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On a generalization of Frenet equations

by S. TOPA (Kraków)

The purpose of this note is to generalize the result of my former work [4] (related to the curve in a 3-dimensional space) to the case of an n-dimensional Euclidean space ($n > 3$).

I. The vectors of a Frenet n-hedron for a curve. Let C be a curve in an n-dimensional Euclidean space R_n given by the vectorial equation

(1) $$\boldsymbol{r} = \boldsymbol{r}(\sigma), \quad \sigma \in I \text{ }(^1)$$

where σ is the length of the arc of the curve C taken from a fixed point.

We assume the following properties for the curve C:

1) For any j-dimensional plane L_j ($j < n$) through the point $M(\sigma_0)$ there exists a neighbourhood of this point such that the plane L_j and the curve C meet only at the point $M(\sigma_0)$ in it.

2) The function $\boldsymbol{r}(\sigma)$ is of the class C^{n-2} in the interval I.

3) The derivative of the order $n-1$ of the function $\boldsymbol{r}(\sigma)$ exists at the point σ_0.

4) The following condition holds:

$$\operatorname{ord}(\boldsymbol{r}_0^{(1)}, \boldsymbol{r}_0^{(2)}, \ldots, \boldsymbol{r}_0^{(n-1)}) = n-1$$

where $\boldsymbol{r}_0^{(i)}$ denotes the value of the derivative of order i of the function $\boldsymbol{r}(\sigma)$ at the point σ_0.

Now we shall give a certain construction of the vectors $\boldsymbol{t}_i$ of a Frenet n-hedron, different from the classical one.

On the basis of [3] the curve C with equation (1) has a system $H_1, \ldots, H_{n-1}$ of osculating planes at the point $M(\sigma_0)$, where H_i is the i-dimensional plane given by the equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \sum_{\nu=1}^{i} \lambda_\nu^i \boldsymbol{r}_0^{(\nu)} \quad (i = 1, \ldots, n-1).$$

We shall find an orthonormal system of vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n-1}$ such that $T_i = H_i$, where T_i denotes the i-dimensional plane through the point $M(\sigma_0)$ with the vectorial base $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_i$.

(1) I is the neighbourhood of the point σ_0.

For this purpose we shall prove the following

THEOREM 1. *The system of vectors* $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_i$ $(i = 1, \ldots, n-1)$ *given by the formulas*

$$\boldsymbol{t}_1 = \boldsymbol{r}_0^{(1)}, \tag{2}$$
$$\boldsymbol{t}_m = \operatorname{vers}\Big(\sum_{j=1}^{m-1} \mu_j^m \boldsymbol{t}_j + \lambda_m \boldsymbol{r}_0^{(m)}\Big) \qquad (m = 2, \ldots, n-1),$$

where μ_j^m *and* λ_m $(\lambda_m > 0)$ *are arbitrary real numbers, is the vectorial base for the plane* H_i.

Proof. We use the method of mathematical induction with respect to i $(i = 1, \ldots, n-1)$. For $i = 1$ the theorem is true.

Now let us assume that the theorem is true for $i = s$ and let us show that is true for $i = s+1$. From the inductional assumption it follows that each of the vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_s$ depends linearly on the vectors $\boldsymbol{r}_0^{(1)}, \ldots, \boldsymbol{r}_0^{(s)}$.

In view of (2) we see that

$$\boldsymbol{t}_{s+1} = \operatorname{vers}\Big(\sum_{j=1}^{s} \mu_j^{s+1} \boldsymbol{t}_j + \lambda_{s+1} \boldsymbol{r}_0^{(s+1)}\Big),$$

and we conclude that the vector $\boldsymbol{t}_{s+1}$ depends linearly on the vectors $\boldsymbol{r}_0^{(1)}, \ldots, \boldsymbol{r}_0^{(s+1)}$.

From this follows our theorem.

In formulas (2) we can put $\lambda_m = 1$ $(m = 2, \ldots, n-1)$.

Now we have to choose the coefficients μ_j^m in (2) according to the conditions

$$\boldsymbol{t}_k \cdot \boldsymbol{t}_j = \delta_{kj} \qquad (k, j = 1, \ldots, m). \tag{3}$$

So let us multiply the second formula in (2) scalarly by the vectors $\boldsymbol{t}$ $(1 \leqslant j \leqslant m-1)$. By conditions (3) we get

$$0 = \mu_j^m + \boldsymbol{t}_j \boldsymbol{r}_0^{(m)},$$

which yields

$$\mu_j^m = -\boldsymbol{t}_j \boldsymbol{r}_0^{(m)} \qquad (j = 1, \ldots, m-1;\ m = 2, \ldots, n-1).$$

Therefore we have

$$\boldsymbol{t}_m = \operatorname{vers}\Big(\boldsymbol{r}_0^{(m)} - \sum_{\nu=1}^{m-1} (\boldsymbol{t}_\nu \boldsymbol{r}_0^{(m)}) \boldsymbol{t}_\nu\Big).$$

The vector $\boldsymbol{t}_n$ will be defined by the formula

$$\boldsymbol{t}_n = \boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{n-1},$$

where the symbol on the right-hand side denotes the vector product of the vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n-1}$.

