes the form $|2A_4-5A_3^2/4| \leqslant 4-|A_3|^2/4$. We can, therefore, write

$$(13) \qquad 2A_4-5A_3^2/4 = \varepsilon(4-|A_3|^2/4),$$

where $|\varepsilon| \leqslant 1$. Using (13) in (10), it remains to prove that

$$(14) \qquad \operatorname{Re}\left\{\frac{2+A_3/2+\big(5A_3^2+\varepsilon(16-|A_3|^2)\big)/72}{1+A_3/6+\big(5A_3^2+\varepsilon(16-|A_3|^2)\big)/288}\right\} > 0.$$

This fraction, regarded as a function of ε, is analytic for $|\varepsilon| \leqslant 1$ because

$$\left|\frac{A_3}{6}+\frac{5A_3^2}{288}+\frac{\varepsilon(16-|A_3|^2)}{288}\right| \leqslant \frac{|A_3|}{6}+\frac{5|A_3|^2+16-|A_3|^2}{288} < 1,$$

since $|A_3| \leqslant 4$. Thus the proof of (14) is reduced to that of the following inequality for $|\varepsilon| = 1$, namely,

$$(15) \qquad \operatorname{Re}\{\big(u+\varepsilon(16-|A_3|^2)/72\big)\big(\bar{v}+\bar{\varepsilon}(16-|A_3|^2)/288\big)\} > 0,$$

where we have set

$$u = 2+A_3/2+5A_3^2/72,$$
$$v = 1+A_3/6+5A_3^2/288.$$

Then the left-hand member of (15) becomes

$$\operatorname{Re}(u\bar{v})+(16-|A_3|^2)^2/144^2+(16-|A_3|^2)\operatorname{Re}\{(u+4v)\bar{\varepsilon}\}/288$$
$$\geqslant \operatorname{Re}(u\bar{v})+\big((16-|A_3|^2)/144\big)^2-(16-|A_3|^2)|u+4v|/288$$
$$=(|u+4v|^2-|u-4v|^2)/16+\big((16-|A_3|^2)/144\big)^2-(16-|A_3|^2)|u+4v|/288$$
$$=T_1T_2,$$

where

$$T_1=(|u+4v|+|u-4v|)/4-(16-|A_3|^2)/144,$$
$$T_2=(|u+4v|-|u-4v|)/4-(16-|A_3|^2)/144.$$

Noting that $T_1>T_2$ we have only to show that $T_2>0$. Now

$$|u+4v|=|6+7A_3/6+5A_3^2/36|,$$
$$|u-4v|=|2+A_3/6|.$$

Inequality (15) follows as soon as we show that

$$|6+7A_3/6+5A_3^2/36|-|2+A_3/6|-(16-|A_3|^2)/36>0. \tag{16}$$

Putting $2+A_3/6=re^{i\psi}$, we have $4/3\leqslant r\leqslant 8/3$ since $|A_3|\leqslant 4$. Also, for an arbitrary fixed r in the interval $4/3\leqslant r\leqslant 8/3$, ψ satisfies the inequality

$$|\psi|<\psi_0(r), \tag{17}$$

where $\psi_0(r)$ is determined by the equation $|-2+re^{i\psi}|=2/3$, $0<\psi<\pi/2$, that is,

$$\cos\psi_0(r)=(9r^2+32)/(36r). \tag{18}$$

We can rewrite (16) substituting for A_3 as follows:

$$|6+7(re^{i\psi}-2)+5(re^{i\psi}-2)^2|-r-1/9+(r^2-4r\cos\psi+4)/4>0,$$

that is,

$$36\,|12-13re^{i\psi}+5r^2e^{2i\psi}|+32+9r^2-36r(1+\cos\psi)>0.$$

Putting $Q=-9r^2+36r(1+\cos\psi)$, we have to show

$$36\,|12-13re^{i\psi}+5r^2e^{2i\psi}|>Q-32.$$

Also, since $\cos\psi>\cos\psi_0(r)$, we have

$$Q\geqslant -9r^2+36r+9r^2+32=36r+32.$$

Let us set

$$\varphi(r,Q)=(36\,|12-13re^{i\psi}+5r^2e^{2i\psi}|)^2-(Q-32)^2$$
$$=36^2\big(25r^4+49r^2+144+240r^2\cos^2\psi-26r\cos\psi(12+5r^2)\big)-(Q-32)^2.$$

Remembering that $Q = -9r^2+36r(1+\cos\psi)$, we can express $\varphi(r, Q)$ in terms of r and Q alone by substituting for $\cos\psi$ and compute $\partial\varphi/\partial Q$, $\partial^2\varphi/\partial Q^2$ as follows.

$$\partial\varphi/\partial Q = 480(Q+9r^2-36r)-936(12+5r^2)-2(Q-32),$$
$$\partial^2\varphi/\partial Q^2 = 478 > 0.$$

Hence $\partial\varphi/\partial Q$ is an increasing function of Q. Also the value of $\partial\varphi/\partial Q$ for $Q = 36r+32$ is found to be $-360r^2-72r+4128 > 0$ for $4/3 \leqslant r \leqslant 8/3$.

