ANNALES
POLONICI MATHEMATICI
XLII (1983)

On Max Müller's existence-comparison theorem for infinite systems of ordinary differential equations

by WOLFGANG WALTER (Karlsruhe)

Dedicated to the memory of Jacek Szarski

Abstract. The functions $v, w\colon J = [0, T] \to \boldsymbol{R}^n$ are said to be sub- and superfunctions for the initial value problem (1) $u'(t) = f(t, u(t))$, $u(0) = c$, if $v(0) \leqslant c$, $v_i' \leqslant f_i(t, z)$ whenever $v(t) \leqslant z \leqslant w(t)$ and $z_i = v_i(t)$, and similarly for w. Max Müller's existence-comparison theorem [5] states that if such a pair of function (v, w) exists, then (1) has a solution existing in J and lying between v and w. This result is extended to infinite systems of differential equations considered as differential equations in Banach spaces or locally convex spaces. In particular, our results cover the case l_∞ (which was treated before without success). Methodically, our approach is based on an elementary comparison argument.

Introduction. Throughout this paper J denotes the compact interval $[0, T]$, and $D \in \{D^+, D_+, D^-, D_-\}$ is a Dini derivative. In 1927 M. Müller [5] has proved the following theorem which extends a result of Perron [6] from one ordinary differential equation to systems of such equations. In formulating the theorem we make use of the order relation in $\boldsymbol{R}^n$ induced by the natural positive cone. In other words, for $x, y \in \boldsymbol{R}^n$ the inequality $x \leqslant y$ means that $x_i \leqslant y_i$ for $i = 1, \ldots, n$.

THEOREM A. *Let $J = [0, T]$; $v, w \in C^0(J, \boldsymbol{R}^n)$ and $v(t) \leqslant w(t)$ for $t \in J$. Suppose that $f \in C^0(K, \boldsymbol{R}^n)$, where $K = \{(t, x)\colon t \in J$ and $v(t) \leqslant x \leqslant w(t)\} \subset \boldsymbol{R}^{n+1}$, and that, for $t \in J$ and $i = 1, \ldots, n$,*

$$Dv_i(t) \leqslant f_i(t, x) \quad \text{whenever } v(t) \leqslant x \leqslant w(t) \text{ and } x_i = v_i(t),$$
$$Dw_i(t) \geqslant f_i(t, x) \quad \text{whenever } v(t) \leqslant x \leqslant w(t) \text{ and } x_i = w_i(t).$$

Then the initial value problem

$$u'(t) = f\big(t, u(t)\big), \quad u(0) = c, \tag{1}$$

where $v(0) \leqslant c \leqslant w(0)$, has a solution u existing in J and satisfying $v(t) \leqslant u(t) \leqslant w(t)$ in J.

Recently, Volkmann [7] has extended Müller's theorem to the Banach spaces c_0 (real null sequences) and l_p $(1 \leqslant p < \infty)$. He remarks that a corresponding result for the space l_∞ would be desirable (footnote 2 on page 89). Deimling and Lakshmikantham [3] have further generalized Theorem A to certain Banach spaces with a Schauder base (see Example (c) below). They also formulate the theorem for the space l_∞ (Theorem 3 in [3]), but the proof indicated there works only in the finite-dimensional case; see Example (d) below.

We give here an alternative approach to the problem in which the comparison aspect is separated from the existence question. This approach is both elementary and far-reaching. It yields, *cum grano salis*, a theorem of the above type in all cases where a local existence theorem is available, for l_∞ in particular.