Thus we have the formulas

$$\begin{aligned} &\boldsymbol{t}_1 = \boldsymbol{r}_0^{(1)}, \\ (4) \qquad &\boldsymbol{t}_m = \operatorname{vers}\Big(\boldsymbol{r}_0^{(m)} - \sum_{\nu=1}^{m-1}(\boldsymbol{t}_\nu \boldsymbol{r}_0^{(m)})\boldsymbol{t}_\nu\Big) \qquad (m = 2, \ldots, n-1), \\ &\boldsymbol{t}_n = \boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{n-1}. \end{aligned}$$

DEFINITION 1. The vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ given by formulas (4) will be named *the vectors of a Frenet n-hedron for the curve C at the point* $M(\sigma_0)$.

The following theorem will give a certain geometrical interpretation of the vectors $\{\boldsymbol{t}_i\}$ $(i = 1, \ldots, n)$ given by formulas (4):

THEOREM 2. *Assume that* $m \geqslant 2$. *Let us consider an* $(n-m+1)$-*dimensional plane* H_{n-m+1} *through the point* $M(\sigma_0)$ *which is perpendicular to the plane* H_{m-1}. *Let us project* (perpendicularly) *the curve C on the plane* H_{n-m+1}. *Let us denote the projection by* C^*. *The projection of the point* $M(\sigma)$ *of the curve C will be denoted by* $M^*(\sigma)$ *and its radius-vector by* $\boldsymbol{r}^*(\sigma)$. *Next let us form the ratio*

$$\boldsymbol{s}_m(h) = \frac{(\boldsymbol{r}^* - \boldsymbol{r}_0)}{|\boldsymbol{r}^* - \boldsymbol{r}_0|} = \operatorname{vers}(\boldsymbol{r}^* - \boldsymbol{r}_0),$$

where $h = \sigma - \sigma_0$, $\boldsymbol{r}^* = \boldsymbol{r}^*(\sigma)$, $\boldsymbol{r}_0 = \boldsymbol{r}(\sigma_0)$.

We shall prove that there exists a limit

$$\lim_{h \to 0} \boldsymbol{s}_m(h) = \boldsymbol{t}_m, \qquad h > 0.$$

Proof. Let us write the equation of the plane H_{n-m+1} in the form

$$\boldsymbol{r} = \boldsymbol{r}_0 + \sum_{\nu=1}^{n-m+1} a_\nu \boldsymbol{a}_\nu, \qquad a_\nu = \text{const},$$

where the system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-m+1}$ is an orthonormal system such that

$$\boldsymbol{a}_j \boldsymbol{t}_k = 0 \qquad (j = 1, \ldots, n-m+1;\ k = 1, \ldots, m).$$

The equation of the projecting plane H_{m-1} can be written in the form

$$\hat{\boldsymbol{r}} = \boldsymbol{r} + \sum_{\nu=1}^{m-1} \lambda_\nu \boldsymbol{t}_\nu.$$

In order to get the equation of the curve C^* we put

$$(5) \qquad \boldsymbol{r} + \sum_{\nu=1}^{m-1} \lambda_\nu \boldsymbol{t}_\nu = \boldsymbol{r}_0 + \sum_{j=1}^{n-m+1} a_j \boldsymbol{a}_j.$$

Multiplying (5) scalarly by $\boldsymbol{t}_\nu$ we obtain

$$\lambda_\nu = -(\boldsymbol{r} - \boldsymbol{r}_0)\boldsymbol{t}_\nu \qquad (\nu = 1, \ldots, m-1),$$

so the equation of the curve C^* has the form

$$\boldsymbol{r}^* = \boldsymbol{r} - \sum_{\nu=1}^{m-1} [(\boldsymbol{r}-\boldsymbol{r}_0)\boldsymbol{t}_\nu]\boldsymbol{t}_\nu . \tag{6}$$

In consequence of (6) we have

$$\boldsymbol{s}_m(h) = \operatorname{vers}\Big\{(\boldsymbol{r}-\boldsymbol{r}_0) - \sum_{\nu=1}^{m-1} [(\boldsymbol{r}-\boldsymbol{r}_0)\boldsymbol{t}_\nu]\boldsymbol{t}_\nu\Big\} .$$

Let us apply Peano's formula with derivatives up to the order m to the difference $\boldsymbol{r}-\boldsymbol{r}_0$; we get

$$\begin{aligned}\boldsymbol{s}_m(h) &= \operatorname{vers}\Big\{\sum_{j=1}^{m} \frac{h^j}{j!}\boldsymbol{r}_0^{(j)} + \frac{h^m}{m!}\boldsymbol{e} - \sum_{\nu=1}^{m-1}\Big[\Big(\Big(\sum_{j=1}^{m}\frac{h^j}{j!}\boldsymbol{r}_0^{(j)} + \frac{h^m}{m!}\boldsymbol{e}\Big)\boldsymbol{t}_\nu\Big]\boldsymbol{t}_\nu\Big\} \\ &= \operatorname{vers}\Big\{\sum_{j=1}^{m-1}\frac{h^j}{j!}\Big[\boldsymbol{r}_0^{(j)} - \sum_{\nu=1}^{m-1}(\boldsymbol{t}_\nu\boldsymbol{r}_0^{(j)})\boldsymbol{t}_\nu\Big] + \\ &\qquad + \frac{h^m}{m!}\Big[\boldsymbol{r}_0^{(m)} + \boldsymbol{e} - \sum_{\nu=1}^{m-1}\big((\boldsymbol{r}_0^{(m)} + \boldsymbol{e})\boldsymbol{t}_\nu\big)\boldsymbol{t}_\nu\Big]\Big\} .\end{aligned}$$