This shows that $\varphi(r, Q)$ is a monotone increasing function of Q for a fixed r in the interval $4/3 \leqslant r \leqslant 8/3$ and so attains its minimum for $Q = 36r-32$. The condition $Q = 36r+32$, however, implies

$$\cos\psi = (9r^2+32)/36r = \cos\psi_0(r)$$

in virtue of (18). In other words, $\varphi(r, Q)$ attains its minimum when $|A_3| = 4$. Hence putting $A_3 = 4e^{i\theta}$, we have to show that

$$|6+14e^{i\theta}/3+20e^{2i\theta}/9|-|2+2e^{i\theta}/3| > 0,$$

which would imply (16).

Equivalently we have to prove that

$$(19) \qquad |27+21e^{i\theta}+10e^{2i\theta}|^2-9|3+e^{i\theta}|^2 > 0.$$

The left-hand side of (19) is, on simplification, found to be

$$\begin{aligned}(37\cos\theta+21)^2+289\sin^2\theta-9(10+6\cos\theta)\\ &= 20(54\cos^2\theta+75\cos\theta+32)\\ &= 1080(\cos\theta+75/108)^2+20(32-75^2/216) > 0.\end{aligned}$$

Thus inequality (19) holds and we have, therefore, proved that $\mathrm{Re}\{zS_4'(z)/S_4(z)\} > 0$ for $|z| < 1/6$. Also $S_4(z)$ has precisely two zeros in $|z| < 1/6$ and none on $|z| = 1/6$. Hence $S_4(z)$ is two-valent in $|z| < 1/6$ by Ozaki's theorem. The theorem is, therefore, proved for $n = 4$ also. To see that our result is sharp, we consider the function $F(z) = z^2/(1-z)^4$ which satisfies the hypothesis of the theorem. The third partial sum $S_3(z)$ satisfies $zS_3'(z)/S_3(z) = 0$ for $z = -1/6$. Thus, for the function of our choice, $S_3(z)$ is not starlike in any disc $|z| < r$ for r exceeding $1/6$. The proof of the theorem is complete.

THEOREM 2. *Let $f(z) = z^2+a_3z^3+\ldots$ be 2-valently close-to-convex relation to the function $F(z)$ in $|z| < 1$, where $F(z) = z^2+A_3z^3+\ldots$ is 2-valently starlike in the unit disc. Then any partial sum $s_n(z) = z^2+a_3z^3+ +\ldots+a_nz^n$ is 2-valently close-to-convex relative to the corresponding partial sum $S_n(z) = z^2+A_3z^3+\ldots+A_nz^n$ in $|z| < 1/6$ and the bound is sharp.*

Proof. By hypothesis, we have for $|z| < 1$

$$\operatorname{Re}\{zf'(z)/F(z)\} > 0. \tag{20}$$

Write $f(z) = s_n(z) + r_n(z)$ and $F(z) = S_n(z) + R_n(z)$. By Theorem 1, we see that $S_n(z)$ is 2-valently starlike in $|z| < 1/6$, $n > 3$. Also, since $S_n(z)$ has a double zero at $z = 0$, it does not vanish elsewhere in $|z| < 1/6$. Therefore, for $n > 3$, $\operatorname{Re}\{zs_n'(z)/S_n(z)\}$ is harmonic in $|z| < 1/6$,

$$\frac{zs_n'(z)}{S_n(z)} = \frac{zf'(z)}{F(z)} + \frac{zf'(z)/F(z)R_n(z) - zr_n'(z)}{F(z) - R_n(z)}. \tag{21}$$

