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# On density concomitants of the covariant curvature tensor in the two- and three-dimensional Riemann space

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**1. Introduction.** One of the basic questions of the theory of geometric objects is to determine the algebraic concomitants of a certain type for a given object.

In the case $n = 2$ the scalar concomitants of the mixed tensor have been determined by Gołąb [8] under the supposition that the functions in question are of closs $C^1$. All the scalar concomitants of the mixed tensor in the $n$-dimensional space have been determined by Aczél and Hosszú [2], while those of the twice covariant tensor in the $n$-dimensional space were determined by Zajtz in [14].

By means of the analytic method Bieszk [3] has determined for the curvature tensor concomitants being either densities or tensors of second order in the two-dimensional space and linear concomitants being two-times covariant tensors in the three-dimensional space.

The same analytic method, reducing the system of functional equations to the system of differential equations of the first order, has been applied by Bieszk and Węgrzynowski in [5] and [6] to determine densities and vector concomitants of the antisymmetric tensor and linear concomitants of a tensor $T^{\gamma}_{\alpha\beta}$ in the two-dimensional space.

Scalar concomitants of a tensor $T^{\gamma}_{\alpha\beta}$ without regularity assumptions in the two-dimensional space were determined by Węgrzynowski in paper [13].

A certain general and uniform method reducing the determination of the concomitants of geometric objects to the question of determinning certain special subgroups of the general linear group $GL_n$ was given by Zajtz and Siwek in [12].

In this paper we shall determine by the analytic method all density (scalar) concomitants of the covariant curvature tensor in the two- and three-dimensional Riemann space.

Another method of solving the above-mentioned problem will be given in a forthcoming paper by S. Topa.

**2. Density (scalar) concomitants in the two-dimensional space $V_2$.** First of all we give some general notations. If the passage from one allowable coordinate system $(\lambda)$ to another $(\lambda')$ is given by the system of functions

$$\xi^{\lambda'} = \varphi^{\lambda'}(\xi^{\lambda}), \quad \lambda = 1, 2, \ldots, n, \ \lambda' = 1', 2', \ldots, n', \tag{1}$$

where

$$A_{\lambda}^{\lambda'} = \frac{\partial \varphi^{\lambda'}(\xi^{\lambda})}{\partial \xi^{\lambda}}, \tag{2}$$

$$J \overset{\mathrm{df}}{=} \det(A_{\lambda}^{\lambda'}) \neq 0, \tag{3}$$

then for the inverse transformation

$$\xi^{\lambda} = \psi^{\lambda}(\xi^{\lambda'}), \tag{4}$$

we introduce the notation

$$A_{\lambda'}^{\lambda} \overset{\mathrm{df}}{=} \frac{\partial \psi^{\lambda}(\xi^{\lambda'})}{\partial \xi^{\lambda'}}. \tag{5}$$

Between $A_{\lambda}^{\lambda'}$ and $A_{\lambda'}^{\lambda}$ the following relations occur:

$$A_{\lambda'}^{\lambda} = \frac{\operatorname{minor} A_{\lambda}^{\lambda'}}{J}. \tag{6}$$

In a Riemann space $V_n$ the induced connexion is given by means of the Christoffel symbols of second kind

$$\left\{ {\gamma \atop \alpha\beta} \right\} \overset{\mathrm{df}}{=} \tfrac{1}{2} g^{\gamma\varrho}(\partial_{\alpha} g_{\beta\varrho} + \partial_{\beta} g_{\alpha\varrho} - \partial_{\varrho} g_{\alpha\beta}), \tag{7}$$

where $g^{\alpha\beta}$ is the inverse tensor to the metric tensor $g_{\alpha\beta}$.

The curvature tensor (Riemann–Christoffel tensor) is defined as follons:

$$R_{\alpha\beta\gamma}^{\delta} \overset{\mathrm{df}}{=} 2\partial_{[\alpha} \left\{ {\delta \atop \beta]\gamma} \right\} + 2 \left\{ {\delta \atop [\alpha|\varrho|} \right\} \left\{ {\varrho \atop \beta]\gamma} \right\}. \tag{8}$$

The so-called covariant curvature tensor is defined by

$$R_{\alpha\beta\gamma\delta} \overset{\mathrm{df}}{=} R_{\alpha\beta\gamma}^{\varrho} g_{\delta\varrho}. \tag{9}$$

Tensor (9) has the well-known properties

$$\begin{aligned} &1^{\circ} \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}, \\ &2^{\circ} \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}. \end{aligned} \tag{10}$$

The number $N$ of the so-called essential components of the tensor $R_{\alpha\beta\gamma\delta}$ in space $V_n$ is defined by the formula

$$N = \frac{n^2(n^2-1)}{12}, \tag{11}$$

(see [11], 90.9).