It can be verified by using formulas (4) that

$$\sum_{j=1}^{m-1}\frac{h^j}{j!}\Big[\boldsymbol{r}_0^{(j)} - \sum_{\nu=1}^{m-1}(\boldsymbol{t}_\nu\boldsymbol{r}_0^{(j)})\boldsymbol{t}_\nu\Big] = 0 .$$

In consequence we obtain

$$\boldsymbol{s}_m(h) = \operatorname{vers}\Big\{\boldsymbol{r}_0^{(m)} + \boldsymbol{e} - \sum_{\nu=1}^{m-1}[(\boldsymbol{r}_0^{(m)} + \boldsymbol{e})\boldsymbol{t}_\nu]\boldsymbol{t}_\nu\Big\} .$$

If $h \to 0$, this vector tends to the limit vector equal to $\boldsymbol{t}_m$.
Then the theorem is proved.

II. The definition of curvatures for a curve. Now we make use of a lemma of S. Gołąb [2] which states that for a curve with equation (1) of such regularity that Frenet's equations are valid (in the classical sense) we have relations of the form

$$\boldsymbol{r}_0^{(m+1)} = \sum_{j=1}^{m} a_j^m(\sigma)\boldsymbol{t}_j + \prod_{j=1}^{m}\varkappa_j \boldsymbol{t}_{m+1} \qquad (m = 1, \ldots, n-2) , \tag{7}$$

where the coefficients a_j^m are certain algebraical functions of the curvatures $\varkappa_j$ and their derivatives.

Multiplying (7) scalarly by $\boldsymbol{t}_{m+1}$ we obtain

$$\boldsymbol{t}_{m+1}\boldsymbol{r}_0^{(m+1)} = \prod_{j=1}^{m} \varkappa_j \qquad (m = 1, \ldots, n-2)\,. \tag{8}$$

Formulas (8) can be used to give a formal definition of successive curvatures $\varkappa_j$ $(j = 1, \ldots, n-2)$ for a curve with weaker assumptions of regularity than the above ones.

Let the curve C satisfy conditions 1)-4), which have been stated in chapter I.

Then there exists a system of vectors $\{\boldsymbol{t}_i\}$ given by formulas (4) which satisfies the conditions

$$\boldsymbol{t}_m\boldsymbol{r}_0^{(m)} > 0 \qquad (m = 1, \ldots, n-1)\,.$$

This implies that all the functions $\varkappa_1, \ldots, \varkappa_{n-2}$ defined formally by (8) are positive. So we can write the formulas

$$\begin{aligned} \varkappa_1 &= \boldsymbol{t}_2\boldsymbol{r}_0^{(2)}\,, \\ \varkappa_m &= \frac{\boldsymbol{t}_{m+1}\boldsymbol{r}_0^{(m+1)}}{\varkappa_1 \ldots \varkappa_{m-1}} \qquad (m = 2, \ldots, n-2)\,. \end{aligned} \tag{9}$$

THEOREM 3. *The formulas in* (9) *may be represented in the simpler form*

$$\begin{aligned} \varkappa_1 &= \boldsymbol{t}_2\boldsymbol{r}_0^{(2)}, \\ \varkappa_m &= \frac{\boldsymbol{t}_{m+1}\boldsymbol{r}_0^{(m+1)}}{\boldsymbol{t}_m\boldsymbol{r}_0^{(m)}} \qquad (m = 2, \ldots, n-2)\,. \end{aligned}$$

Proof. We shall use the method of mathematical induction. We have

$$\varkappa_2 = \frac{\boldsymbol{t}_3\boldsymbol{r}_0^{(3)}}{\varkappa_1} = \frac{\boldsymbol{t}_3\boldsymbol{r}_0^{(3)}}{\boldsymbol{t}_2\boldsymbol{r}_0^{(2)}}\,.$$

Assuming that

$$\varkappa_{k-1} = \frac{\boldsymbol{t}_k\boldsymbol{r}_0^{(k)}}{\boldsymbol{t}_{k-1}\boldsymbol{r}_0^{(k-1)}} \qquad (k = 2, 3, \ldots, n-2)\,,$$

we obtain

$$\varkappa_k = \frac{\boldsymbol{t}_{k+1}\boldsymbol{r}_0^{(k+1)}}{(\varkappa_1 \ldots \varkappa_{k-2})\,\varkappa_{k-1}} = \frac{\boldsymbol{t}_{k+1}\boldsymbol{r}_0^{(k+1)}}{\boldsymbol{t}_{k-1}\boldsymbol{r}_0^{(k-1)} \cdot \dfrac{\boldsymbol{t}_k\boldsymbol{r}_0^{(k)}}{\boldsymbol{t}_{k-1}\boldsymbol{r}_0^{(k-1)}}} = \frac{\boldsymbol{t}_{k+1}\boldsymbol{r}_0^{(k+1)}}{\boldsymbol{t}_k\boldsymbol{r}_0^{(k)}}\,,$$

which was to be proved.

