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Existence and uniqueness of solutions of the Darboux problem for the equation $\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = f\left(x_1, x_2, x_3, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_3}, \frac{\partial^2 u}{\partial x_2 \partial x_3}\right)$

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Publication [3] was dealing with conditions which guarantee uniqueness and the existence of solutions for the equation

$$\text{(1)} \qquad \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = f\left(x_1, x_2, x_3, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_3}, \frac{\partial^2 u}{\partial x_2 \partial x_3}\right)$$

under the conditions

$$u(0, x_2, x_3) = \varphi_1(x_2, x_3), \quad u(x_1, 0, x_3) = \varphi_2(x_1, x_3), \quad u(x_1, x_2, 0) = \varphi_3(x_1, x_2).$$

The problem of solving equation (1) under these conditions was called in the above-mentioned publication—problem (A).

This problem is obviously an analogon of the well-known Darboux problem for the case of two independent variables, i.e., for the equation

$$\text{(1')} \qquad \frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with the conditions

$$z(x, 0) = \sigma(x), \qquad z(0, y) = \tau(y).$$

Problem (A) we shall call also *the Darboux problem for equation* (1) or shortly, *the Darboux problem*, if it is known what equation is in question.

Paper [3], and especially its second part, contains some sufficient conditions for the existence of solutions for problem (A), but considering the method used there and the relatively extensive class of comparative functions $g(x, z)$ (see [5] or [3])—the results do not deal with the case where the right part of equation (1) is dependent on the full set of variables $(\xi, u, p_1, \ldots, q_{23})$ or is not a bounded function in the area under consideration. It is obviously possible to complete those results in some way using the method and the results contained in the works of J. Kisyński ([1] and [2]) and W. Walter ([5] and [6]) concerning especially the Darboux problem for equation (1′).

1. Formulation of the problem and auxiliary notations. We shall say that the function $\varphi(\xi)$ belongs to the class $C^*(V)$ if on the set V there are defined and continuous functions $\varphi(\xi)$, $\varphi_{x_i}(\xi)$, $\varphi_{x_j x_k}(\xi)$ and $\varphi_{x_1x_2x_3}(\xi)$ for $i, j, k = 1, 2, 3$, $j < k$, where

$$V = \{\xi\colon\ 0 \leqslant x_i \leqslant a_i,\ a_i > 0,\ i = 1, 2, 3,\ \xi = (x_1, x_2, x_3)\}\,.$$

Further, for rectangulars $R_{jk} = \{x_j, x_k)\colon\ 0 \leqslant x_i \leqslant a_i,\ i = j, k\}$ $(j, k = 1, 2, 3;\ j < k)$ let us determine functions $\varphi_1(x_2, x_3)$, $\varphi_2(x_1, x_3)$ and $\varphi_3(x_1, x_2)$ continuous, including second mixed derivatives, and fulfilling the conditions

$$\varphi_1(0, x_3) = \varphi_2(0, x_3), \qquad \varphi_1(x_2, 0) = \varphi_3(0, x_2), \qquad \varphi_2(x_1, 0) = \varphi_3(x_1, 0)$$

for $x_i \in \langle 0, a_i\rangle$, $i = 1, 2, 3$.

Darboux problem. Let us seek the function $u(\xi) \in C^*(V)$ fulfilling equation (1) and the conditions

$$(2)\qquad u(0, x_2, x_3) = \varphi_1(x_2, x_3), \qquad u(x_1, 0, x_3) = \varphi_2(x_1, x_3), \qquad u(x_1, x_2, 0) = \varphi_3(x_1, x_2)$$

for $(x_j, x_k) \in R_{jk}$, where $j, k = 1, 2, 3;\ j < k$.

It is easy to show that the solution of the Darboux problem is equivalent to the solution of the integro-differential equation

$$(3)\qquad u(x_1, x_2, x_3) = \psi_0(x_1, x_2, x_3) + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} f[t_1, t_2, t_3, u(t_1, t_2, t_3), P(t_1, t_2, t_3), Q(t_1, t_2, t_3)]\,dt_1\,dt_2\,dt_3\,,$$

where

$$(4)\quad \psi_0(x_1, x_2, x_3) = \varphi_1(0, 0) + \sum_{\nu=1}^{3} \varphi_\nu(x_j, x_k) - [\varphi_1(0, x_3) + \varphi_2(x_1, 0) + \varphi_3(0, x_2)]\,,$$

but

$$P = (p_1, p_2, p_3)\,, \qquad Q = (q_{12}, q_{13}, q_{23})\,, \qquad p_i(\xi) = \frac{\partial u}{\partial x_i}\,, \qquad q_{jk}(\xi) = \frac{\partial^2 u}{\partial x_j \partial x_k}\,.$$

Further, we shall write the right side of equation (1) shortly as $f(\xi, u, P, Q)$.

