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Leja's type polynomial condition and polynomial approximation in Orlicz spaces

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Abstract. Bernstein's "lethargy" theorem on polynomial approximation in Orlicz spaces is given. The theorem is used to characterize holomorphic and entire functions in spaces of μ-integrable functions on a subset E of C^n such that the pair (E, μ) satisfies Leja's type polynomial condition.

1. Condition L^*. Let E be a subset of the space C^n and let μ be a non-negative function defined on the family of Borel subsets of E such that $\mu(\varphi) = 0$. In the sequel such a function will be simply called a measure. The pair (E, μ) is said to satisfy *condition* L^* at a point $a \in \bar{E}$, closure of E, if for every family $\mathscr{F}$ of polynomials in C^n such that

$$\mu\big(\{z \in E: \sup_{f \in \mathscr{F}} |f(z)| = \infty\}\big) = 0,$$

and for every $b > 1$ there exists a constant $M > 0$ and a neighbourhood U of a such that for each f in $\mathscr{F}$,

$$\sup |f|(U) \leqslant M b^{\deg f}.$$

By the well-known polynomial lemma of Leja [7] every subset E of the complex plane C^1 satisfies L^* at a point $a \in \bar{E}$ with respect to the measure

$$\mu(A) = m_1\big(\{t > 0: A \cap C(a, t) \neq \emptyset\}\big)$$

where $C(a, t)$ denotes the circle with centre a and radius t and m_1 is the Lebesgue linear measure provided that $\mu(\{z \in E: |z-a| \leqslant \delta\}) = \delta$ for some $\delta > 0$. This remarkable result appeared a very useful tool in complex analysis especially in the theory of polynomial approximation (for references see [8], [12], [13], [14] and [9]). It also yields the following

1.1. EXAMPLE. If E is a rectifiable Jordan arc in C^1 and μ is length measure over E, then (E, μ) satisfies L^* at every point $a \in E$.

By Fubini's theorem, from Example 1.1 we derive

1.2. Example. Let E be a subset of the space $\boldsymbol{R}^n$ ($\boldsymbol{R}^n$ is treated as a subset of $\boldsymbol{C}^n$ such that $\boldsymbol{C}^n = \boldsymbol{R}^n + i\boldsymbol{R}^n$). If, for a point $a \in \bar{E}$, there exists a non-singular affine mapping l in $\boldsymbol{R}^n$ such that $a \in l(I^n) \subset E \cup \{a\}$, where I^n is the n-th Cartesian power of $I = [0, 1]$, then (E, m_n) satisfies L^* at a, m_n denoting the Lebesgue measure in $\boldsymbol{R}^n$. In particular, for every bounded, convex set E in $\boldsymbol{R}^n$ or else for every bounded Lipschitz domain (of class Lip 1), the pair (E, m_n) satisfies L^* at every point $a \in \bar{E}$.

The next two examples are related to the *complex Monge–Ampère operator theory* developed by Bedford and Taylor (see [2] and previous articles by same authors).

1.3. Example ([3]). Suppose E is a compact set in $\boldsymbol{C}^n$. If E is L-regular at a point $a \in E$ (see Section 2) and μ is the counting measure over E, then (E, μ) satisfies L^* at a. (In this case L^* reduces to the classical *polynomial condition* of Leja [7].)

1.4. Example. Let E be a bounded subset of $\boldsymbol{C}^n$ that is L-regular at a point $a \in \bar{E}$. If μ is a measure over E that dominates L-capacity c (see Section 2), then (E, μ) satisfies L^* at a (see e.g. [6]). Actually, we may take μ to be a measure that dominates the Borel measure $(dd^c u_E^*)^n$ related to the complex Monge–Ampère equation (see [9] and [20]). (Observe that $(dd^c u_E^*)^n$ vanishes on pluripolar sets).

