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Asymptotic analyses of two fourth order linear differential equations

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Abstract. With r and q positive and continuous, the equations $(ru'')''+qu = 0$ and $(ru'')''-qu = 0$ are studied with respect to oscillation and the asymptotic behavior of non-oscillatory solutions.

I. Introduction. Let each of r and q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and suppose

(1) $$\int_0^\infty r(s)^{-1}ds = \infty.$$

In [4], we showed that if

(2) $$\int_0^\infty \left(\int_0^t (t-s)r(s)^{-1}ds\right) q(t)dt = \infty,$$

then every solution of

(3) $$(ru'')''+qu = 0$$

is oscillatory, i.e., has an unbounded set of zeros. In Section II we shall refine this result. In particular, we shall exhibit two functions φ and ψ such that if

(4) $$z''+\varphi z = 0$$

is oscillatory then (3) is oscillatory, and such that if

(5) $$z''+\psi z = 0$$

is non-oscillatory, then (3) is non-oscillatory.

There is no hypothesis, on the other hand, which will ensure that every solution of

(6) $$(ru'')''-qu = 0$$

is oscillatory. We shall prove, in Section III the existence of two classes of non-oscillatory solutions of (6), and then find conditions under which all non-oscillatory solutions are in these two classes. Also, it will be shown that if all non-oscillatory solutions are in the two classes, then the solution space of (6) has a two-dimensional subspace consisting of only oscillatory solutions. It will follow, for example, from the results of Section III that (1) and

$$\int_0^\infty t^2 q(t)\,dt = \infty \tag{7}$$

imply that (6) has non-trivial oscillatory solutions.

II. Equation (3). Throughout this section we shall assume that (2) fails, for otherwise the oscillatory properties of (3) are known. From (1), we can find $c \geqslant 0$ such that

$$\int_0^c r(s)^{-1}ds = 1.$$

Now

$$\int_0^\infty \Big(\int_0^t (t-s)r(s)^{-1}ds\Big)\, q(t)\,dt = \int_0^\infty \Big(\int_0^t r(s)^{-1}ds\Big)\Big(\int_t^\infty q(s)\,ds\Big)dt$$

$$\geqslant \int_c^\infty \Big(\int_0^t r(s)^{-1}ds\Big)\Big(\int_t^\infty q(s)\,ds\Big)dt \geqslant \int_c^\infty \Big(\int_t^\infty q(s)\,ds\Big)dt = \int_c^\infty (t-c)\,q(t)\,dt.$$

In particular, the failure of (2) implies

$$\int_0^\infty tq(t)\,dt < \infty.$$

Let φ and ψ be defined on $[0, \infty)$ by

$$\varphi(t) = r(t)^{-1}\int_t^\infty (s-t)\,q(s)\,ds$$

and

$$\psi(t) = q(t)\int_0^t (t-s)r(s)^{-1}ds.$$

The following theorem is the main result of this section.

THEOREM 1. *If* (4) *is oscillatory, then every solution of* (3) *is oscillatory, and if* (5) *is non-oscillatory, then every non-trivial solution of* (3) *is non-oscillatory.*

There are a great many results known with respect to oscillation and non-oscillation in second order equations, and Theorem 1, together with these results, permits the drawing of many corollaries. We cite an example.

COROLLARY 1. *If*

$$\limsup_{t\to\infty} t \int_t^\infty \left(\int_t^s (s-\xi) r(\xi)^{-1} d\xi\right) q(s)\,ds > 1,$$

then every solution of (3) *is oscillatory, and if*

$$\limsup_{t\to\infty} t \int_t^\infty \left(\int_0^s (s-\xi) r(\xi)^{-1} d\xi\right) q(s)\,ds < \tfrac{1}{4},$$

then every non-trivial solution of (3) *is non-oscillatory.*

Corollary 1 is clear from Theorem 1 and classical results of E. Hille [2] (see C. A. Swanson [5], Theorem 2.1, p. 45, for a summary of the relevant portions of Hille's work). It should be noted here that two application of integration-by-parts yield

$$\int_t^\infty \varphi(s)\,ds = \int_t^\infty \left(\int_t^s (s-\xi) r(\xi)^{-1} d\xi\right) q(s)\,ds$$

f $t \geqslant 0$. This last computation also shows that

$$\int_t^\infty \varphi(s)\,ds \leqslant \int_t^\infty \psi(s)\,ds$$

if $t \geqslant 0$, thus yeilding independent proof of a consequence of Theorem 1: If (4) is oscillatory, then (5) is oscillatory. (See Hille [2] and A. Wintner [7], also [5], Theorem 2.12, p. 60.)