Class $G'(0, a\rangle$ (see W. Walter [5]).

We say that the function $g(x, z)$ defined for $x \in (0, a\rangle$, $z \geqslant 0$ belongs to class $G'(0, a\rangle$ if it fulfils the conditions:

1° $g(x, z) \geqslant 0$ and $g(x, 0) = 0$,

2° $g(x, z)$ is continuous for $x \in (0, a\rangle$ and $z \geqslant 0$,

3° every solution $z(x)$ of equation

$$(*) \qquad \frac{dz}{dx} = g(x, z)$$

fulfils the condition $z(x) \equiv 0$ or $z(x) \geqslant \delta \cdot x$ $(\delta = \delta(z) > 0)$,

4° every solution of equation $(*)$ is bounded,

5° $g(x, z) \leqslant g(x, \bar{z})$ for $z \leqslant \bar{z}$.

After these notes we are going to give some theorems proved for the problem which interests us.

2. Convergence of successive approximations for equation (1). Here we shall occupy ourselves with the formulation of certain conditions sufficient to ensure the convergence of a sequence of successive approximations for equation (1).

Therefore we determine the sequence of functions for equation (3) using the recurrent equation

$$(5) \qquad u^{(n+1)}(x_1, x_2, x_3) = \psi_0(x_1, x_2, x_3) + \\ + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} f[t_1, t_2, t_3, u^{(2)}(t_1, t_2, t_3), P^{(n)}(t_1, t_2, t_3), Q^{(n)}(t_1, t_2, t_3)]\,dt_1 dt_2 dt_3$$

for $n = 0, 1, 2, \ldots$, where

$$u^{(0)}(x_1, x_2, x_3) = \psi_0(x_1, x_2, x_3), \qquad P^{(0)}(x_1, x_2, x_3) = \left(\frac{\partial u^{(0)}}{\partial x_1}, \frac{\partial u^{(0)}}{\partial x_2}, \frac{\partial u^{(0)}}{\partial x_3}\right),$$

$$Q^{(0)}(x_1, x_2, x_3) = \left(\frac{\partial^2 u^{(0)}}{\partial x_1 \partial x_2}, \frac{\partial^2 u^{(0)}}{\partial x_1 \partial x_3}, \frac{\partial^2 u^{(0)}}{\partial x_2 \partial x_3}\right).$$

Now we can write the following

THEOREM 1. *If the function $f(\xi, u, P, Q)$ is defined, continuous and bounded for $\xi \in V$, $-\infty < u, P, Q < +\infty$ and for $0 < x_i \leqslant a_i$, $i = 1, 2, 3$, $-\infty < u, \bar{u}, P, \bar{P}, Q, \bar{Q} < +\infty$, we have the inequality*

$$(6) \qquad \prod_{s=1}^{3} x_s \cdot |f(\xi, u, P, Q) - f(\xi, \bar{u}, \bar{P}, \bar{Q})| \\ \leqslant a_0|u - \bar{u}| + \sum_{\substack{\nu=1 \\ \nu+j+k=6,\, j<k}}^{3} (a_\nu x_\nu |p_\nu - \bar{p}_\nu| + \beta_\nu x_j x_k |q_{jk} - \bar{q}_{jk}|),$$

where $a_0, a_\nu, \beta_\nu \geqslant 0$ and $a_0 + \sum_{\nu=1}^{3}(a_\nu + \beta_\nu) = 1$, then the sequence of approximations determined by formula (5) *is convergent to the unique solution of problem* (A).