We note that the condition L^* is invariant under non-degenerate holomorphic mappings from $\boldsymbol{C}^n$ to $\boldsymbol{C}^m$ ($m \leqslant n$) (see [13] and [6]). We also quote a convenient geometrical criterion for L^* (see [14]):

1.5. Analytic accessibility criterion. Given $a \in \bar{E}$, suppose there exists an analytic mapping h: $[0, 1] \to \bar{E}$ such that $h(0) = a$. If, for each $t \in (0, 1]$, the pair (E, μ) satisfies L^* at $h(t)$, then (E, μ) also satisfies L^* at a.

This criterion extends that for the L-regularity proved (independently) by Cegrell [4] and Sadullaev [17]. Actually, if $E \subset \boldsymbol{R}^n$, the analytic curve h can be replaced by a semianalytic arc (see [15]). Hence in particular, by Examples 1.2, 1.3 and 1.4, we get (see [15]).

1.6. Example. If E is a bounded open subanalytic set in $\boldsymbol{R}^n$ and μ is either the counting measure or the L-capacity c, or else the Lebesgue n-dimensional measure, then (E, μ) satisfies L^* at every point $a \in \bar{E}$.

We add that if (E, μ) satisfies L^* and ν is a measure over E that dominates μ (i.e. $\nu(A) = 0$ implies $\mu(A) = 0$), then (E, ν) also satisfies L^*.

In this note we shall show how the condition L^* is involved in problems of polynomial approximation in Orlicz's type spaces. In particular we shall give a version of Bernstein's lethargy theorem for such spaces as well as a characterization of the Bernstein–Walsh type of holomorphic and entire functions in Orlicz spaces. An important role will be played by an extremal plurisubharmonic function associated with a subset of $\boldsymbol{C}^n$ which can be considered as a multidimensional counterpart of Green's function.

2. L-extremal function. For any subset E of C^n, we define an *L-extremal function* by

$$V_E(z) = \sup\{u(z):\ u \in \mathscr{L},\ u \leqslant 0 \text{ on } E\} \quad \text{for } z \in C^n,$$

where $\mathscr{L} = \mathscr{L}(C^n)$ is the set of all plurisubharmonic functions u on C^n such that $u(z) - \log(1+|z|) = O(1)$, as $|z| \to \infty$, $|\cdot|$ being any norm in C^n (see [19], [18]). If E is a compact subset of C^1, V_E is the generalized *Green function* of the unbounded component of $C^1 \setminus E$. Set

$$V_E^*(z) = \limsup_{w \to z} V_E(w), \qquad z \in C^n,$$

and

$$c(E) = \liminf_{|z| \to \infty} \left[|z| \exp\left(-V_E^*(z)\right)\right].$$

The quantity $c(E)$ is called *L-capacity* of E. If $n = 1$, $c(E)$ is *logarithmic capacity* of E. It is known ([18], Theorem 3.10), that $c(E) = 0$ if and only if E is *pluripolar*, i.e. if there exists a plurisubharmonic function u on C^n such that $E \subset \{u = -\infty\}$. We note that if $c(E) > 0$ then $V_E^* \in \mathscr{L}$ (see [18], Corollary 3.9).

We recall that E is said to be *L-regular* at a point $a \in \bar{E}$ if $V_E^*(a) = 0$; E is L-regular if $V_E^*(z) = 0$ for each $z \in \bar{E}$. We quote an analogue of one-dimensional Kellogg's lemma which follows from [1], Theorem 2: If E is an F_σ-set in C^n, then the set $\{z \in E:\ V_E^*(z) > 0\}$ is pluripolar.