Proof of Theorem 1. First we shall show that if (3) has a non-oscillatory solution, then (4) is non-oscillatory. Let u be a non-oscillatory solution of (3). If u is eventually negative, we may replace u by $-u$, so we assume that u is eventually positive. Let $w = ru''$, and find $a \geqslant 0$ such that $u(t) > 0$ if $t \geqslant a$. Now $w'' = -qu < 0$ on $[a, \infty)$, so w' is decreasing on $[a, \infty)$. If w' is ever negative on $[a, \infty)$, then $w(t) \to -\infty$ as $t \to \infty$, since w' is decreasing. This and (1) say that $u'(t) \to -\infty$ and $u(t) \to -\infty$ as $t \to \infty$, a contradiction. Thus $w' \geqslant 0$ on $[a, \infty)$. If $b > a$ and $w'(b) = 0$, then $w'(t) = 0$ whenever $t \geqslant b$, contradicting the fact that w' is decreasing. Thus $w'(t) > 0$ if $t \geqslant a$. Now w is increasing on $[a, \infty)$, and it is clear that either $w < 0$ on $[a, \infty)$, or there is $b \geqslant a$ such that $w > 0$ on $[b, \infty)$. We take cases.

Case 1. Suppose $w < 0$ on $[a, \infty)$. Now $u'' < 0$ on $[a, \infty)$, and reasoning similar to the above says $u' > 0$ on $[a, \infty)$. In particular, u is increasing on $[a, \infty)$. Since $w'' < 0$ and $w' > 0$ on $[a, \infty)$, $w'(\infty) = \lim_{t\to\infty} w'(t)$ exists and $w'(\infty) \geqslant 0$. Since $w' > 0$ and $w < 0$ on $[a, \infty)$,

$w(\infty)$ exists and $w(\infty) \leqslant 0$. The joint existence of $w(\infty)$ and $w'(\infty)$ says $w'(\infty) = 0$. Also,

$$u'(t) = u'(a) + \int_a^t r(s)^{-1} w(s)\,ds \leqslant u'(a) + w(\infty) \int_a^t r(s)^{-1}\,ds$$

if $t \geqslant a$, so (1) says that $w(\infty) < 0$ would contradict $u' > 0$ on $[a, \infty)$. Thus $w(\infty) = 0$. Now, if $\tau \geqslant t \geqslant a$,

$$w'(\tau) - w'(t) = -\int_t^\tau q(s)u(s)\,ds,$$

so

$$w'(t) = \int_t^\infty q(s)u(s)\,ds.$$

Also, if $\tau \geqslant t \geqslant a$,

$$\begin{aligned} w(\tau) - w(t) &= \int_t^\tau w'(s)\,ds = \int_t^\tau \Big(\int_s^\infty q(\xi)u(\xi)\,d\xi\Big)\,ds \\ &= (\tau - t)\int_\tau^\infty q(s)u(s)\,ds + \int_t^\tau (s-t)q(s)u(s)\,ds \\ &\geqslant \int_t^\tau (s-t)q(s)u(s)\,ds, \end{aligned}$$

so

$$-w(t) \geqslant \int_t^\infty (s-t)q(s)u(s)\,ds$$

and

$$\begin{aligned} -u''(t) &= -r(t)^{-1}w(t) \geqslant r(t)^{-1}\int_t^\infty (s-t)q(s)u(s)\,ds \\ &\geqslant r(t)^{-1}u(t)\int_t^\infty (s-t)q(s)\,ds, \end{aligned}$$

since u is increasing. Thus

$$u''(t)/u(t) \leqslant r(t)^{-1}\int_t^\infty (s-t)q(s)\,ds$$

whenever $t \geqslant a$. Let v be given on $[a, \infty)$ by $v(t) = u'(t)/u(t)$ and note that $v(t) > 0$ if $t \geqslant a$. Now

$$v'(t) = u''(t)/u(t) - v(t)^2$$

and

$$v'(t) + v(t)^2 \leqslant -r(t)^{-1}\int_t^\infty (s-t)q(s)\,ds \tag{8}$$

whenever $t \geqslant a$. A classical result of A. Wintner [6] (see also [5], Theorem 2.15, p. 63) says that the existence of a positive solution of (8) on $[a, \infty)$ implies that (4) is non-oscillatory, so the proof is complete for Case 1.