Proof. (See also W. Walter [6].) First, for any natural number m by the fact that the function f is bounded $\bigl(K = \sup_{\xi\epsilon V, -\infty<u,P,Q<\infty} |f(\xi, u, P, Q)|\bigr)$ and by equation (5) we have

$$|u^{(m)}(\xi)-u^{(0)}(\xi)| \leqslant K\prod_{s=1}^{3} x_s, \qquad |p_i^{(m)}(\xi)-p_i^{(0)}(\xi)| \leqslant K\prod_{l=1, l\neq 1}^{3} x_l,$$

$$|q_{jk}^{(m)}(\xi)-q_{jk}^{(0)}(\xi)| \leqslant Kx_\nu .$$

Further, since

$$\psi(t) = \sup_{A_t}\bigl\{|f(\xi, u, P, Q)-f\bigl(\xi, u^{(0)}(\xi), P^{(0)}(\xi), Q^{(0)}(\xi)\bigr)|\bigr\}$$

for $0 \leqslant t \leqslant \min(a_1, a_2, a_3)$ and $\psi(t) = \psi\bigl(\min(a_1, a_2, a_3)\bigr)$ for $t > \min(a_1, a_2, a_3)$,

$$\begin{aligned} A_t = \Bigl\{(\xi, u, P, Q)\colon\ \xi \in V,\ 0 \leqslant \min(x_1, x_2, x_3) \leqslant t, \\ |u-u^{(0)}(\xi)| \leqslant K\prod_{1}^{3} x_s, |p_i-p_i^{(0)}(\xi)| \leqslant K\prod_{l=1, l\neq i}^{3} x_l, \\ |q_{jk}-q_{jk}^{(0)}(\xi)| \leqslant Kx_\nu, i, j, k = 1, 2, 3, j < k, j+k+\nu = 6\Bigr\} . \end{aligned}$$

Assuming $\bar{\psi}(t) = \sqrt[3]{\psi(t)}$ we choose a function ψ_0 for the function $\bar{\psi}$. This function should be continuous, non-decreasing, convex and fulfilling the conditions $\psi_0(t) \geqslant \psi(t)$ and $\psi_0(0) = 0$. By relation (5) we then obtain for $m = 0, 1, 2, \ldots$

$$(7) \qquad |f(\xi, u^{(m)}, P^{(m)}, Q^{(m)})-f(\xi, u^{(0)}, P^{(0)}, Q^{(0)})| \leqslant \prod_{i=1}^{3}\psi_0(x_i),$$

$$\text{where} \qquad \xi = (x_1, x_2, x_3) .$$

Putting $\psi_n(t) = \psi_0(2^{-n}t)$ for $n = 0, 1, 2, \ldots$ we obtain the inequality

$$\int_0^t \psi_n(\tau)d\tau \leqslant t\psi_{n+1}(t) \qquad \text{for} \qquad n = 0, 1, 2, \ldots,$$

which together with (6) and (7) permits us for $n = 1, 2, 3, \ldots$ and for $m = 0, 1, 2, \ldots$ to confirm the validity of the inequalities

$$(8') \quad \Delta_{mn}^{(0)}(\xi) = |u^{(m+n)}(\xi)-u^{(n)}(\xi)| \leqslant x_1x_2x_3 \sum_{\substack{r,s=0 \\ r\leqslant s, r+s\leqslant n}}^{n} a_{r,s}\psi_r(x_1)\psi_s(x_2)\psi_{n-r-s}(x_3),$$

$$a_{r,s} \geqslant 0, \quad \sum_{r,s} a_{r,s} = 1 ,$$

$$(8'')\quad \Delta_{mn}^{(i)}(\xi) = |p_i^{(m+n)}(\xi) - p_i^{(n)}(\xi)| \leqslant \frac{x_1x_2x_3}{x_i} \sum_{\substack{r,s=0\\ r\leqslant s,\, r+s\leqslant n}}^{n} \beta_{r,s}^{(i)}\psi_r(x_1)\psi_s(x_2)\psi_{n-r-s}(x_3),$$

$$\beta_{r,s}^{(i)} \geqslant 0, \qquad \sum_{r,s} \beta_{r,s}^{(i)} = 1\,,$$

$$(8''')\quad \Delta_{mn}^{(jk)}(\xi) = |q_{jk}^{(m+n)}(\xi) - q_{jk}^{(n)}(\xi)| \leqslant \frac{x_1x_2x_3}{x_jx_k} \sum_{\substack{r,s=0\\ r\leqslant s,\, r+s\leqslant n}}^{n} \gamma_{r,s}^{(jk)}\psi_r(x_1)\psi_s(x_2)\psi_{n-r-s}(x_3),$$

$$\gamma_{r,s}^{(jk)} \geqslant 0, \qquad \sum_{r,s} \gamma_{r,s}^{(jk)} = 1$$

$$\text{for}\quad i,j,k = 1,2,3, \quad j<k\,,$$

where $p_i^{(n)}(\xi) = \dfrac{\partial u^{(n)}}{\partial x_i}$, $q_{jk}^{(n)}(\xi) = \dfrac{\partial^2 u^{(n)}}{\partial x_j \partial x_k}$.