Now let us make the additional assumption that the vector $\boldsymbol{t}_{n-1}$ has a derivative of order 1 at the point σ_0.

Then we can put

$$\varkappa_{n-1} = [\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-1}, \boldsymbol{t}'_{n-1}], \qquad \boldsymbol{t}'_{n-1} \stackrel{\text{def}}{=} \frac{d\boldsymbol{t}_{n-1}}{d\sigma},$$

where the symbol on the right side denotes a mixed product of the vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-1}, \boldsymbol{t}'_{n-1}$, which is simply equal to $\det(\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-1}, \boldsymbol{t}'_{n-1})$.

DEFINITION 2. The functions $\varkappa_1, \dots, \varkappa_{n-1}$ given by the formulas

$$\begin{aligned} \varkappa_1 &= \boldsymbol{t}_2 \boldsymbol{r}_0^{(2)}, \\ \varkappa_m &= \frac{\boldsymbol{t}_{m+1} \boldsymbol{r}_0^{(m+1)}}{\boldsymbol{t}_m \boldsymbol{r}_0^{(m)}} \qquad (m = 2, \dots, n-2), \\ \varkappa_{n-1} &= [\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-1}, \boldsymbol{t}'_{n-1}] \end{aligned}$$

will be named *the seccessive curvatures for the curve C at the point $M(\sigma_0)$*.

III. Frenet equations. We assume that the curve C has vectors of a Frenet n-hedron in a certain neighbourhood of the point $M(\sigma_0)$ in the sense of definition 1. Furthermore all the curvatures of the curve C at the point $M(\sigma_0)$ exist in the sense of definition 2.

THEOREM 4. *If the above assumptions are satisfied then the following Frenet equations for $\sigma = \sigma_0$ hold:*

$$\begin{aligned} \boldsymbol{t}'_1 &= \varkappa_1 \boldsymbol{t}_2, \\ \boldsymbol{t}'_m &= -\varkappa_{m-1} \boldsymbol{t}_{m-1} + \varkappa_m \boldsymbol{t}_{m+1} \qquad (m = 2, \dots, n-1), \\ \boldsymbol{t}'_n &= -\varkappa_{n-1} \boldsymbol{t}_{n-1}. \end{aligned} \tag{10}$$

Remark 1. In the above theorem we do not assume the differentiability of the vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_n$ at the point σ_0, because we can easily verify that:

THEOREM. *The necessary and sufficient condition for the existence of the successive curvatures for the curve C at the point $M(\sigma_0)$ is that the vectors of a Frenet n-hedron exist in a certain neighbourhood of the point $M(\sigma_0)$ and be differentiable at the point σ_0.*

Proof. The sufficiency does not require a proof; so let us pass to the necessity.

If there exist successive curvatures for the curve C at the point $M(\sigma_0)$, then using the method of mathematical induction we can prove the differentiability of the vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-2}$. The differentiability of the vector $\boldsymbol{t}_{n-1}$ was assumed in definition of the curvature $\varkappa_{n-1}$. Finally the differentiability of the vector $\boldsymbol{t}_n$ follows by the theorem on the derivative of the vector product.

Now we shall prove theorem 4.

Proof of theorem 4. The first equality follows simply by the calculation

$$\varkappa_1 t_2 = (t_2 r_0^{(2)}) t_2 = \left(\frac{r_0^{(2)}}{|r_0^{(2)}|} r_0^{(2)}\right) \frac{r_0^{(2)}}{|r_2^{(2)}|} = r_0^{(2)} = t_1' .$$

The next equalities except the last one, will be proved by means of the method of mathematical induction.

We have

$$\varkappa_1 = t_2 r_0^{(2)} = \frac{r_0^{(2)}}{|r_0^{(2)}|} r_0^{(2)} = \frac{r_0^{(2)2}}{|r_0^{(2)}|} = |r_0^{(2)}| ,$$

$$\varkappa_2 = \frac{t_3 r_0^{(3)}}{t_2 r_0^{(2)}} = \frac{[r_0^{(3)} - (t_1 r_0^{(3)}) t_1 - (t_2 r_0^{(3)}) t_2] r_0^{(3)}}{|[r_0^{(3)} - (t_1 r_0^{(3)}) t_1 - (t_2 r_0^{(3)}) t_2]| \cdot |r_0^{(2)}|}$$

$$= \frac{\left[r_0^{(3)} - (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - \left(\frac{r_0^{(2)}}{|r_0^{(2)}|} r_0^{(3)}\right) \frac{r_0^{(2)}}{|r_0^{(2)}|}\right] r_0^{(3)}}{\left|r_0^{(3)} - (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - \left(\frac{r_0^{(2)}}{|r_0^{(2)}|} r_0^{(3)}\right) \frac{r_0^{(2)}}{|r_0^{(2)}|}\right| \cdot |r_0^{(2)}|}$$