Case 2. Suppose there is $b \geqslant a$ such that $w > 0$ on $[b, \infty)$. Now

$$u'(t) = u'(b) + \int_b^t r(s)^{-1} w(s)\, ds \geqslant u'(b) + w(b) \int_b^t r(s)^{-1} ds$$

if $t \geqslant b$, so (1) says there is $c \geqslant b$ such that $u' > 0$ on $[c, \infty)$. If $t \geqslant c$, then

$$u(t) = u(c) + \int_c^t u'(s)\, ds \geqslant \int_c^t u'(s)\, ds = \int_c^t \Big(u'(c) + \int_c^s u''(\xi)\, d\xi\Big)\, ds$$

$$\geqslant \int_c^t \Big(\int_c^s u''(\xi)\, d\xi\Big)\, ds = \int_c^t (t-s) u''(s)\, ds = \int_c^t (t-s) r(s)^{-1} w(s)\, ds.$$

Also, $w' > 0$ and $w'' < 0$ on $[c, \infty)$, so $w'(\infty)$ exists, $w'(\infty) \geqslant 0$. If $\tau \geqslant t \geqslant c$, then

$$w'(t) = w'(\tau) + \int_t^\tau q(s) u(s)\, ds,$$

so

$$w'(t) = w'(\infty) + \int_t^\infty q(s) u(s)\, ds \geqslant \int_t^\infty q(s) u(s)\, ds.$$

Thus

$$w'(t) \geqslant \int_t^\infty q(s) \Big(\int_c^s (s-\xi) r(\xi)^{-1} w(\xi)\, d\xi\Big)\, ds \tag{9}$$

$$\geqslant \int_t^\infty q(s) \Big(\int_t^s (s-\xi) r(\xi)^{-1} w(\xi)\, d\xi\Big)\, ds$$

if $t \geqslant c$. Now (9) and standard iteration methods say that there is a continuously differentiable function z from $[c, \infty)$ to $[w(c), \infty)$ such that $z(c) = w(c)$, $z(t) \leqslant w(t)$ if $t \geqslant c$, and

$$z'(t) = \int_t^\infty q(s) \Big(\int_t^s (s-\xi) r(\xi)^{-1} z(\xi)\, d\xi\Big)\, ds \tag{10}$$

if $t \geqslant c$. Differentiating (10) yields

$$z''(t) = -\int_t^\infty q(s)(s-t) r(t)^{-1} z(t)\, ds,$$

$$z''(t) = -\Big(r(t)^{-1} \int_t^\infty (s-t) q(s)\, ds\Big) z(t).$$

Thus z solves (4) on $[c, \infty)$. Clearly z can be extended to a solution of (4) on $[0, \infty)$ and, since z has no zeros in $[c, \infty)$, this solution is non-oscillatory. This completes the proof of Case 2, and we have shown that if (4) is oscillatory, then (3) is oscillatory.

Suppose (5) is non-oscillatory, and let z be an eventually positive solution of (5). Find $a \geqslant 0$ such that $z(t) > 0$ if $t \geqslant a$. Now $z' > 0$ on $[a, \infty)$. If $\tau \geqslant t \geqslant a$, then

$$z'(t) = z'(\tau) + \int_t^\tau \psi(t) z(s)\, ds \geqslant \int_t^\tau \psi(s) z(s)\, ds,$$

so

$$z'(t) \geqslant \int_t^\infty \psi(s) z(s)\, ds = \int_t^\infty \Big(\int_0^s (s-\xi) r(\xi)^{-1} d\xi\Big) q(s) z(s)\, ds$$

$$\geqslant \int_t^\infty \Big(\int_t^s (s-\xi) r(\xi)^{-1} d\xi\Big) q(s) z(s)\, ds$$

if $t \geqslant a$. Thus, as before, there is a continuously differentiable function u from $[a, \infty)$ to $[z(a), \infty)$ such that $u(a) = z(a)$, $u(t) \leqslant z(t)$ if $t \geqslant a$, and

$$u'(t) = \int_t^\infty \Big(\int_t^s (s-\xi) r(\xi)^{-1} d\xi\Big) q(s) u(s)\, ds$$

if $t \geqslant a$. Now, if $t \geqslant a$,

$$u''(t) = -\int_t^\infty (s-t) r(t)^{-1} q(s) u(s)\, ds,$$

$$r(t) u''(t) = -\int_t^\infty (s-t) q(s) u(s)\, ds,$$

$$\big(r(t) u''(t)\big)' = \int_t^\infty q(s) u(s)\, ds,$$

$$\big(r(t) u''(t)\big)'' = -q(t) u(t).$$

Thus u solves (3) on $[a, \infty)$. Clearly u can be extended to a solution of (3) on $[0, \infty)$, and, since u has no zeros in $[a, \infty)$, this solution is non-oscillatory. But W. Leighton and Z. Nehari [3], Corollary 9.10, p. 367, have shown that the non-trivial solutions of (3) are either all oscillatory or all non-oscillatory, so the proof of Theorem 1 is complete.