3. Orlicz spaces. In the sequel the term "measure" is used in proper sense. Let E be a subset of C^n and let μ be a Borel measure over E. We adopt notations of Rolewicz [16], p. 18. Let N be a continuous, strictly increasing function defined on $[0, \infty)$ such that $N(t) = 0$ iff $t = 0$ and $N(s+t) \leqslant M[N(s)+N(t)]$ for $s, t \geqslant 0$ with a constant $M \geqslant 1$ idependent of s and t. Denote by X the linear space of all μ-measurable complex functions defined on E and set $I = \{x \in X:\ x(z) = 0\ \mu\text{-almost everywhere on } E\}$. For $x \in X/I$, we define

$$\varrho_N(x) = \int_E N(|x(z)|)\, d\mu(z).$$

Let X_N be the space of all x in X/I such that $\varrho_N(kx) < \infty$ for some $k > 0$ (depending on x). If $x \in X_N$, we set

$$\|x\| = \inf\{\varepsilon > 0:\ \varrho_N(x/\varepsilon) < \varepsilon\}. \tag{3.1}$$

It is proved (see [16], Theorem I.2.3) that $\|\cdot\|$ is an *F-norm* in X_N (i.e. $\|x\| = 0$ iff $x = 0$, $\|\alpha x\| = \|x\|$ for all scalars α with $|\alpha| = 1$, $\|x+y\| \leqslant \|x\| + \|y\|$ and $\|\alpha_k x\| \to 0$ as $\alpha_k \to 0$) such that $\|x_k\| \to 0$ iff $\varrho_N(x_k) \to 0$. The space X_N endowed with this F-norm is an *F-space* (a complete, metric linear space) denoted further on by $N(L(E, \mu))$. If $E = (0, 1) \subset \boldsymbol{R}^1$ and $\mu = m_1$, $N(L(E, \mu))$ is called *Orlicz space* and N is called *Orlicz function.*

4. Bernstein's "lethargy" theorem. For $k = 0, 1, \ldots,$ we set

$$\mathscr{P}_k = \{p_{|E}\colon p \text{ is a polynomial in } \boldsymbol{C}^n \text{ of degree at most } k\}.$$

In the sequel we shall assume that E and μ are such that for each k, $\mathscr{P}_k$ is a linear subspace of $N(L(E, \mu))$ and $\dim \mathscr{P}_k < \dim \mathscr{P}_{k+1}$. (This is the case if e.g. $\mu(A) > 0$ implies $c(\bar{A}) > 0$ and for each $m \geqslant 0$, $\int_E N(|z|^m)\, d\mu(z) < \infty$; we obviously assume $\mu(E) > 0$.)

An important problem of constructive function theory is to classify elements x in $N(L(E, \mu))$ with respect to the range of convergence to zero of the sequence

$$d_N(x, \mathscr{P}_k) = \inf\{\|x - p\|\colon p \in \mathscr{P}_k\}, \quad k = 0, 1, \ldots$$

In general, every speed can be attained, since we have, similarly to the classical Bernstein "lethargy" theorem for the polynomial approximation of continuous functions on compact intervals in $\boldsymbol{R}^1$ (see e.g. [5]), the following

4.1. Proposition. *Suppose μ vanishes on every subset of E whose closure is pluripolar. Then for any decreasing null-sequence $\{\varepsilon_k\}$ of non-negative numbers there is an element x in $N(L(E, \mu))$ such that*

$$d_N(x, \mathscr{P}_k) = \varepsilon_k \quad \textit{for } k \geqslant k_0.$$

The proof of the proposition is based on two following lemmas and a general version of Bernstein's theorem for F-spaces (see Theorem 4.5).

4.2. Lemma. *Set $M(N, E) = \mu(E)\sup\{N(t)\colon t > 0\}$. With the assumptions of Proposition 4.1, for each $\varepsilon \in (0, M(N, E))$, the sets*

$$B_k = \{x \in \mathscr{P}_k\colon \varrho_N(x) \leqslant \varepsilon\}$$

are compacts ($k = 0, 1, \ldots$).