$$= \frac{r_0^{(2)2} r_0^{(3)2} - r^{(2)2} (r_0^{(1)} r_0^{(3)})^2 - (r_0^{(2)} r_0^{(2)})^2}{|r_0^{(2)2} r_0^{(3)} - r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}| \cdot |r_0^{(2)}|} ,$$

$$t_3 = \frac{r_0^{(2)2} r_0^{(3)} - r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}}{|r_0^{(2)2} r_0^{(3)} - r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}|} ,$$

and

$$t_2' = \left(\frac{r^{(2)}}{|r^{(2)}|}\right)'\Bigg|_{\sigma=\sigma_0} = \frac{|r_0^{(2)}| r_0^{(3)} - r_0^{(2)} \cdot \frac{1}{2\,|r_0^{(2)}|} 2 (r_0^{(2)} r_0^{(3)})}{r_0^{(2)2}} \tag{11}$$

$$= \frac{r_0^{(2)2} r_0^{(3)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}}{|r_0^{(2)}|^3} .$$

Hence

$$-\varkappa_1 t_1 + \varkappa_2 t_3 = -|r_0^{(2)}| r_0^{(1)} +$$

$$+ \frac{[r_0^{(2)2} r_0^{(3)2} - (r_0^{(1)} r_0^{(3)})^2 r_0^{(2)2} - (r_0^{(2)} r_0^{(3)})^2}{[r_0^{(2)2} r_0^{(3)} - (r_0^{(1)} r_0^{(3)}) r_0^{(2)2} r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}] \cdot |r_0^{(2)}|} \times$$

$$\times \frac{[r_0^{(2)2} r_0^{(3)} - r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}]}{[r_0^{(2)2} r_0^{(3)} - (r_0^{(1)} r_0^{(3)}) r_0^{(2)2} r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}]}$$

$$= -|r_0^{(2)}| r_0^{(1)} + \frac{1}{|r_0^{(2)}|^3} [r_0^{(2)2} r_0^{(3)} - r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)} - (r_0^{(2)} r_0^{(3)}) r_2^{(2)}]$$

$$= \left(-|r_0^{(2)}| r_0^{(1)} - \frac{1}{|r_0^{(2)}|^3} r_0^{(2)2} (r_0^{(1)} r_0^{(3)}) r_0^{(1)}\right) + \frac{r_0^{(2)2} r_0^{(3)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}}{|r_0^{(2)}|^3}$$

$$= \frac{r_0^{(2)2} r_0^{(3)} - (r_0^{(2)} r_0^{(3)}) r_0^{(2)}}{|r_0^{(2)}|^3} ,$$

and in view of (11) we have

$$-\varkappa_1 \boldsymbol{t}_1 + \varkappa_2 \boldsymbol{t}_3 = \boldsymbol{t}_2' .$$

In the foregoing calculation we have made use of the formulas:

a) $$[\boldsymbol{r}_0^{(2)^2}\boldsymbol{r}_0^{(3)} - \boldsymbol{r}_0^{(2)^2}(\boldsymbol{r}_0^{(1)}\boldsymbol{r}_0^{(3)})\boldsymbol{r}_0^{(1)} - (\boldsymbol{r}_0^{(2)}\boldsymbol{r}_0^{(3)})\boldsymbol{r}_0^{(2)}]^2$$
$$= \boldsymbol{r}_0^{(2)^2}[\boldsymbol{r}_0^{(2)^2}\boldsymbol{r}_0^{(3)^2} - \boldsymbol{r}_0^{(2)^2}(\boldsymbol{r}_0^{(1)}\boldsymbol{r}_0^{(3)}) - (\boldsymbol{r}_0^{(2)}\boldsymbol{r}_0^{(3)})^2] .$$

Furthermore

$$\boldsymbol{r}^{(1)}\boldsymbol{r}^{(2)} = 0 ,$$

which after differentiation yields

$$\boldsymbol{r}^{(2)^2} + \boldsymbol{r}^{(1)}\boldsymbol{r}^{(3)} = 0 .$$

Hence it follows that

$$\boldsymbol{r}^{(1)}\boldsymbol{r}^{(3)} = -\boldsymbol{r}^{(2)^2} ,$$

and particularly at the point σ_0 we have

b) $$\boldsymbol{r}_0^{(1)}\boldsymbol{r}_0^{(3)} = -\boldsymbol{r}_0^{(2)^2} .$$

Therefore we see that the second equation holds.

Suppose now that the equation with the number $m = k$ is valid. We shall prove that the equation with the number $m = k+1$ remains true.