III. Equation (6). Throughout this section, if u is a solution of (6) we let $w_u = ru''$. If $i = 1, 2, 3, 4$ let y_i be the solution of

$$y_i(t) = f_i(t) + \int_0^t (t-s) r(s)^{-1} \Big(\int_0^s (s-\xi) q(\xi) y_i(\xi)\, d\xi\Big) ds \tag{11}$$

on $[0, \infty)$, where $f_1(t) = 1$, $f_2(t) = t$,

$$f_3(t) = \int_0^t (t-s)r(s)^{-1}ds, \quad \text{and} \quad f_4(t) = \int_0^t (t-s)sr(s)^{-1}ds$$

if $t \geqslant 0$. Now $\{y_1, y_2, y_3, y_4\}$ is a basis for the solution space of (6). Since each f_i is positive on $(0, \infty)$, (11) says that each y_i is positive on $(0, \infty)$. Again using (11), this says that if u is in $\{y_1, y_2, y_3, y_4\}$, then u', w_u, and w_u' are positive on $(0, \infty)$. So we see that there exist eventually positive solutions u of (6) with u', w_u, and w_u' all eventually positive. There is also another class of eventually positive solutions.

THEOREM 2. *There exists a solution u of* (6) *with $u(t) > 0$, $u'(t) < 0$, $w_u(t) > 0$, and $w_u'(t) < 0$ whenever $t \geqslant 0$.*

If u is an eventually positive solution of (6) and each of u', w_u, and w_u' is eventually positive, we shall call u *strongly increasing*. If u is an eventually positive solution of (6) with w_u eventually positive and u' and w_u' eventually negative, we shall call u *strongly decreasing*. Before proving Theorem 2 we need a lemma.

LEMMA 1. *If u is a solution of* (6), *$c \geqslant 0$, $u(c) > 0$, $u'(c) < 0$, $w_u(c) > 0$, and $w_u'(c) < 0$, then $u(t) > 0$, $u'(t) < 0$, $w_u(t) > 0$, and $w_u'(t) < 0$ whenever $0 \leqslant t \leqslant c$.*

Lemma 1 is so similar to a result of S. P. Hastings and A. C. Lazer [1], Lemma 1.2, that we shall not include a proof.

Proof of Theorem 2. If n is a positive integer let v_n be a solution of (6) such that $v_n(n) > 0$, $v_n'(n) < 0$, $w_{v_n}(n) > 0$ and $w_{v_n}'(n) < 0$. Find numbers a_n, b_n, c_n, d_n such that $u_n = a_n y_1 + b_n y_2 + c_n y_3 + d_n y_4$. By multiplying u_n by a positive constant, if necessary, we may assume

$$a_n^2 + b_n^2 + c_n^2 + d_n^2 = 1. \tag{12}$$

But (12) says there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that

$$\alpha = \lim_{k\to\infty} a_{n_k}, \quad \beta = \lim_{k\to\infty} b_{n_k}, \quad \gamma = \lim_{k\to\infty} c_{n_k}, \quad \text{and} \quad \delta = \lim_{k\to\infty} d_{n_k}$$

exist. Clearly

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1. \tag{13}$$

Let $u_0 = \alpha y_1 + \beta y_2 + \gamma y_3 + \delta y_4$. Now (13) says u_0 is non-trivial. Also, Lemma 1 and the construction of $\{u_n\}_{n=1}^{\infty}$ ensure that $u_0 \geqslant 0$, $u_0' \leqslant 0$, $w_{u_0} \geqslant 0$, $w_{u_0}' \leqslant 0$ on $[0, \infty)$. If $t \geqslant 0$ and $u_0(t) = 0$, then $u_0(s) = 0$ whenever $s \geqslant t$, a contradiciton. Thus $u_0(t) > 0$ whenever $t \geqslant 0$. Now $w_{u_0}'' > 0$, so w_{u_0}' is increasing. Thus if $t \geqslant 0$ and $w_{u_0}'(t) = 0$, then $w_{u_0}' > 0$ on (t, ∞), a contradiction. Thus $w_{u_0}' < 0$ on $[0, \infty)$. Similarly, $w_{u_0} > 0$ and $u_0' < 0$ on $[0, \infty)$, and the proof is complete.