Proof. Since $\dim \mathscr{P}_k$ is finite, it suffices to show that for each bounded Borel subset F of E with $\mu(F) > 0$ the set B_k is bounded with respect to the uniform norm $\|\cdot\|_F$ (observe that $\mu(F) > 0$ implies $c(\bar{F}) > 0$, whence $\|\cdot\|_F$ is indeed a norm in $\mathscr{P}_k$). Choose the set F and suppose there exists a sequence $\{p_m\} \subset B_k$ such that $\lim_{m\to\infty} \|p_m\|_F = \infty$. Without loss of generality we may assume that the sequence $\{|p_m(z)|\}$ is increasing for each $z \in \bar{F}$. Then for each $\delta > 0$, the sequence of sets

$$F_m = \{z \in \bar{F}\colon |p_m(z)| \leqslant \delta\}, \quad m = 1, 2, \ldots,$$

is decreasing. We claim that $\mu(F_0) = 0$, where $F_0 = \bigcap_{m=1}^{\infty} F_m$. For if $\mu(F_0) > 0$, then $c(F_0) > 0$ and for $d = c(F_0)/2$ we could find $r > 0$ such that $\bar{F} \subset B(0, r)$ and

$$\exp V_{F_0}^*(z) \leqslant r/d \quad \text{as } |z| \leqslant r.$$

Hence, since for every polynomial $p \neq 0$, $(1/\deg p)\log|p| \in \mathscr{L}$, we would have

$$\|p_m\|_F \leqslant \|p_m\|_{F_m} \exp[k \sup_F V_{F_m}^*(z)] \leqslant \|p_m\|_{F_m} \exp[k \sup_{B(0,r)} V_{F_0}^*(z)] \leqslant \delta (r/d)^k$$

for $m = 1, 2, \ldots$, which is impossible. Thus, $\lim_{m \to \infty} \mu(F_m) = 0$. Consequently, by choosing F and δ such that $\mu(F)N(\delta) > \varepsilon$, we get

$$\varepsilon \geqslant \varrho_N(p_m) \geqslant \int_F N(|p_m(z)|)\, d\mu(z) \geqslant \int_{F \setminus F_m} N(\delta)\, d\mu = N(\delta)[\mu(F) - \mu(F_m)] > \varepsilon$$

for sufficiently large m, contradiction.

4.3. Lemma. *With the assumptions of the preceding lemma, for each k we have*

$$\tilde{\eta}_k := \sup_{x \in \mathscr{P}_{k+1}} \{\inf_{p \in \mathscr{P}_k} \varrho_N(x-p)\} = M(N, E).$$

Proof. Choose a basis $\{e_1, \ldots, e_r, \ldots, e_s\}$ of $\mathscr{P}_{k+1}$ such that $\{e_1, \ldots, e_r\}$ is a basis of $\mathscr{P}_k$. Let F be a bounded Borel subset of E with $\mu(F) > 0$. Since all norms in $\mathscr{P}_{k+1}$ are equivalent, there is a constant $d > 0$ such that for each $x = \alpha_1 e_1 + \ldots + \alpha_s e_s \in \mathscr{P}_{k+1}$,

$$d^{-1}\|x\|_F \geqslant |x|_{k+1} := |\alpha_1| + \ldots + |\alpha_s|.$$

Hence, for every $p \in \mathscr{P}_k$ and $m = 1, 2, \ldots$, we have

$$\|me_s - p\|_F \geqslant d \cdot \inf_{p \in \mathscr{P}_k} |me_s - p|_{k+1} = dm.$$

For any $x \in \mathscr{P}_{k+1}$, we set

$$\mathscr{P}_k(x) = \{p \in \mathscr{P}_k \colon \inf_{q \in \mathscr{P}_k} \|x - q\|_F = \|x - p\|_F\}.$$