At first we shall find the derivative of the function $\boldsymbol{t}_{k+1}$. If we use the notation

$$\boldsymbol{v}_s = \boldsymbol{r}^{(s)} - \sum_{j=1}^{s-1} (\boldsymbol{t}_j \boldsymbol{r}^{(s)})\boldsymbol{t}_j ,$$

then

$$\boldsymbol{t}_{k+1} = \operatorname{vers} \boldsymbol{v}_{k+1} ,$$

and at the point σ_0 we have

$$\boldsymbol{t}_{k+1}' = \frac{\boldsymbol{v}_{k+1}'|\boldsymbol{v}_{k+1}| - \boldsymbol{v}_{k+1}\left[\dfrac{1}{2|\boldsymbol{v}_{k+1}|} 2(\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+1}')\right]}{\boldsymbol{v}_{k+1}^2}$$
$$= \frac{1}{|\boldsymbol{v}_{k+1}|}\left[\boldsymbol{v}_{k+1}' - \frac{(\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+1}')\boldsymbol{v}_{k+1}}{\boldsymbol{v}_{k+1}^2}\right]$$
$$= \frac{1}{|\boldsymbol{v}_{k+1}|}\left[\boldsymbol{v}_{k+1}' - \left(\frac{\boldsymbol{v}_{k+1}}{|\boldsymbol{v}_{k+1}|}\boldsymbol{v}_{k+1}'\right)\frac{\boldsymbol{v}_{k+1}}{|\boldsymbol{v}_{k+1}|}\right]$$
$$= \frac{1}{|\boldsymbol{v}_{k+1}|}[\boldsymbol{v}_{k+1}' - (\boldsymbol{t}_{k+1}\boldsymbol{v}_{k+1}')\boldsymbol{t}_{k+1}] .$$

Now we shall find $\boldsymbol{v}_{k+1}'$ and further $\boldsymbol{t}_{k+1}\boldsymbol{t}_{k+1}'$.

We have

$$\begin{aligned}
\boldsymbol{v}'_{k+1} &= \boldsymbol{r}^{(k+2)} - \sum_{j=1}^{k} [(\boldsymbol{t}_j \boldsymbol{r}^{(k+2)} + \boldsymbol{t}'_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j + (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_j] \\
&= \boldsymbol{r}^{(k+2)} - \sum_{j=1}^{k} (\boldsymbol{t}_j \boldsymbol{r}^{(k+2)}) \boldsymbol{t}_j - \sum_{j=1}^{k} [(\boldsymbol{t}'_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j + (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_j] \\
&= \boldsymbol{r}^{(k+2)} - \sum_{j=1}^{k} (\boldsymbol{t}_j \boldsymbol{r}^{(k+2)}) \boldsymbol{t}_j - \{(\boldsymbol{t}'_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_1 + (\boldsymbol{t}_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_1\} - \\
&\qquad - \sum_{j=2}^{k} [(\boldsymbol{t}'_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j + (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_j] .
\end{aligned}$$

Let us consider the following expressions:

$$\begin{aligned}
1)\ (\boldsymbol{t}'_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j &= [-\varkappa_{j-1} (\boldsymbol{t}_{j-1} \boldsymbol{r}^{(k+1)}) + \varkappa_j (\boldsymbol{t}_{j+1} \boldsymbol{r}^{(k+1)})] \boldsymbol{t}_j \\
&= -\varkappa_{j-1} (\boldsymbol{t}_{j-1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j + \varkappa_j (\boldsymbol{t}_{j+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j , \\
(\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_j &= (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) [-\varkappa_{j-1} \boldsymbol{t}_{j-1} + \varkappa_j \boldsymbol{t}_{j+1}] \\
&= -\varkappa_{j-1} (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j-1} + \varkappa_j (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j+1} ,
\end{aligned}$$

for $j = 2, \ldots, k$.

$$\begin{aligned}
2)\ &\sum_{j=2}^{k} [(\boldsymbol{t}'_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j + (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_j] \\
&= \sum_{j=2}^{k} [\varkappa_j (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j+1} - \varkappa_{j-1} (\boldsymbol{t}_{j-1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j] + \\
&\quad + \sum_{j=2}^{k} [\varkappa_j (\boldsymbol{t}_{j+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j - \varkappa_{j-1} (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j-1}] \\
&= \sum_{j=2}^{k} \varkappa_j (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j+1} - \sum_{j=1}^{k-1} \varkappa_j (\boldsymbol{t}_j \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{j+1} + \\
&\quad + \sum_{j=2}^{k} \varkappa_j (\boldsymbol{t}_{j+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j - \sum_{j=1}^{k-1} \varkappa_j (\boldsymbol{t}_{j+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_j \\
&= \varkappa_k (\boldsymbol{t}_k \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{k+1} + \varkappa_k (\boldsymbol{t}_{k+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_k - \\
&\quad - \{\varkappa_1 (\boldsymbol{t}_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_2 + \varkappa_1 (\boldsymbol{t}_2 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_1\} .
\end{aligned}$$

$$3)\ (\boldsymbol{t}'_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_1 + (\boldsymbol{t}_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}'_1 = \varkappa_1 (\boldsymbol{t}_2 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_1 + \varkappa_1 (\boldsymbol{t}_1 \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_2 .$$

As a result of 1), 2) and 3) we have

$$\boldsymbol{v}'_{k+1} = \boldsymbol{r}^{(k+2)} - \sum_{j=1}^{k} (\boldsymbol{t}_j \boldsymbol{r}^{(k+2)}) \boldsymbol{t}_j - \varkappa_k (\boldsymbol{t}_k \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_{k+1} - \varkappa_k (\boldsymbol{t}_{k+1} \boldsymbol{r}^{(k+1)}) \boldsymbol{t}_k$$

and

$$\boldsymbol{t}_{k+1}\boldsymbol{v}'_{k+1} = \boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+2)} - \varkappa_k(\boldsymbol{t}_k\boldsymbol{r}^{(k+1)}) .$$