Since we now know of two classes of eventually positive solutions of (6), namely strongly increasing and strongly decreasing, the best oscillation theorem one can hope for is one which restricts all eventually positive solutions to these classes.

THEOREM 3. *If*

$$\int_0^\infty t^2 q(t)\,dt = \infty, \tag{14}$$

or if (14) *fails and*

$$(rz')' + \omega z = 0, \tag{15}$$

is oscillatory, where ω *is given by*

$$\omega(t) = \int_t^\infty (s-t)q(s)\,ds,$$

then every eventually positive solution of (6) *is either strongly increasing or strongly decreasing.*

It might appear that (14) gives and hypothesis which is independent of r, but it should be recalled that throughout we are assuming (1). Before proving Theorem 3, we need another lemma. This is an easy extension of an earlier cited classical result of A. Wintner [6] (see also [5], Theorem 2.15, p. 65). We suspect the lemma is known, but the author has not been able to find it in the literature. The proof of the lemma is so standard that we shall not include it.

LEMMA 2 *Equation* (15) *is non-oscillatory if and only if there is* $c \geqslant 0$, *and a positive continuously differentiable function* v *on* $[c, \infty)$, *such that*

$$r(t)v'(t) + v(t)^2 \leqslant -r(t)\omega(t)$$

whenever $t \geqslant c$.

Proof of Theorem 3. Suppose that u an eventually positive solution of (6) which is neither strongly increasing or strongly decreasing. Find $a \geqslant 0$ such that $u(t) > 0$ if $t \geqslant a$. Now $w_u'' > 0$ on $[a, \infty)$, so w_u' is increasing on $[a, \infty)$. If w_u' is ever non-negative, then, since w_u' is increasing, $w_u(t) \to +\infty$, $u'(t) \to +\infty$, and $u(t) \to +\infty$, as $t \to \infty$, i.e., u is strongly increasing. Thus $w_u' < 0$ on $[a, \infty)$. Now w_u is decreasing on $[a, \infty)$, and if w_u is ever negative on $[a, \infty)$, then $u'(t) \to -\infty$ and $u(t) \to -\infty$ as $t \to \infty$, contradicting the eventual positivity of u. Thus $w_u > 0$ on $[a, \infty)$. Now u' is increasing on $[a, \infty)$. If $u' < 0$ on $[a, \infty)$, then u is strongly decreasing, so there is $b \geqslant a$ such that $u' > 0$ on $[b, \infty)$.

Since $w_u'' > 0$ and $w_u' < 0$ on $[b, \infty)$, $w_u'(\infty)$ exists and $w_u'(\infty) \leqslant 0$. Since $w_u' < 0$ and $w_u > 0$ on $[b, \infty)$, $w_u(\infty)$ exists and $w_u(\infty) \geqslant 0$. Also, the joint existence of $w_u(\infty)$ and $w'_u(\infty)$ says $w'_u(\infty) = 0$. If $\tau \geqslant t \geqslant b$, then

$$w'_u(\tau) - w'_u(t) = \int_t^\tau q(s)u(s)\,ds,$$

so

$$w'_u(t) = -\int_t^\infty q(s)u(s)\,ds. \tag{16}$$

Also,

$$w_u(t) \geqslant \int_t^\infty (s-t)q(s)u(s)\,ds \tag{17}$$

if $t \geqslant b$. If $s \geqslant t \geqslant b$, then

$$u(s) = u(b) + \int_b^s u'(\xi)\,d\xi \geqslant \int_t^s u'(\xi)\,d\xi \geqslant (s-t)u'(t), \tag{18}$$

since u' is increasing. This and (17) say

$$w_u(b) \geqslant u'(b)\int_b^\infty (s-b)^2 q(s)\,ds.$$

Thus, (14) fails. Now (16) and (18) says that if $t \geqslant b$, then

$$\begin{gathered} -w'_u(t) \geqslant u'(t)\int_t^\infty (s-t)q(s)\,ds, \\ w'_u(t)/u'(t) \leqslant -\int_t^\infty (s-t)q(s)\,ds = -\omega(t). \end{gathered} \tag{19}$$

Let v be given on $[b, \infty)$ by $v(t) = w_u(t)/u'(t)$, and note that v is positive. Also,

$$v'(t) = w'_u(t)/u'(t) = r(t)^{-1}v(t)^2,$$

so (19) says

$$r(t)v'(t) + v(t)^2 \leqslant -r(t)\,\omega(t)$$

if $t \geqslant b$. Thus (15) is non-oscillatory and the proof is complete.