We omit here the simple inductive proof of inequalities (8')-(8'''). We can now state that for $n = 1, 2, 3, \ldots$ and for $m = 0, 1, 2, \ldots$ we have

$$\Delta_{mn}(\xi) = \Delta_{mn}^{(0)}(\xi) + \sum_{i=1}^{3} \Delta_{mn}^{(i)}(\xi) + \sum_{j,k=1}^{3} \Delta_{mn}^{(jk)}(\xi) \leqslant \text{const}\cdot\psi_{N(n)}(\mu)\,,$$

where $\mu = \max(a_1, a_2, a_3)$, $N(n) = E(n/3)$. Since $\lim\limits_{n\to\infty}\psi_{N(n)}(\mu) = 0$, we have $\Delta_{mn}(\xi) \underset{n}{\rightrightarrows} 0$ for $m = 0, 1, 2, \ldots$, which permits us to state that there is a function $u(\xi)$ for which

$$u^{(n)}(\xi) \underset{n}{\rightrightarrows} u(\xi), \qquad \frac{\partial u^{(n)}}{\partial x_i} \underset{n}{\rightrightarrows} \frac{\partial u}{\partial x_i}, \qquad \frac{\partial^2 u^{(n)}}{\partial x_j\partial x_k} \underset{n}{\rightrightarrows} \frac{\partial^2 u}{\partial x_j \partial x_k}$$

and which at the same time fulfils equation (3) and solves the Darboux problem.

In a similar way it is possible to prove the theorems given further on, so we omit the proofs for them.

THEOREM 2. *Let $f(\xi, u, p_r, p_s, q_{rs})$ be a function defined, continuous and bounded for $\xi \in V$, $-\infty < u, p_r, p_s, q_{rs} < +\infty$, where (r, s), $r < s$, is one of the three pairs formed of numbers $1, 2, 3$. If this function for $0 < x_i \leqslant a_i$, $-\infty < u, \bar{u}, p_r, \bar{p}_r, \ldots, \bar{q}_{rs} < +\infty$, $i = 1, 2, 3$, fulfils the inequality*

$$(9)\quad |f(\xi, u, p_r, p_s, q_{rs}) - f(\xi, \bar{u}, \bar{p}_r, \bar{p}_s, \bar{q}_{rs})|$$
$$\leqslant \alpha g\Big(\prod_1^3 x_i, |u-\bar{u}|\Big) + \beta_s g\Big(\frac{1}{x_s}\prod_1^3 x_i, |p_s - \bar{p}_s|\Big) +$$
$$+ \gamma_r g\Big(\frac{1}{x_r}\prod_1^3 x_i, |p_r - \bar{p}_r|\Big) + \delta_{rs} g\Big(\frac{1}{x_r x_s}\prod_1^3 x_i, |q_{rs} - \bar{q}_{rs}|\Big)\,,$$

where $\alpha, \beta_s, \gamma_r, \delta_{rs} \geqslant 0$, $\alpha+\beta_s+\gamma_r+\delta_{rs} = 1$, $g(x, z) \in G'(0, a_{rs}\rangle$, $a_{rs} = \prod_{i=1}^{3} a_i \cdot \max\left(1, \frac{1}{a_r}, \frac{1}{a_s}, \frac{1}{a_r a_s}\right)$ *and the function* $g(x, xz)$ *is non-decreasing with respect to* x *for every settled* $z \geqslant 0$, *then the sequence of successive approximations* (5) *is convergent to the unique solution of the Darboux problem.*

THEOREM 3. *If the function* $f(\xi, u, P, Q)$ *is defined, continuous and bounded for* $\xi \in V$, $-\infty < u, P, Q < +\infty$ *and for* $0 < x_i \leqslant a_i$, $i = 1, 2, 3$, $-\infty < u, \bar{u}, \ldots, \bar{Q} < \infty$ *fulfils the inequality*

$$|f(\xi, u, P, Q) - f(\xi, \bar{u}, \bar{P}, \bar{Q})| \leqslant g\Big(x_1+x_2+x_3, |u-\bar{u}| + \sum_{\nu=1}^{3} |p_\nu - \bar{p}_\nu| + \sum_{j,k=1, j<k}^{3} |q_{jk} - \bar{q}_{jk}|\Big), \tag{10}$$

where $g(x, z)$ *fulfils the condition* $(3 + \sum_1^3 a_i + a_1 a_2) g(x, z) \in G'(0, \sum_1^3 a_i\rangle$, *then it is obvious that problem* (A) *has exactly one solution. It is possible to obtain that unique solution by using the method of successive approximations* (5).