Hence we obtain

$$\boldsymbol{t}'_{k+1} = \frac{1}{|\boldsymbol{v}_{k+1}|}\Big[\boldsymbol{r}^{(k+2)} - \sum_{j=1}^{k}(\boldsymbol{t}_j\boldsymbol{r}^{(k+2)})\boldsymbol{t}_j - \varkappa_k(\boldsymbol{t}_k\boldsymbol{r}^{(k+1)})\boldsymbol{t}_{k+1} -$$

$$- \varkappa_k(\boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+1)})\boldsymbol{t}_k - (\boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+2)})\boldsymbol{t}_{k+1} + \varkappa_k(\boldsymbol{t}_k\boldsymbol{r}^{(k+1)})\boldsymbol{t}_{k+1}\Big]$$

$$= \frac{|\boldsymbol{v}_{k+2}|}{|\boldsymbol{v}_{k+1}|}\boldsymbol{t}_{k+2} - \varkappa_k\frac{\boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+1)}}{|\boldsymbol{v}_{k+1}|}\boldsymbol{t}_k .$$

Because of

$$\frac{\boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+1)}}{|\boldsymbol{v}_{k+1}|} = \frac{\boldsymbol{v}_{k+1}\boldsymbol{r}^{(k+1)}}{\boldsymbol{v}_{k+1}^2} = \frac{\boldsymbol{r}^{(k+1)2} - \sum_{j=1}^{k}(\boldsymbol{t}_j\boldsymbol{r}^{(k+1)})^2}{\boldsymbol{r}^{(k+1)2} - \sum_{j=1}^{k}(\boldsymbol{t}_j\boldsymbol{r}^{(k+1)})^2} = 1$$

and

$$\frac{|\boldsymbol{v}_{k+2}|}{|\boldsymbol{v}_{k+1}|} = \frac{\boldsymbol{t}_{k+2}\boldsymbol{r}^{(k+2)}}{\boldsymbol{t}_{k+1}\boldsymbol{r}^{(k+1)}} = \varkappa_{k+1}$$

we obtain finally

$$\boldsymbol{t}'_{k+1} = -\varkappa_k\boldsymbol{t}_k + \varkappa_{k+1}\boldsymbol{t}_{k+2} .$$

Just that was to be proved.

We have yet to verify the last equation in (10). Applying the formula for the derivative of the vector product of the vectors and using the first $n-1$ equations of (10) we obtain

$$\begin{aligned}\boldsymbol{t}'_n &= \{\boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{n-1}\}' \\ &= \sum_{j=1}^{n-1}\{\boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{j-1} \wedge \boldsymbol{t}'_j \wedge \boldsymbol{t}_{j+1} \wedge \ldots \wedge \boldsymbol{t}_{n-1}\} \\ &= \varkappa_{n-1}\{\boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{n-2} \wedge \boldsymbol{t}_n\} \\ &= -\varkappa_{n-1}\{\boldsymbol{t}_n \wedge \boldsymbol{t}_1 \wedge \ldots \wedge \boldsymbol{t}_{n-2}\} = -\varkappa_{n-1}\boldsymbol{t}_{n-1} .\end{aligned}$$

The theorem is proved.

The fundamental theorem. Suppose we are given a system of scalar functions of the variable τ

(12) $$\varkappa_1, \ldots, \varkappa_{n-1} , \quad \tau \in I \ (^2) ,$$

which are summable in I.

(²) I denotes an interval containing the point τ_0.

We consider an orthonormal system of constant vectors

$$\boldsymbol{t}_1^0, \boldsymbol{t}_2^0, \dots, \boldsymbol{t}_n^0$$

with an orientation consistent with that of the system of coordinates. Furthermore let us fix a point M_0.

THEOREM 5. *Under the above assumptions there exists a unique curve C with the equation*

$$(13) \qquad \boldsymbol{r} = \boldsymbol{r}(\tau), \quad \tau \in I_1, \quad I_1 \subset I, \quad \tau_0 \in I_1$$

and the following properties:

1) *the curve C contains the point M_0 which corresponds to the value τ_0 of parameter τ;*

2) *$\sigma = \tau - \tau_0$ is the length of arc of the curve C taken from the point M_0;*

3) *there is a system of vectors*

$$\boldsymbol{t}_1(\tau), \dots, \boldsymbol{t}_n(\tau), \quad \tau \in I_1$$

absolutely continuous in I_1 and such that

$$\boldsymbol{t}_i(\tau_0) = \boldsymbol{t}_i^0 \quad (i = 1, \dots, n);$$

4) *the following system of equations holds:*

$$\begin{aligned} &\boldsymbol{t}_1 = \frac{d\boldsymbol{r}}{d\tau}, \\ &\boldsymbol{t}_1' \doteq \varkappa_1 \boldsymbol{t}_2, \\ &\boldsymbol{t}_m' \doteq -\varkappa_{m-1}\boldsymbol{t}_{m-1} + \varkappa_m \boldsymbol{t}_{m+1} \qquad (m = 2, \dots, n-1), \\ &\boldsymbol{t}_n' \doteq -\varkappa_{n-1}\boldsymbol{t}_{n-1}. \end{aligned}$$

(The symbol $\doteq$ denotes that the relations are satisfied almost everywhere in I_1).