If we write $zf'(z)/F(z) = 2H(z)$, $H(z)$ is regular in $|z| < 1$ and satisfies $\operatorname{Re} H(z) > 0$ for $|z| < 1$, $H(0) = 1$. Hence we have

$$\operatorname{Re}\{zf'(z)/F(z)\} \geqslant 2(1-r)/(1+r), \tag{22}$$

$$|zf'(z)/F(z)| \leqslant 2(1+r)/(1-r), \qquad r = |z| < 1.$$

The estimates for $|R_n(z)|$ and for $|F(z)| - |R_n(z)|$ given by (5) and (7) hold. Also ([6], Theorem 2, Corollary 1)

$$|a_n| \leqslant (n+1)C_3, \qquad n = 3, 4, \ldots,$$

and

$$|r_n'(z)| \leqslant (n+1)|a_{n+1}|r^n + (n+2)|a_{n+2}|r^{n+1} + \ldots$$

Hence estimate (6) for $|R_n'(z)|$ also holds for $|r_n'(z)|$. Using the above estimates in

$$\operatorname{Re}\{zs_n'(z)/S_n(z)\} \geqslant \operatorname{Re}\{zf'(z)/F(z)\} - \frac{|z|\,|r_n'(z)| + |zf'(z)/F(z)|\,|R_n(z)|}{|F(z)| - |R_n(z)|}$$

obtained from (21), and proceeding as in Theorem 1, we get

$$\operatorname{Re}\{zs_n'(z)/S_n(z)\} > 0, \qquad n \geqslant 5 \tag{23}$$

for $|z| = r = 1/6$. This, of course, implies that inequality (23) holds also for $|z| < 1/6$. The theorem is, therefore, proved for $n \geqslant 5$. Let us now write

$$zf'(z)/F(z) = \frac{2 + 3a_3z + 4a_4z^2 + \ldots}{1 + A_3z + A_4z^2 + \ldots}$$

$$= 2 + c_1z + c_2z^2 + \ldots, \qquad \text{say}.$$

Then we have

$$\begin{aligned} c_1 + 2A_3 &= 3a_3, \\ c_2 + A_3c_1 + 2A_4 &= 4a_4 \qquad \text{etc.} \end{aligned} \tag{24}$$

Again, since $\mathrm{Re}\{zf'(z)/F(z)\} > 0$, for $|z| < 1$, we have, by Carathéodory's theorem (Bieberbach [1])

$$(25) \qquad |4c_2 - c_1^2| \leqslant 16 - |c_1|^2, \quad |c_n| \leqslant 4, \quad n = 1, 2, \ldots$$

We can also express

$$zF'(z)/F(z) = \frac{2 + 3A_3 z + 4A_4 z^2 + \ldots}{1 + A_3 z + A_4 z^2 + \ldots}$$

$$= 2 + d_1 z + d_2 z^2 + \ldots, \quad \text{say}.$$

Then we get

$$(26) \qquad \begin{aligned} A_3 &= d_1, \\ 2A_4 &= d_2 + A_3 d_1 = d_2 + d_1^2. \end{aligned}$$

Again, since $\mathrm{Re}\{zF'(z)/F(z)\} > 0$ for $|z| < 1$, we get by an application of Carathéodory-Toeplitz's theorem referred to earlier

$$(27) \qquad \begin{aligned} &|4d_2 - d_1^2| \leqslant 16 - |d_1|^2, \\ &|d_n| \leqslant 4, \quad n = 1, 2, \ldots \end{aligned}$$

This enables us to write

$$(28) \qquad 4d_2 - d_1^2 = \varepsilon(16 - |d_1|^2),$$

where $|\varepsilon| \leqslant 1$. Using (26) in (28), we can rewrite the latter as

$$(29) \qquad 8A_4 = 5A_3^2 + \varepsilon(16 - |A_3|^2), \quad |\varepsilon| \leqslant 1.$$

We can now proceed to prove the case $n = 4$ of the theorem. Since $\mathrm{Re}\{zs_4'(z)/S_4(z)\}$ is harmonic for $|z| < 1/6$, we have only to prove that for $|z| = 1/6$

$$(30) \quad \mathrm{Re}\{zs_4'(z)/S_4(z)\} = \mathrm{Re}\{(2 + 3a_3 z + 4a_4 z^2)/(1 + A_3 z + A_4 z^2)\} > 0.$$

By considering $\varepsilon^2 f(\varepsilon z)$ in place of $f(z)$, with a suitable ε, $|\varepsilon| = 1$, the proof of (30) can be reduced to that of the same with $z = 1/6$. Thus we have only to prove that

$$(31) \qquad \mathrm{Re}\{(2 + a_3/2 + a_4/9)/(1 + A_3/6 + A_4/36)\} > 0.$$

Using (24) and (29) in the left-hand side of (31), the proof of (31) is reduced to that of the following

$$(32) \quad \mathrm{Re}\left\{\frac{2 + (c_1 + 2A_3)/6 + \big(c_2 + A_3 c_1 + 5A_3^2/4 + \varepsilon(16 - |A_3|^2)/4\big)/36}{1 + A_3/6 + \big(5A_3^2 + \varepsilon(16 - |A_3|^2)\big)/288}\right\} > 0,$$

where $|\varepsilon| < 1$.