3. Method of fixed point for equation (1). Theorem 3 given above for a certain subclass of G'-class functions can be formulated in the same way as in [2] without the assumption that the function f is bounded. At the moment we shall occupe ourselves with the theorem concerning the existence of a solution for problem (A) when the right side of equation (1) fulfils the Osgood type condition for P and Q and the increase condition for u.

For analogical conditions concerning the right side of equation (1′), we have the results contained in [1].

Further, we shall investigate the following integral equation, equivalent to problem (A)

$$\begin{aligned} s(\xi) = f\Big[&\xi, \psi_0(\xi) + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} s(t_1, t_2, t_3)\,dt_1\,dt_2\,dt_3, \psi_1(\xi) + \int_0^{x_2}\int_0^{x_3} s(x_1, t_2, t_3)\,dt_2\,dt_3, \\ &\psi_2(\xi) + \int_0^{x_1}\int_0^{x_3} s(t_1, x_2, t_3)\,dt_1\,dt_3, \psi_3(\xi) + \int_0^{x_1}\int_0^{x_2} s(t_1, t_2, x_3)\,dt_1\,dt_2, \\ &\psi_{12}(\xi) + \int_0^{x_3} s(x_1, x_2, t_3)\,dt_3, \psi_{13}(\xi) + \int_0^{x_2} s(x_1, t_2, x_3)\,dt_2, \\ &\psi_{23}(\xi) + \int_0^{x_1} s(t_1, x_2, x_3)\,dt_1\Big], \end{aligned} \tag{11}$$

where $\psi_0(\xi)$ is determined by (4), but $\psi_i = \dfrac{\partial \psi_0}{\partial x_i}$, $\psi_{jk} = \dfrac{\partial^2 \psi_0}{\partial x_j \partial x_k}$ and $s(\xi) = \dfrac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}$.

Let us note the following lemma important for the further part of this paper:

LEMMA. *If the functions $\omega(\eta)$ and $\tilde{\omega}(\eta)$ are defined, non-negative, continuous and non-decreasing for $\eta \geqslant 0$, $\omega(0) = \tilde{\omega}(0) = 0$ and $\omega(\eta) > 0$ for $\eta > 0$ and if, moreover, the function $\omega(\eta)$ fulfils the conditions*

$$(12) \qquad \omega(\eta_1 + \eta_2) \leqslant \omega(\eta_1) + \omega(\eta_2) \quad \textit{for} \quad \eta_i \geqslant 0, \quad i = 1, 2$$

and

$$(13) \qquad \int_0^\eta \frac{dt}{\omega(t)} = +\infty \quad \textit{for} \quad \eta > 0,$$

then the equation

$$(14) \quad \varepsilon(x, y; \eta) = \omega\Big[\int_0^x\int_0^y \varepsilon(t, \tau; \eta)\, dt\, d\tau + \int_0^x \varepsilon(t, y; \eta)\, dt + \int_0^y \varepsilon(x, \tau; \eta)\, d\tau\Big] + \tilde{\omega}(\eta)$$

has for every $\eta \geqslant 0$ exactly one solution, continuous for $(x, y, \eta) \in R \times \langle 0, c\rangle$ such that $\varepsilon(x, y, 0) \equiv 0$, where $R = \{0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\}$ c – any constant > 0.

We obtain the proof of this lemma by applying theorem 2 from paper [1] to the function $\omega(z + p + q) + \tilde{\omega}(\eta)$ and repeating the argumentation contained in the proofs of lemmas 9-12 of that paper for equation (13).

Now we can introduce the theorem announced, proving only that part of it which relates to the existence of the solution of problem (A) since its uniqueness is included in theorem 3 if we substitute in it $g(x, z) = \omega(z)$. We have

THEOREM 4. (see [1], theorem 2). 1° *If the function $f(\xi, u, P, Q)$ is defined and continuous for $\xi \in V$, $-\infty < u, P, Q < +\infty$ and fulfils the condition*

$$(15) \quad |f(\xi, u, P, Q) - f(\xi, \bar{u}, \bar{p}, \bar{Q})| \leqslant \omega\Big(|u - \bar{u}| + \sum_{i=1}^{3} |p_i - \bar{p}_i| + \sum_{\substack{j,k=1 \\ j<k}}^{3} |q_{jk} - \bar{q}_{jk}|\Big),$$

where $\omega(z)$ has the properties enumerated in the lemma, then problem (A) has exactly one solution.