After having passed to the new coordinate system $(\lambda')$ the tensor $R_{\alpha\beta\gamma\delta}$ has the components $R_{\alpha'\beta'\gamma'\delta'}$ connected with $R_{\alpha\beta\gamma\delta}$ by the formula

(12) $$R_{\alpha'\beta'\gamma'\delta'} = A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} A^{\gamma}_{\gamma'} A^{\delta}_{\delta'} R_{\alpha\beta\gamma\delta}.$$

Let us proceed to determine the density concomitants of $R_{\alpha\beta\gamma\delta}$ in the two-dimensional Riemann space.

For $n = 2$, $N = 1$, i.e. $R_{\alpha\beta\gamma\delta}$ has only one essential component

(13) $$x \overset{\text{df}}{=} R_{1212}.$$

For the matrix $A^{\lambda}_{\lambda'}$ we introduce the shorter notation

(14) $$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \overset{\text{df}}{=} \begin{bmatrix} A^1_{1'} & A^2_{1'} \\ A^1_{2'} & A^2_{2'} \end{bmatrix}.$$

Hence we have

(15) $$J = (p_1 p_4 - p_2 p_3)^{-1},$$

and according to [11] (formula 92.1)

$$x' \overset{\text{df}}{=} R_{1'2'1'2'} = (p_1 p_4 - p_2 p_3)^2 x,$$

or more briefly

(16) $$x' = J^{-2} x.$$

We seek an algebraic concomitant $H$ of $R_{\alpha\beta\gamma\delta}$, which is a density of weight $(-r)$.

For $n = 2$, $H$ is a function of $x$ fulfilling the equation

(17) $$H(x') = \varepsilon |J|^r H(x)$$

or

(18) $$H[(p_1 p_4 - p_2 p_3)^2 x] = \varepsilon |J|^r H(x),$$

where

(19) $$\varepsilon = \begin{cases} 1 & \text{for a Weyl density,} \\ \operatorname{sgn} J & \text{for an ordinary density.} \end{cases}$$

Assuming that $H(x)$ is of class $C^1$, we reduce equation (17) to an ordinary differential one. We differentiate (18) with respect to $p_1, p_2, p_3, p_4$ and next substitute

(20) $$p_1 = p_4 = 1, \quad p_2 = p_3 = 0.$$

After this operation we get four equations reducing to

(21) $$2xH'(x) = -rH(x),$$

in which there is no intervention of $\varepsilon$.

We have to distinguish two cases, I and II.

I. Let us assume that $r = 0$.

Ia. If $x = 0$, the problem is trivial, because $R_{1212} = 0$ in every coordinate system.

Ib. If $x \neq 0$, then $H'(x) = 0$; hence $H(x) = \text{const}$ and in this case only arbitrary scalars could be concomitants of the tensor $R_{1212}$.

II. Let us assume that $r \neq 0$.

IIa. If $x = 0$, then we have again the trivial case.

IIb. If $x \neq 0$, then the general solution of (21) has the form

$$H = \begin{cases} C_1 |x|^{-r/2} & \text{for } x > 0, \\ C_2 |x|^{-r/2} & \text{for } x < 0, \end{cases} \tag{22}$$

where $C_1$, $C_2$ are arbitrary constants different from zero (different or equal).

It is easy to prove that if $\varepsilon = 1$, $H(x)$ defined by (22) fulfils (17), while for $\varepsilon = \operatorname{sgn} x$, formula (22) is no solution of (17) in the whole domain but for $x > 0$ only.