A comparison theorem. Let I be a non-empty set (index set), and let $\boldsymbol{R}^I$ be the set of all functions $y\colon I \to \boldsymbol{R}$. We use index notation, $y = (y_i)_{i\in I}$. The set $\boldsymbol{R}^I$ is ordered componentwise (natural ordering), i.e.,

$$y \leqslant z \quad \text{iff} \quad y_i \leqslant z_i \quad \text{for all } i \in I.$$

Furthermore, $u = y \vee z$ and $v = y \wedge z$ are the elements of $\boldsymbol{R}^I$ defined componentwise by $u_i = y_i \vee z_i = \max(y_i, z_i)$, $v_i = y_i \wedge z_i = \min(y_i, z_i)$ for $i \in I$. Let $v(t)$, $w(t)$ be two functions from $J = [0, T]$ into $\boldsymbol{R}^I$ satisfying $v(t) \leqslant w(t)$ in J and let, for $t \in J$ and $z \in \boldsymbol{R}^I$,

$$P(t, z) = v(t) \vee \big(z \wedge w(t)\big). \tag{2}$$

For fixed t, $P(t, z)$ is a projection of z onto the "interval" $[v(t), w(t)] = \{y \in \boldsymbol{R}^I\colon v(t) \leqslant y \leqslant w(t)\}$.

THEOREM 1. *Let u, v, w be functions from J into $\boldsymbol{R}^I$ and assume that v_i, w_i are continuous and u_i is differentiable in J for all $i \in I$ and that $v(t) \leqslant w(t)$ in J. Let $f\colon K \to \boldsymbol{R}^I$ be given, where $K = \{(t, z)\colon t \in J, v(t) \leqslant z \leqslant w(t)\}$ is the region between v and w. Let D be a Dini derivative and let P be defined by* (2). *If, for $t \in J$ and $i \in I$,*

$$\begin{aligned} Dv_i(t) &\leqslant f_i(t, z) \quad \textit{whenever } v(t) \leqslant z \leqslant w(t) \textit{ and } z_i = v_i(t), \\ Dw_i(t) &\geqslant f_i(t, z) \quad \textit{whenever } v(t) \leqslant z \leqslant w(t) \textit{ and } z_i = w_i(t) \end{aligned} \tag{3}$$

and

$$u'(t) = f\big(t, P\big(t, u(t)\big)\big) \quad \textit{in } J,\ u(0) = c, \tag{4}$$

where $v(0) \leqslant c \leqslant w(0)$, then

$$v(t) \leqslant u(t) \leqslant w(t) \quad \textit{in } J. \tag{5}$$

Here, $u'(t)$ is, by definition, the function $u_i'(t)$, i.e., the differential equation is componentwise satisfied.

Proof. Assume that the conclusion is false and that, e.g., $w_j(t_1) < u_j(t_1)$ for some index j and some t_1, $0 < t_1 \leqslant T$. Since, by assumption, $u_j(0) \leqslant w_j(0)$, there is a largest interval (t_0, t_1) to the left of t_1 such that $u_j(t_0) = w_j(t_0)$ and $w_j(t) < u_j(t)$ for $t_0 < t \leqslant t_1$. Let $t \in (t_0, t_1]$ be fixed and let $z = P\big(t, u(t)\big)$. Obviously, $v(t) \leqslant z \leqslant w(t)$ and $z_j = w_j(t)$. Hence z satisfies the conditions in the second inequality of (3), and we have

$$u_j'(t) = f_j(t, z) \quad \text{and} \quad Dw_j(t) \geqslant f_j(t, z).$$

Hence the real-valued function $\varphi(t) = w_j - u_j$ has the properties $\varphi(t_0) = 0$ and $D\varphi(t) \geqslant 0$ for any $t \in (t_0, t_1)$. It follows that φ is increasing, in particular $\varphi(t_1) \geqslant 0$ or $u_j(t_1) \leqslant w_j(t_1)$. This is contradiction to our initial assumption that $w_j(t_1) < u_j(t_1)$ proves the theorem.

Remarks. If the real-valued function φ is continuous and satisfies $D\varphi(t) \geqslant 0$ except in a countable set of values t, or if φ is absolutely continuous and $\varphi'(t) \geqslant 0$ almost everywhere, then φ is increasing. This remark leads immediately to the following corollaries.