The denominator of the above fraction does not vanish for $|\varepsilon| < 1$ as we have seen in proving Theorem 1. Hence, the fraction on the left-hand side of (32), regarded as a function of ε, is analytic for $|\varepsilon| < 1$. Thus the left-hand side of (32) is a harmonic function of ε in $|\varepsilon| < 1$ and it suffices to prove (32) for $|\varepsilon| = 1$. Putting

$$u = 2+(c_1+2A_3)/6+(c_2+A_3c_1+5A_3^2/4)/36,$$
$$v = 1+A_3/6+5A_3^2/288,$$

and proceeding as in the proof of Theorem 1, we find that the required inequality holds, provided we show that

$$|u+2v|-|u-2v|-(16-|A_3|^2)/72 > 0. \tag{33}$$

Now we have, since $|c_1| \leqslant 4$, $|c_2| \leqslant 4$,

$$|u+2v| \geqslant |4+2A_3/3+5A_3^2/72|-(|c_1|\,|6+A_3|-|c_2|)/36$$
$$\geqslant |4+2A_3/3+5A_3^2/72|-|6+A_3|/9-1/9.$$

Also, $|u-2v| \leqslant |6+A_3|/9+1/9$.

So we have only to prove that

$$|4+2A_3/3+5A_3^2/72|-2(6+A_3)/9-2/9-(16-A_3|^2)/72 > 0. \tag{34}$$

Here again, arguing as in the proof of Theorem 1, we observe that it is sufficient to prove inequality (34) when $|A_3| = 4$. In other words, we have to prove that

$$|4+8e^{i\theta}/3+10e^{2i\theta}/9|-2|6+4e^{i\theta}|/9-2/9 > 0,$$

that is,

$$|18+12e^{i\theta}+5e^{2i\theta}|-|6+4e^{i\theta}|-1 > 0.$$

The above inequality is implied by

$$|18e^{-i\theta}+12+5e^{i\theta}|^2-(1+|6+4e^{i\theta}|)^2 > 0. \tag{35}$$

The left-hand side of (35) is

$$(23\cos\theta+12)^2+169\sin^2\theta-\big(53+48\cos\theta+4\sqrt{(13+12\cos\theta)}\big)$$
$$= 4\big(65+90\cos^2\theta+126\cos\theta-\sqrt{(13+12\cos\theta)}\big).$$

Now $65+90\cos^2\theta+126\cos\theta = 90(\cos\theta+7/10)^2+20\cdot 90 \geqslant 20\cdot 90$, while $\sqrt{(13+12\cos\theta)} \leqslant 5$. Hence inequality (35) holds and the proof of the theorem for the case $n = 4$ is complete. Then we consider the case $n = 3$. Using relations (24), we get

$$\begin{aligned}\operatorname{Re}\{zs_3'(z)/S_3(z)\} &= \operatorname{Re}\{(2+3a_3)/(1+A_3)\}\\ &= \operatorname{Re}\{(2+(2A_3+c_1)z)/(1+A_3z)\}\\ &= 2+\operatorname{Re}\{c_1z/(1+A_3z)\}\\ &\geqslant 2-\{|c_1|/(6-|A_3|)\}\end{aligned}$$

for $|z| < 1/6$. Since $|c_1| \leqslant 4$, $|A_3| \leqslant 4$ this implies that $\mathrm{Re}\{zs_3'(z)/S_3(z)\} \geqslant 0$ for $|z| < 1/6$. This proves the theorem for $n = 3$. To see that our result is sharp, we consider

$$f(z) = z^2/(1-z)^4 = z^2+4z^3+10z^4+\ldots, \qquad |z| < 1.$$

$f(z)$ is 2-valently starlike and hence 2-valently close-to-convex relative to itself in $|z| < 1$. For this function $zs_3'(z)/s_3(z) = 0$ for $z = -1/6$. Hence $s_3(z)$ is not close-to-convex in any disc $|z| < R$ if R exceeds $1/6$. The proof of the theorem is complete.

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Reçu par la Rédaction le 6. 11. 1968