Since $\dim \mathscr{P}_k < \infty$, $\mathscr{P}_k(x) \neq \emptyset$. Choose $p_0 \in \mathscr{P}_k(e_s)$. Then, for each $m = 1, 2, \ldots$, $mp_0 \in \mathscr{P}_k(me_s)$ and the sequence of the sets

$$G_m = \{z \in F \colon m|e_s(z) - p_0(z)| \leqslant \delta\}$$

is decreasing. If $r > 0$ is such that $\bar{F} \subset B(0, r)$, we have

$$\begin{aligned} dm &\leqslant \|me_s - mp_0\|_F \leqslant \|me_s - mp_0\|_{B(0,r)} \\ &\leqslant \|me_s - mp_0\|_{F_m} \cdot \sup_{z \in B(0,r)} \exp[(k+1) V_{F_m}^*(z)] \\ &\leqslant \delta \exp[(k+1) \sup_{z \in B(0,r)} V_{F_m}^*(z)], \end{aligned}$$

and by a similar argument to that of the proof of the preceding lemma, we get $\lim_{m \to \infty} \mu(G_m) = 0$. Hence

$$\begin{aligned} M(N, E) \geqslant \tilde{\eta}_k &\geqslant \sup_m \int_F N(|me_s - mp_0|)\, d\mu \geqslant \sup_m \int_{F \setminus G_m} N(\delta)\, d\mu \\ &= \sup_m N(\delta)[\mu(F) - \mu(G_m)] = N(\delta)\mu(F). \end{aligned}$$

By the arbitrariness of the choice of $\delta > 0$ and $F \subset E$, the proof of the lemma is concluded.

By the definition of the F-norm (3.1) and by Lemma 4.3, we get

4.4. COROLLARY. *There exists $\gamma > 0$ such that for each $k = 0, 1, \ldots,$*

$$\eta_k := \sup_{x \in \mathscr{P}_{k+1}} \{\inf_{p \in \mathscr{P}_k} \|x - p\|\} \geq \gamma.$$

Now Proposition 4.1 immediately follows from

4.5. THEOREM ([11]). *Suppose $\{\mathscr{V}_k\}$ is an increasing sequence of distinct, finite dimensional linear subspaces of an F-space X. Suppose that*

(i) *there exists $\varepsilon_0 > 0$ such that the sets $B(0, \varepsilon_0) \cap \mathscr{V}_k$ are compact* $(k = 1, 2, \ldots)$.

Then for every nullsequence $\{\varepsilon_k\}$ of real numbers such that

(ii) $\frac{1}{2}\varepsilon_0 > \varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq 0$

and

(iii) $2\varepsilon_k \leq \eta_k := \sup_{x \in \mathscr{V}_{k+1}} \{\inf_{v \in \mathscr{V}_k} \{\|x - v\|\}$ $(k = 1, 2, \ldots)$ *there is an element $x \in X$ such that*

$$\operatorname{dist}(x, \mathscr{V}_k) = \varepsilon_k \quad \textit{for } k = 1, 2, \ldots,$$

$\|\cdot\|$ *denoting an F-norm in X.*

5. A criterion of analyticity in Orlicz spaces. Our goal is to describe these x in $N(L(E, \mu))$ that are the restrictions to E of holomorphic functions in a neighbourhood of $\bar{E}$. With this aim we shall additionally assume that $\lim_{t \to \infty} N(t) = \infty$ and

(5.1) For each $a \in (0, 1)$, $\limsup_{k \to \infty} [N(a^k)]^{1/k} < 1$.

The typical examples of such functions N are: $N(t) = t^p$ $(p > 0)$ and $N(t) = \log_a(1+t)$ $(a > 1)$. Observe also that if N_1 and N_2 satisfy (5.1) so do the functions $\max(N_1, N_2)$, $N_1 + N_2$, $N_1 N_2$, and $N_1 \circ N_2$ (superposition). We note a simple technical

5.1. LEMMA. *For every $b > 1$ there is $c > 1$ such that*

$$b^k N(t) \geq N(c^k t)$$

for $t \geq 0$ and $k \geq k_0$.