Corollaries to Theorem 1. (a) In Theorem 1 it is sufficient to assume that for each $i \in I$ there is a countable subset C_i of J such that (3) holds for $t \in J \setminus C_i$; the Dini derivatives for v_i and w_i may be different, and they may change with i (but not with t for fixed i).

(b) Theorem 1 remains true if the components u_i, v_i, w_i are assumed to be absolutely continuous and if inequalities (3) (where Dv_i, Dw_i can be replaced by v_i', w_i') and the differential equations $u_i' = f_i\big(t, P(t, u)\big)$ hold almost everywhere in J. The exceptional set of Lebesgue measure 0 may change with i (this remark is substantial only if the index set I is not countable).

It is an essential feature of our method that the comparison result is separated from the existence problem for the initial value problem $u' = f\big(t, P(t, u)\big)$, $u(0) = c$. There are no assumptions about f; on the other hand, the existence of a solution u is assumed.

If conditions are imposed on f such that the initial value problem (4) has a solution, then Theorem 1 leads immediately to an existence-comparison theorem of the type of Theorem A.

The following setting is used. Let $X \subset \boldsymbol{R}^I$ be a Banach space or, more generally, a (separated) locally convex topological vector space with the property that X is closed under the operations $\vee$ and $\wedge$, that these operations are continuous and that the coordinate functionals ϱ_i defined by $\varrho_i(z) = z_i$ are continuous. These properties ensure that the operator P given by (2) is defined and continuous and that for a continuous or differentiable function $v\colon J \to X$ the components v_i are continuous or differentiable real-valued functions. In view of these remarks and Theorem 1, the next theorem is self-evident.

THEOREM 2. *Let $X \subset \boldsymbol{R}^I$ be a locally convex space such that $z \to 0 \vee z$ and $z \to z_i$ $(i \in I)$ are continuous maps from X into X and X into $\boldsymbol{R}$, respectively. Let $v, w \in C^0(J, X)$, $v(t) \leqslant w(t)$ in J, $v(0) \leqslant c \leqslant w(0)$ and $K = \{(t, z) \in J \times X\colon v(t) \leqslant z \leqslant w(t)\}$. Assume that $f\colon G \to X$ is continuous (with respect to the product topology in $J \times X$), where $K \subset G \subset J \times X$, and that f satisfies condition* (E),

(E) *The initial value problem* (4) *has a solution in J.*

Then there exists a solution $u \in C^1(J, X)$ of the initial value problem

$$u'(t) = f\big(t, u(t)\big) \quad \text{in } J, \quad u(0) = c \tag{1}$$

satisfying (5).

Note that the assumption regarding the map $z \to z \vee 0$ implies that for any $a \in X$ the maps $z \to a \vee z = a + (z-a) \vee 0$ and $z \wedge a = -\big((-z) \vee (-a)\big)$ are defined and continuous. Hence $P(t, z)$ as given by (2) is a continuous map from $J \times X$ into X.

EXAMPLES. We shall consider a few special cases in which condition (E) is satisfied.

(a) I is finite. In this case, Peano's existence theorem applies and M. Müller's Theorem A follows.

(b) Let $I = \boldsymbol{N}$ and $X = c_0$ or $X = l_p$, $1 \leqslant p < \infty$. In these cases, K is compact, and the function $g(t, z) = f\big(t, P(t, z)\big)\colon J \times X \to X$ is continuous and compact. Hence condition (E) holds. This is well known and is usually proved by applying Schauder's fixed point theorem to the map $T\colon Y \to Y$, where $Y = C^0(J, X)$ and

$$(Tu)(t) = c + \int_0^t f\big(s, u(s)\big)\,ds. \tag{6}$$

Case (b) has first been given by Volkmann [7].