2° *If the function $f(\xi, u, P, Q)$ fulfils the conditions*

$$(16) \qquad |f(\xi, u, P, Q) - f(\xi, u, \bar{P}, \bar{Q})| \leqslant \omega\Big(\sum_{i=1}^{3} |p_i - \bar{p}_i| + \sum_{\substack{j,k=1 \\ j<k}}^{3} |q_{jk} - \bar{q}_{jk}|\Big)$$

and

(17) $$|f(\xi, u, 0, 0)| \leqslant c_1 + c_2|u| \quad \textit{(increase condition)},$$

where c_i, $i = 1, 2$, are equal to any constant >0, the function $\omega(z)$ has the properties as above—then there exists at least one solution of problem (A).

Proof. It is sufficient to show that equation (11), when fulfilling conditions (16) and (17), has a continuous solution on the set V. Basing ourselves on (12) we have $\omega(\eta) \leqslant (1+\eta)\omega(1)$, which permits us to state the following appreciation for the function f

$$|f(\xi, u, P, Q)| \leqslant c_1 + c_2|u| + \omega(1)\Big(1 + \sum_{i=1}^{3}|p_i| + \sum_{j,k=1,j<k}^{3}|q_{jk}|\Big).$$

Let us establish the following notations:

$$\mu_i = \max_V|\psi_i(\xi)|, \quad \mu_{jk} = \max_V|\psi_{jk}(\xi)| \quad \text{for} \quad i = 0, 1, 2, 3, \quad j, k = 1, 2, 3, \quad j<k,$$

$$\varkappa = 1 + c_1 + c_2(1+\mu_0) + \omega(1)\Big(7 + \sum_{i=1}^{3}\mu_i + \sum_{j,k=1,j<k}^{3}\mu_{jk}\Big),$$

$$A = \{(\xi, u, P, Q)\colon \xi \in V,\ |u| \leqslant \mu_0 + \varkappa^{-2}\exp[\varkappa(a_1+a_2+a_3)],$$
$$|p_i| \leqslant \mu_i + \varkappa^{-1}\exp[\varkappa(a_1+a_2+a_3)],\ |q_{jk}| \leqslant \mu_{jk} + \exp[\varkappa(a_1+a_2+a_3)],$$
$$i, j, k = 1, 2, 3,\ j<k\},$$

$$\omega_1(\eta) = \sup_{\substack{(\xi,u,P,Q),(\bar{\xi},\bar{u},P,Q)\in A \\ |u-\bar{u}|+\sum_1^3|\bar{x}_i-x_i|\leqslant\eta}} \{|f(\xi, u, P, Q) - f(\xi, \bar{u}, P, Q)|\}.$$

Furthermore, let $\omega_2(\eta)$ be for $\eta \geqslant 0$ a continuous, non-decreasing function, $\omega_2(0) = 0$, such that

$$|\psi_i(\xi) - \psi_i(\bar{\xi})|,\ |\psi_{jk}(\xi) - \psi_{jk}(\bar{\xi})| \leqslant \omega_2\Big(\sum_1^3|\bar{x}_s - x_s|\Big)$$
$$\text{for} \quad i, j, k = 1, 2, 3,\ j<k$$

Next let $\omega_3(\eta)$ be a function defined as follows

$$\omega_3(\eta) = \omega_1\{[1+\exp[\varkappa(a_1+a_2+a_3)]]\eta + \omega_2(\eta)\} +$$
$$+ \omega\{(2+\varkappa)\exp[\varkappa(a_1+a_2+a_3)]\eta + 6\omega_2(\eta)\}.$$

Finally, let $C(V)$ be the Banach space of continuous functions $\varphi(\xi)$ defined on the set V with norm $\|\varphi\| = \sup_V|\varphi(\xi)|$.