By Siciak's version of a Bernstein–Walsh theorem (see [18], Theorem 8.5), if E is a bounded subset of $\mathbf{C}^n$ such that $\bar{E}$ is polynomially convex, then for every holomorphic function x in a neighbourhood of $\bar{E}$,

$$\limsup_{k \to \infty} [\inf\{\|x - p\|_E \colon p \in \mathscr{P}_k\}]^{1/k} < 1$$

whence we also have

$$\limsup_{k \to \infty} [d_N(x, \mathscr{P}_k)]^{1/k} < 1.$$

Conversely, suppose $x \in N(L(E, \mu))$ is such that for a strictly increasing sequence $\{k_j\}$ of positive integers, we have

$$\limsup_{j \to \infty} [d_N(x, \mathscr{P}_{k_j})]^{1/k_j} < 1. \tag{5.2}$$

Then we have

5.2. THEOREM. 1° *If (E, μ) satisfies L^* at each point $a \in \bar{E}$ and*

$$\limsup_{j \to \infty} k_{j+1}/k_j < \infty, \tag{5.3}$$

then every function $x \in N(L(E, \mu))$ that satisfies (5.2) is equal μ-almost everywhere on E to a function $\tilde{x}$ holomorphic in a neighbourhood of $\bar{E}$.

2° *If μ satisfies the assumptions of Proposition 4.1, $\bar{E}$ is polynomially convex, compact set in C^n, and*

$$\limsup_{j \to \infty} k_{j+1}/k_j = \infty, \tag{5.4}$$

then there is a function $x \in N(L(E, \mu))$ that satisfies (5.2) and is not the restriction of a holomorphic function in any neighbourhood of $\bar{E}$.

Proof. 1° By (5.2), for each j there is a polynomial p_j of degree at most k_j such that

$$\|x - p_j\| \leqslant M a^{k_j}$$

with some constants $M > 0$ and $a \in (0, 1)$ that are independent of j, $\|\cdot\|$ denoting the F-norm (3.1). Hence by (5.3),

$$\|p_{j+1} - p_j\| \leqslant M(a^{k_{j+1}} + a^{k_j}) \leqslant 2M b^{k_{j+1}}$$

for $j \geqslant j_0 = j_0(c)$, where $c > \limsup_{j \to \infty} k_{j+1}/k_j$ and $b = a^{1/c}$. Consequently,

$$\varrho_N(p_{j+1} - p_j) \leqslant 3M b^{k_{j+1}} \tag{5.5}$$

for $j \geqslant j_1$, where $j_1 \geqslant j_0$ is chosen such that $3M b^{k_{j+1}} \leqslant 1$. Now we adopt a reasoning of Nguyen Thanh Van [8]. Take any $c \in (b, 1)$ and set

$$E_{m,j} = \{z \in E: c^{-k_j} N(|p_{j+1}(z) - p_j(z)|) > m\}.$$

Then by (5.5), $\mu(E_{m,j}) \leqslant 3M(b/c)^{k_{j+1}}/m$ for $j \geqslant j_1$, whence by setting

$$E_m = \bigcup_{j=j_1}^{\infty} E_{m,j}$$

we get

$$\mu\Big(\bigcap_{m=1}^{\infty} E_m\Big) = \lim_{m \to \infty} \mu(E_m) = 0.$$

This means that for each $c \in (b, 1)$ we have

$$\sup_{j \geqslant 1} c^{-k_{j+1}} N(|p_{j+1}(z) - p_j(z)|) < \infty$$

μ-a.e. on E and hence, by Lemma 5.1, we can find $d > 1$ such that the sequence

$$d^{k_{j+1}} |p_{j+1}(z) - p_j(z)| \quad (j = 1, 2, \ldots)$$

is bounded μ-a.e. on E. Since (E, μ) satisfies L^*, for each $r \in (1, d)$, there exists a constant $C > 0$ and a neighbourhood U of $\bar{E}$ such that

$$\sup_U |p_{j+1} - p_j| \leqslant C r^{k_{j+1}}, \quad j \geqslant 1,$$

whence the series

$$p_1 + \sum_{j=1}^{\infty} (p_{j+1} - p_j)$$

is uniformly convergent on U to a holomorphic function $\tilde{x}$ such that $\tilde{x}(z) = x(z)$ μ-a.e. on E.