(c) Let $I = \boldsymbol{N}$ and assume that (e_i) is a Schauder base in X, where e_i is the element $(0, \ldots, 0, 1, 0, \ldots)$ with 1 at the ith place. If the positive cone $C = \{z \in X\colon z_i \geqslant 0 \text{ for } i \in \boldsymbol{N}\}$ is normal (i.e., if there exists $M > 0$ such that $0 \leqslant y \leqslant z$ implies $\|y\| \leqslant M\|z\|$), then K is again compact, and (E) holds. This case which includes the cases in (b) was given by Deimling and Lakshmikantham [3].

(d) Let I be an arbitrary index set and $X = l_\infty(I)$ normed by means of $\|z\| = \sup\{|z_i|\colon i \in I\}$. Let f satisfy a γ-Lipschitz condition

$$\gamma\big(f(B)\big) \leqslant L\gamma(B) \quad \text{for } B \subset K,$$

where γ is the Hausdorff (or ball) measure of non-compactness and L is a constant. It is easily seen that $g(t, z) = f\big(t, P(t, z)\big)$ satisfies the same condition, hence (E) holds; see. e.g., [2].

This case has particular importance due to its connection with the method of lines for parabolic differential equations (cf. [8], §§ 35, 36). We shall take up this matter in a future paper.

We remark that the above Lipschitz condition can be replaced by a weaker condition of the Kamke type.

(e) Let $X = \mathbf{R}^I$ endowed with the locally convex topology of componentwise convergence, which is generated by the seminorms $p_i(z) = |z_i|$, $i \in I$. This topology is the product topology of $\mathbf{R}^I$, hence K is compact by Tychonoff's theorem. Condition (E) is satisfied without further assumptions on f. Note that in this case the continuity of f signifies that for any sequence $(t_n, z^n) \subset G$, $z^n = (z_i^n)_{i\in I}$,

$$\lim_{n\to\infty} t_n = t_0, \quad \lim_{n\to\infty} z_i^n = z_i \quad (i \in I)$$

implies

$$\lim_{n\to\infty} f_i(t_n, z^n) = f_i(t_0, z) \quad \text{for } i \in I,$$

where $z = (z_i)$. It is not true that a function $f \in C^0\big(J \times l_\infty(I), l_\infty(I)\big)$ (the Banach space $l_\infty(I)$ was defined in (d)) is continuous in the topology considered here. On the other hand, if f has the property that each component f_i depends only (on t and) on a finite number of z_i and is continuous as a function of these variables, then f is continuous in the topology of $\mathbf{R}^I$.

Therefore, Theorem 2 applies, whenever f_i depends only on finitely many z_i and is continuous. A more general situation is considered in the next paragraph.

(f) Let f be continuous and such that $f(K) \subset C$, where C is a convex, compact subset of the locally convex space X which is not assumed to be complete. Then condition (E) is satisfied.

We sketch the proof. Let $\{p_a\}_{a\in A}$ be a system of seminorms in X which generates the topology of X. Let $Y = C^0(J, X)$. The space Y becomes a locally convex space by means of the system of seminorms $\{q_a\}$,

$$q_a(u) = \max\big\{p_a\big(u(t)\big)\colon t \in J\big\} \quad \text{for } u \in Y.$$

We need the following

LEMMA (Ascoli–Arzelà). *A subset M of Y is relatively compact if*

(i) *$M(t) = \{u(t)\colon u \in M\}$ is relatively compact in X for each $t \in J$;*

(ii) *M is equicontinuous, i.e., for any $a \in A$ there is a modulus of continuity $\delta_a(s)$ such that*

$$p_a\big(u(t) - u(s)\big) \leqslant \delta_a(|s-t|) \quad \textit{for } u \in M.$$

(*A modulus of continuity is a function δ which is continuous and increasing for $s \geqslant 0$ and for which $\delta(s) = 0$.*)

For a proof of this lemma, see [4], p. 34, or [1], Chap. X, §2, Theorem 2.