In $C(V)$ we shall consider W—the convex, closed and compact set of form

$$W = \Big\{\varphi\colon\ \varphi \in C,\ |\varphi(\xi)| \leqslant \varkappa \exp\Big(\varkappa \sum_{i=1}^{3} x_i\Big),\ |\varphi(\bar\xi)-\varphi(\xi)| \leqslant \varepsilon(\bar x_1, \bar x_2; |\bar x_3 - x_3|) +$$

$$+\varepsilon(\bar x_1, x_3; |\bar x_2 - x_2|) + \varepsilon(x_2, x_3; |\bar x_1 - x_1|),\ \xi = (x_1, x_2, x_3),\ \bar\xi = (\bar x_1, \bar x_2, \bar x_3) \in V\Big\};$$

besides $\varepsilon(x, y; \eta)$ denotes here the only solution of equation (14) for $\widetilde\omega(\eta) = \omega_3(\eta)$, fulfilling the condition $\varepsilon(x, y, 0) \equiv 0$. It is easy to verify that the operation $\psi = F\varphi$, $\psi, \varphi \in C(V)$, where F is determined by the right side of equation (11), is continuous on the set W and $F(W) \subset W$. Indeed, if $\varphi \in W$, then

$$|F\varphi(\xi)| \leqslant c_1 + c_2\Big[\mu_0 + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} |\varphi(t_1, t_2, t_3)|\, dt_1 dt_2 dt_3\Big] + \omega(1)\Big[1 + \sum_{i=1}^{3} \mu_i + \sum_{i,k=1}^{3} \mu_{jk} +$$

$$+\int_0^{x_2}\int_0^{x_3} |\varphi(x_1, t_2, t_3)|\, dt_2 dt_3 + \int_0^{x_1}\int_0^{x_3} |\varphi(t_1, x_2, t_3)|\, dt_1 dt_3 +$$

$$+\int_0^{x_1}\int_0^{x_2} |\varphi(t_1, t_2, x_3)|\, dt_1 dt_2 + \int_0^{x_3} |\varphi(x_1, x_2, t_3)|\, dt_3 +$$

$$+\int_0^{x_2} |\varphi(x_1, t_2, x_3)|\, dt_2 + \int_0^{x_1} |\varphi(t_1, x_2, x_3)|\, dt_1\Big]$$

$$\leqslant (c_1 + c_2\mu_0 + \varkappa^{-2} c_2) \exp\Big(\varkappa \sum_{1}^{3} x_i\Big) + \omega(1)\Big(1 + 3\varkappa^{-1} + 3 + \sum_{i=1}^{3} \mu_i +$$

$$+\sum_{j,k=1}^{3} \mu_{jk}\Big) \exp\Big(\varkappa \sum_{1}^{3} x_i\Big) \leqslant \varkappa \exp\Big(\varkappa \sum_{1}^{3} x_i\Big),$$

but

$$|F\varphi(\xi) - F\varphi(\bar\xi^{(1)})| \leqslant \omega_1\Big[|x_1 - \bar x_1| + \omega_2(|x_1 - \bar x_1|) + |x_1 - \bar x_1|\varkappa^{-1} \exp\Big(\varkappa \sum_{1}^{3} a_i\Big)\Big] +$$

$$+\Big| f\Big[\bar\xi^{(1)}, \psi_0(\bar\xi^{(1)}) + \int_0^{\bar x_1}\int_0^{x_2}\int_0^{x_3} \varphi(t_1, t_2, t_3)\, dt_1 dt_2 dt_3,\ \psi_1(\xi) +$$

$$+\int_0^{x_2}\int_0^{x_3} \varphi(x_1, t_2, t_3)\, dt_2 dt_3, \ldots, \psi_{23}(\xi) + \int_0^{x_1} \varphi(t_1, x_2, x_3)\, dt_1\Big] -$$

$$+ f\Big[\bar\xi^{(1)}, \psi_0(\bar\xi^{(1)}) + \int_0^{\bar x_1}\int_0^{x_2}\int_0^{x_3} \varphi(t_1, t_2, t_3)\, dt_1 dt_2 dt_3,\ \psi_1(\xi^{(1)}) +$$

$$+\int_0^{x_2}\int_0^{x_3} \varphi(\bar x_1, t_2, t_3)\, dt_2 dt_3, \ldots, \psi_{23}(\bar\xi^{(1)}) + \int_0^{\bar x_1} \varphi(t_1, x_2, x_3)\, dt_1\Big]\Big| \leqslant$$

$$\leqslant \omega_1\Big\{\Big[1+\exp\Big(\varkappa\sum_1^3 a_i\Big)\Big]|x_1-\bar{x}_1|+\omega_2(|x_1-\bar{x}_1|)\Big\}+$$

$$+\omega\Big[\int_0^{x_2}\int_0^{x_3}|\varphi(x_1,t_2,t_3)-\varphi(\bar{x}_1,t_2,t_3)|\,dt_2dt_3+$$