2° By (5.4) there is a subsequence $\{k_{j_l}\}$ of $\{k_j\}$ such that $\{k_{j_{l+1}}/k_{j_l}\}$ tends to ∞, as $l \to \infty$. For simplifying the notations we leave $\{k_j\}$ to be the subsequence. By Proposition 4.1 there is a function $x \in N(L(E, \mu))$ such that

$$d_N(x, \mathscr{P}_k) = \varepsilon_k \quad \text{for } k \geqslant k_0,$$

where $\varepsilon_k = \varrho^{k_j}$, as $k_j \leqslant k < k_{j+1}$, for $j = 1, 2, \ldots$, and $\varrho \in (0, 1)$. Thus, x fulfils (5.2) but, according to the Bernstein–Walsh theorem cannot be extended to a holomorphic function in any neighbourhood of $\bar{E}$, since we have

$$\begin{aligned} \limsup_{k \to \infty} [d_N(x, \mathscr{P}_k)]^{1/k} &\geqslant \limsup_{j \to \infty} [d_N(x, \mathscr{P}_{k_{j+1}-1})]^{1/(k_{j+1}-1)} \\ &= \lim_{j \to \infty} \varrho^{k_j/(k_{j+1}-1)} = 1. \end{aligned}$$

5.3. Remark. A weaker version of Theorem 5.2 was proved in [12].

A similar criterion to that of Theorem 5.2 can be proved for entire functions without any restrictions on E (in part 1°).

Consider the set of elements x in $N(L(E, \mu))$ such that for a strictly increasing sequence $\{k_j\}$ of positive integers, we have

$$\lim_{j \to \infty} [d_N(x, \mathscr{P}_{k_j})]^{1/k_j} = 0. \tag{5.6}$$

5.4. Theorem. 1° *For every subset E of $\mathbf{C}^n$, if $x \in N(L(E, \mu))$ satisfies* (5.6) *with a sequence $\{k_j\}$ fulfilling* (5.3), *then x is equal μ-a.e. on E to an entire function.*

2° *If μ fulfils the requirements of Proposition* 4.1 *and $\{k_j\}$ is a sequence that satisfies* (5.4), *then there is a function x in $N(L(E, \mu))$ fulfilling* (5.6) *which cannot be the restriction of an entire function.*

Proof. 1° By a similar argument to that of the proof of part 1° of Theorem 5.2, we show that there is a subset F of E with $\mu(F) > 0$ such that for each $z \in F$,

$$\sup_{j \geq 1} (2\varepsilon_j)^{-k_{j+1}} N(|p_{j+1}(z) - p_j(z)|) < \infty,$$

where $\varepsilon_j = [d_N(x, \mathscr{P}_{k_j})]^{1/k_j} = \|x - p_j\|^{1/k_j} \to 0$ as $j \to \infty$. Then, similarly as in Lemma 5.1, we can find a sequence $\eta_j \to 0$ of positive numbers such that for $z \in F$,

$$\sup_{j \geq 1} N(\eta_j^{-k_{j+1}} |p_{j+1}(z) - p_j(z)|) < \infty. \tag{5.6}$$

Since for each j, $-\log \eta_j + k_{j+1}^{-1} \log |p_{j+1} - p_j|$ is a plurisubharmonic function from the class $\mathscr{L}$, and since $\bar{F}$ is not pluripolar, by [18], Theorem 3.5, it follows from (5.6) that the sequence

$$\eta_j^{-1} |p_{j+1} - p_j|^{1/k_{j+1}}, \quad j = 1, 2, \ldots$$

is uniformly bounded on every compact set in C^n. Therefore the series

$$p_1 + \sum_{j=1}^{\infty} (p_{j+1} - p_j)$$

is uniformly convergent on every compact set in C^n to an entire function $\tilde{x}$ such that $\tilde{x}(z) = x(z)$ μ-a.e. on E.