Next we consider Riemann integration. Let $C \subset X$ be convex and compact and let $D = C^0(J, C) \subset Y$. It is easily seen that D is convex and compact. Let $0 < a \leqslant T$, let π: $t_0 = 0 < t_1 < \ldots < t_p = a$ be a partitioning of the interval $[0, a]$ and let $\tau = (\tau_i)$, $t_{i-1} \leqslant \tau_i \leqslant t_i$. The Riemann sums of a function $u \in D$ defined by

$$\sigma(\pi, \tau) = \sum_{k=1}^{p} (t_k - t_{k-1}) u(\tau_k)$$

are elements of aC. Using the uniform continuity of u, $p_a\big(u(s) - u(t)\big) \leqslant \delta_a(|s-t|)$ (δ_a modulus of continuity), one sees that $\{\sigma(\pi, \tau)\}$ is a Cauchy set (with respect to the order relation defined by subdivision of partitions). Hence the limit

$$\int_0^a u(t)\,dt = \lim_{\pi} \sigma(\pi, \tau) \in aC$$

exists, and we have the usual properties of integrals such as

$$p_a\Big(\int_0^a u(t)\,dt\Big) \leqslant \int_0^a p_a\big(u(t)\big)\,dt.$$

Now let $g(t, z)$: $J \times X \to C$ be continuous and let T: $Y \to Y$ be defined by

$$(Tu)(t) = \int_0^t g\big(s, u(s)\big)\,ds.$$

The set $M = T(Y)$ has the properties

(i) $M(t) \subset tC$ for $0 \leqslant t \leqslant T$;

(ii) $p_a\big(v(t) - v(s)\big) \leqslant L_a |t-s|$ for $v \in M$, where $L_a = \max\big\{p_a\big(g(t, z)\big)$: $(t, z) \in J \times C\big\} < \infty$.

Hence M is relatively compact by the Ascoli–Arzelà lemma. It follows from Tychonoff's fixed point theorem that the equation

$$u = c + Tu$$

has a solution. If this result is applied to the function $g(t, z) = f\big(t, P(t, z)\big)$, condition (E) follows. Our result is summed up in the next theorem.

THEOREM 3. *Let X and v, w, K be as in Theorem 2. If f: $G \to X$ is continuous and $f(K) \subset C$, where C is a convex, compact subset of X, then there exists a solution u of the initial value problem* (1) *in J satisfying $v(t) \leqslant u(t) \leqslant w(t)$ in J.*

This theorem deals with the case where f is a compact map. If f is not assumed to be compact, additional hypotheses have to be introduced which guarantee existence. One such case was treated in Example (d). It is clear that this procedure works in other Banach spaces, too.

COROLLARY. *Assume that X is a Banach space and that the hypotheses of Theorem* 3 *are satisfied with the following changes. Instead of $f(K) \subset C$, we assume that*

$$\gamma\big(f(B)\big) \leqslant L\gamma(B) \qquad \text{for } B \subset J \times K,$$

where γ is the Hausdorff measure of non-compactness. Then the conclusion of Theorem 3 *remains valid.*

Literature

[1] N. Bourbaki, *Topologie générale*, Hermann, Paris 1961.

[2] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes, Vol. 596, Springer-Verlag, 1977.

[3] —, and V. Lakshmikantham, *Existence and Comparison Theorems for Differential Equations in Banach Spaces*, Nonlinear Analysis, Vol. 3, No. 5, (1979), p. 569–575.

[4] R. E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, New York 1965.

[5] M. Müller, *Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen*, Math. Z. 26 (1926), p. 619–645.

[6] O. Perron, *Ein neuer Existenzbeweis für die Integrale der Differentialgleichung $y' = f(x, y)$*, Math. Ann. 76 (1915), p. 471–484.

[7] P. Volkmann, *Ausdehnung eines Satzes von Max Müller auf unendliche Systeme von gewöhnlichen Differentialgleichungen*, Funkcialaj Ekvacioj (Ser. Int.), Vol. 21, No. 2 (1978), p. 81–96.

[8] W. Walter, *Differential and Integral Inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 55, Springer-Verlag, 1970.

Reçu par la Rédaction le 12.11.1979