Although Theorem 3 gives conditions which restrict the possible asymptotic behaviors of non-oscillatory solutions of (6) (if u is non-oscillatory, then one of u and $-u$ is eventually positive), it does not in fact ensure the existence of non-trivial oscillatory solutions. The following theorem not only yields oscillatory solutions, but it confirms that the question partially answered by Theorem 3 is indeed the relevant question to ask.

THEOREM 4. *Suppose that every eventually positive solution of* (6) *is either strongly increasing or strongly decreasing. Then there are three linearly independent oscillatory solutions of* (6), *and there is a two-dimensional subspace of the solution space of* (6) *every member of which is oscillatory.*

Theorems 3 and 4 have an obvious corollary which we state for completeness.

COROLLARY 2. *If* (14) *holds, or if* (14) *fails and* (15) *is oscillatory, then there are three linearly independent solutions of* (6), *and there is a two-dimensional subspace of the solution space of* (6) *every member of which is oscillatory.*

Proof of Theorem 4. If $i = 1, 2, 3$ let $\{a_{in}\}_{n=1}^{\infty}$ and $\{b_{in}\}_{n=1}^{\infty}$ be sequences such that

$$a_{in} y_i(n) + b_{in} y_{i+1}(n) = 0 \tag{20}$$

and such that

$$a_{in}^2 + b_{in}^2 = 1. \tag{21}$$

By (21), there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that

$$a_i = \lim_{k\to\infty} a_{in_k} \quad \text{and} \quad \beta_i = \lim_{k\to\infty} b_{in_k}$$

exist for $i = 1, 2, 3$. For $i = 1, 2, 3$, let $x_i = a_i y_i + \beta_i y_{i+1}$. It follows from (21) that $a_i^2 + \beta_i^2 = 1$, so each x_i is non-trivial. We claim each x_i is oscillatory.

Let i be in $\{1, 2, 3\}$ and suppose x_i is non-oscillatory. There is no loss in assuming that x_i is eventually positive, and we do. Since x_i is a linear combination of two of $\{y_1, y_2, y_3, y_4\}$, at least one of $x_i(0)$, $x_i'(0)$, $w_{x_i}(0)$, and $w_{x_i}'(0)$ is zero. Thus Lemma 1 says that x_i is not strongly decreasing, so, by hypothesis, x_i is strongly increasing. Find $c \geqslant 0$ such that $x_i(t) > 0$, $x_i'(t) > 0$, $w_{x_i}(t) > 0$, and $w_{x_i}'(t) > 0$ if $t \geqslant c$. Clearly there is an integer k such that $n_k > c$ and such that, if $u = a_{in_k} y_i + b_{in_k} y_{i+1}$, $u(c) > 0$, $u'(c) > 0$, $w_u(c) > 0$, and $w_u'(c) > 0$. Now,

$$\begin{aligned} u(t) = u(c) + (t-c)u'(c) + w_u(c) \int_c^t (t-s)r(s)^{-1} ds + \\ + w_u'(c) \int_c^t (t-s) s r(s)^{-1} ds + \int_c^t (t-s) r(s)^{-1} \Big(\int_c^s (s-\xi) q(\xi) u(\xi) d\xi \Big) ds \end{aligned} \tag{22}$$

if $t \geqslant c$. But the solution of (22) is positive on $[c, \infty)$, and $u(n_k) = 0$. This is a contradiction and x_i is oscillatory.

Similar arguments can be used to show that no linear combination of $\{x_1, x_2, x_3\}$ is strongly increasing. Furthermore, a use of Lemma 1 similar to that above says that no linear combination of $\{x_1, x_2\}$ is strongly decreasing, so every member of *span* $\{x_1, x_2\}$ is oscillatory. If, for any i, $a_i \beta_i = 0$, then x_i could not be oscillatory. Thus $a_i \neq 0$ and $\beta_i \neq 0$, and the linear independence of $\{x_1, x_2, x_3\}$ is immediate. This completes the proof.

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