Proof. The system of equations

$$(10') \qquad \begin{aligned} &\boldsymbol{t}_1' = \varkappa_1 \boldsymbol{t}_2, \\ &\boldsymbol{t}_m' = -\varkappa_{m-1}\boldsymbol{t}_{m-1} + \varkappa_m \boldsymbol{t}_{m+1} \qquad (m = 2, \dots, n-1), \\ &\boldsymbol{t}_n' = -\varkappa_{n-1}\boldsymbol{t}_{n-1} \end{aligned}$$

with coefficients (12) has one and only one solution ([1])

$$(14) \qquad \boldsymbol{t}_1(\tau), \dots, \boldsymbol{t}_n(\tau),$$

where the functions $\boldsymbol{t}_i(\tau)$ are absolutely continuous in I_1 and satisfy the conditions

$$\boldsymbol{t}_i(\tau_0) = \boldsymbol{t}_i^0 \quad (i = 1, \dots, n).$$

We shall prove that solution (14) represents in the whole interval I_1 orthonormal system of vectors and has the same orientation as the system of coordinates.

In order to show this we consider the scalar functions λ_{sk} defined by the formulas

$$\lambda_{sk} = \boldsymbol{t}_s \boldsymbol{t}_k \quad (s, k = 1, \dots, n) . \tag{15}$$

In consequence of (10′) these functions (15) satisfy the system of equations

$$\begin{aligned}
\lambda'_{11} &= 2\varkappa_1 \lambda_{12} , \\
\lambda'_{1i} &= \varkappa_1 \lambda_{2i} - \varkappa_{i-1}\lambda_{1,i-1} + \varkappa_i \lambda_{1,i+1} , \\
\lambda'_{ij} &= -\varkappa_{i-1}\lambda_{i-1,j} + \varkappa_i \lambda_{i+1,j} - \varkappa_{j-1}\lambda_{i,j-1} + \varkappa_j \lambda_{i,j+1} , \\
\lambda'_{1n} &= \varkappa_1 \lambda_{2n} - \varkappa_{n-1}\lambda_{1,n-1} , \\
\lambda'_{ni} &= -\varkappa_{i-1}\lambda_{n,i-1} + \varkappa_i \lambda_{n,i+1} - \varkappa_{n-1}\lambda_{n-1,i} , \\
\lambda'_{nn} &= -2\varkappa_{n-1}\lambda_{n-1,n} ,
\end{aligned} \tag{16}$$

for $i, j = 2, \dots, n-1$, and the initial conditions

$$\lambda_{sk}(\tau_0) = \delta_{sk} \quad (s, k = 1, \dots, n) . \tag{17}$$

The conditions for the existence and uniqueness of the solution of system (16), (17) are satisfied.

Since the system of functions (15) and the system defined by

$$\hat{\lambda}_{sk} \equiv \delta_{sk} , \quad \tau \in I_1 \quad (s, k = 1, \dots, n)$$

satisfy both together the system of equations (16) with the initial conditions (17), then we conclude that

$$\boldsymbol{t}_s \boldsymbol{t}_k \equiv \delta_{sk} , \quad \tau \in I_1 \quad (s, k = 1, \dots, n) . \tag{18}$$

In view of (18) we see that the system of vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_n$ represents an orthonormal system with the same orientation as the system of coordinates.

Now we define the function $\boldsymbol{r}(\tau)$ by the formula

$$\boldsymbol{r}(\tau) = \int_{\tau_0}^{\tau} \boldsymbol{t}_1(\tau) d\tau . \tag{19}$$

But we always have

$$|\boldsymbol{t}_1(\tau)| \equiv 1 , \quad \tau \in I_1;$$

so from (19) we obtain $\sigma = \tau - \tau_0$, where σ denotes the length of arc of the curve C defined by function (19), which is taken from the point M_0.

The theorem is proved.

Remark 2. The system of vectors in (14) can formally be named the system of vectors of a Frenet n-hedron at the point $M(\tau)$ of the curve C ($\tau \in I_1$), and the functions in (12) its curvatures.

COROLLARY. If we assume that for the functions $\varkappa_i$ $(i = 1, \ldots, n-3)$ in formulas (12) there exists a derivative of the order $(n-i-1)$ and it is absolutely continuous in I_1, and furthermore that for the function $\varkappa_{n-2}$ there exists a derivative of order 1 almost everywhere in I_1, $\varkappa_j > 0$ $(j = 1, \ldots, n-2)$, then we state that the vectors of system (14) obtained by means of the above theorem are exactly the vectors of Frenet n-hedron for the curve C almost everywhere in I_1 in the sense of definition 1. The functions $\varkappa_i$ $(i = 1, \ldots, n-1)$ are the successive curvatures for the curve C almost everywhere in I_1 in the sense of definition 2. The curve C is of the regularity class C^{n-2} in I_1 and $d^{n-2}r/d\tau^{n-2}$ is absolutely continuous in it.

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