$$+\int_0^{x_2}|\varphi(x_1,t_2,x_3)-\varphi(\bar{x}_1,t_2,x_3)|dt_2+\int_0^{x_3}|\varphi(x_1,x_2,t_3)-\varphi(\bar{x}_1,x_2,t_3)|dt_3\Big]$$

$$\leqslant \omega_3(|x_1-\bar{x}_1|)+\omega\Big[\int_0^{x_2}\int_0^{x_3}\varepsilon(t_2,t_3;|x_1-\bar{x}_1|)\,dt_2dt_3+$$

$$+\int_0^{x_2}\varepsilon(t_2,x_3;|x_1-\bar{x}_1|)\,dt_2+\int_0^{x_3}\varepsilon(x_2,t_3;|x_1-\bar{x}_1|)\,dt_3\Big]$$

$$=\varepsilon(x_2,x_3;|x_1-\bar{x}_1|)\,,$$

where $\xi=(x_1,x_2,x_3)$, $\bar{\xi}^{(1)}=(\bar{x}_1,x_2,x_3)\in V$.

In an analogical way we obtain

$$|F\varphi(\bar{x}_1,x_2,x_3)-F\varphi(\bar{x}_1,\bar{x}_2,x_3)|\leqslant\varepsilon(\bar{x}_1,x_3;|x_2-\bar{x}_2|)$$

and

$$|F\varphi(\bar{x}_1,\bar{x}_2,x_3)-F\varphi(\bar{x}_1,\bar{x}_2,\bar{x}_3)|\leqslant\varepsilon(\bar{x}_1,\bar{x}_2;|x_3-\bar{x}_3|)\,,$$

which together signifies that $F\varphi\in W$ and at the same time that $F(W)\subset W$.

Further, if $\varphi,\bar{\varphi}\in W$, then

$$\|F\varphi-F\bar{\varphi}\|\leqslant\omega_1\Big(\prod_{i=1}^3 a_i\|\varphi-\bar{\varphi}\|\Big)+\omega\Big[\Big(\sum_1^3 a_i+\sum_{j,k=1,j<k}^3 a_ja_k\Big)\|\varphi-\bar{\varphi}\|\Big]\,.$$

The above properties permit us, by the Schauder theorem [4], to state the existence of a fixed point φ_0 for operation F, and at the same time the existence of a solution of problem (A).

Note 1. A simple example shows us that conditions (16) and (17) do not guarantee the uniqueness of the solutions of problem (A), e.g. for the equation

$$\frac{\partial^3 u}{\partial x_1\partial x_2\partial x_3}=(1+a)^3\prod_{i=1}^3 f_i(x_i)\cdot|u|^{a(1+a)},$$

where $a>0$, $f_k(x_k)$ are continuous, non-negative functions respectively in $\langle 0,a_k\rangle$, it is possible to give at least two different solutions of $C^*(V)$ class under zero conditions (2):

$$u_1(\xi)=0\quad\text{ and }\quad u_2(\xi)=\Big[\prod_{i=1}^3\int_0^{x_i}f_i(t)\,dt\Big]^{1+a}.$$

Note 2. It is evident that condition (15) alongside with continuity of the function f guarantees not only the existence of a unique solution of the problem (A) but it is also sufficient for the convergence of the sequence of successive approximations which is defined by the recurrent equation

$$(**)\qquad \varphi_{n+1}(\xi) = f\Big[\xi, \psi_0(\xi)+\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\varphi_n(t_1,t_2,t_3)\,dt_1dt_2dt_3, \ldots, \psi_i(\xi)+ \\ +\int_0^{x_j}\int_0^{x_k}\varphi_n(\ldots)\,dt_jdt_k, \ldots, \psi_{jk}(\xi)+\int_0^{x_\nu}\varphi_n(\ldots)\,dt_\nu, \ldots\Big],$$

where $\varphi_0(\xi)\in C(V)$.

The corresponding theorem can be formulated in the following way:

If the function $f(\xi, u, P, Q)$ defined and continuous for $\xi\in V$, $-\infty < u, P, Q < +\infty$ fulfils condition (15), *and besides $\omega(z)$ has the properties enumerated in the lemma, then the sequence of successive approximations determined by equation* ($**$), *is convergent, for any $\varphi_0(\xi)\in C(V)$, to the unique solution of the problem* (A).

The proof of this theorem can be obtained by a suitable modification of the method contained in paper [2].

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