2° Set, for $j = 1, 2, \ldots$,

$$t_j = \inf\{k_i/k_{i+1}:\ 1 \leq i \leq j\}$$

and

$$\varepsilon_k = t_j^{k_j} \quad \text{as } k_j \leq k < k_{j+1}.$$

Then by Proposition 4.1, there is a function $x \in N(L(E, \mu))$ for which

$$d_N(x, \mathscr{P}_k) = \varepsilon_k$$

for almost all k. Now by (5.4), it is easy to check that

$$\limsup_{k \to \infty} [d_N(x, \mathscr{P}_k)]^{1/k} = 1.$$

On the other hand, if x is an entire function on C^n, then for each subset E of C^n,

$$\lim_{k \to \infty} [d_N(x, \mathscr{P}_k)]^{1/k} = 0,$$

and this completes the proof.

The functions in $N(L(E, \mu))$ that satisfy (5.2) are called *quasianalytic* in the sense of Bernstein. (Those which satisfy (5.6) are per analogiam called *quasientire*.) For properties of quasianalytic functions we refer to [10] (quasianalyticity in spaces of continuous functions) or to [12] (case of Orlicz spaces). We end by proving a mutation of an identity principle for quasianalytic functions [12], Theorem 5.4.

5.5. PROPOSITION. *Let E be a connected open set in $\mathbf{R}^n$ and let μ be a Borel measure over E such that*

(i) *For each subset A of E, if $c(\bar{A}) = 0$ then $\mu(A) = 0$.*

(ii) *For each n-dimensional compact interval $I \subset E$, the pair (I, μ) satisfies L^* at every point $a \in I$.*

If then $x \in N(L(E, \mu))$ satisfies (5.2) and $x = 0$ on a subset F of E with $\mu(F) > 0$, then $x = 0$ on E.

Proof. By (5.2) there is a sequence $\{p_j\}$ of polynomials in $\mathbf{C}^n$ with $\deg p_j \leqslant k_j$ satisfying $d_N(x, \mathscr{P}_{k_j}) = \|x - p_j\|$, and a constant $a \in (0, 1)$ such that for each $j \geqslant 1$,

$$\int_F N(|p_j|)\, d\mu \leqslant a^{k_j}.$$

We may suppose F is bounded. Take $b \in (a, 1)$. By the already known argument we can find $m > 0$ and a subset H of F with $\mu(H) > 0$ such that for each $z \in H$,

$$\sup_j b^{-k_j} N(|p_j(z)|) \leqslant m.$$

Hence by Lemma 5.1, there is a constant $d \in (b, 1)$ such that for $z \in H$,

$$\sup_j d^{-k_j} |p_j(z)| \leqslant N^{-1}(m) \tag{5.7}$$

and by the continuity of each p_j, the inequality holds for $z \in \bar{H}$. Since $c(\bar{H}) > 0$, by Kellogg's lemma (see Section 2) there is a point $z_0 \in \bar{H}$ such that $V_{\bar{H}}^*(z_0) = 0$. Hence by (5.7), $p_j(z) \to 0$, as $j \to \infty$, uniformly on an open neighbourhood U of z_0, whence $x = 0$ on $E \cap U$. Now our assertion follows from (ii) which can be shown by the same argument as that of the proof of [12], Theorem 5.4, and therefore we omit the details.

5.6. Remark. Condition (ii) does not imply condition (i). For if μ is taken to be counting measure, then by Example 1.3, for each compact interval I, (I, μ) satisfies L^*, and the same time condition (i) fails to hold.

5.7. QUESTION. Does (i) imply (ii)? We note that the answer is "yes" in case of any measure of Example 1.6.

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Added in proof

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