Now we can state

THEOREM 1. *In the space $V_2$ the only scalar concomitants (of the class $C^1$) of the tensor $R_{\alpha\beta\gamma\delta}$ are arbitrary scalars, while the only density concomitants of the weight $(-r)$ are Weyl-densities of the form*

$$H(x) = C|x|^{-r/2}, \tag{23}$$

*where the arbitrary constant $C$ different from zero is given by*

$$C = \begin{cases} C_1 & \textit{for } x > 0, \\ C_2 & \textit{for } x < 0. \end{cases} \tag{24}$$

**3. Density (scalar) concomitants in the Riemann space $V_3$.** In accordance with (11) (Section 2) we have for $n = 3$, $N = 6$. We introduce the following shorter notations for the six essential components of the tensor $R_{\alpha\beta\gamma\delta}$:

$$\begin{aligned} x_1 &= R_{1212}, \quad & x_2 &= R_{1313}, \quad & x_3 &= R_{2323}, \\ x_4 &= R_{1213}, \quad & x_5 &= E_{1223}, \quad & x_6 &= R_{1323} \end{aligned} \tag{1}$$

and the shorter ones for the elements of the matrix $A_\lambda^{\lambda'}$:

$$[a_{ij}] \overset{\text{df}}{=} [A_j^{i'}], \quad \text{where } i, j = 1, 2, 3; \tag{2}$$

$$J = \det(a_{ij}) \neq 0. \tag{3}$$

In accordance with (6) (Section 2) and (2) we have

$$[A_{i'}^j] = J^{-1} \cdot \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{13}a_{32} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}. \tag{4}$$

In the new coordinate system $(\lambda')$ the coordinates $x_{i'}$ $(i = 1, 2, 3, 4, 5, 6)$ of the tensor $R_{\alpha\beta\gamma\delta}$ have the following form:

$$
(5)\quad \begin{cases}
x_{1'} = J^{-2}[a_{33}^2x_1 + a_{23}^2x_2 + a_{13}^2x_3 - 2a_{23}a_{33}x_4 + 2a_{13}a_{33}x_5 - 2a_{13}a_{23}x_6], \\
x_{2'} = J^{-2}[a_{32}^2x_1 + a_{22}^2x_2 + a_{12}^2x_3 - 2a_{22}a_{32}x_4 + 2a_{12}a_{32}x_5 - 2a_{12}a_{22}x_6], \\
x_{3'} = J^{-2}[a_{31}^2x_1 + a_{21}^2x_2 + a_{11}^2x_3 - 2a_{21}a_{31}x_4 + 2a_{11}a_{31}x_5 - 2a_{11}a_{21}x_6], \\
x_{4'} = J^{-2}[-a_{32}a_{33}x_1 - a_{22}a_{23}x_2 - a_{12}a_{13}x_3 + (a_{22}a_{33} + a_{23}a_{32})x_4 - \\
\qquad - (a_{12}a_{33} + a_{13}a_{32})x_5 + (a_{12}a_{23} + a_{13}a_{22})x_6], \\
x_{5'} = J^{-2}[a_{31}a_{33}x_1 + a_{21}a_{23}x_2 + a_{11}a_{13}x_3 - (a_{21}a_{33} + a_{23}a_{31})x_4 + \\
\qquad + (a_{11}a_{33} + a_{13}a_{31})x_5 - (a_{11}a_{23} + a_{13}a_{21})x_6], \\
x_{6'} = J^{-2}[-a_{31}a_{32}x_1 - a_{21}a_{22}x_2 - a_{11}a_{12}x_3 + (a_{21}a_{32} + a_{22}a_{31})x_4 - \\
\qquad - (a_{11}a_{32} + a_{12}a_{31})x_5 + (a_{11}a_{22} + a_{12}a_{21})x_6].
\end{cases}
$$

Similarly to Section 2 the sought density $H$ is a function of $x_1, \ldots, x_6$ fulfilling the functional equation:

$$(6)\qquad H(x_1', \ldots, x_6') = \varepsilon |J|^r H(x_1, \ldots, x_6),$$

where

$$(7)\qquad \varepsilon = \begin{cases} 1 & \text{for a Weyl density,} \\ \operatorname{sgn} J & \text{for an ordinary density.} \end{cases}$$

Let us assume that $H(x_1, \ldots, x_6)$ is of class $C^1$. For simplicity we introduce the following notation:

$$(8)\qquad H_i \overset{\text{df}}{=} \frac{\partial H}{\partial x_i}, \quad i = 1, 2, \ldots, 6.$$

First we differentiate the functional equation (6) with respect to the parameters $a_{ij}$, $i, j = 1, 2, 3$, and next we substitute

$$(9)\qquad [a_{ij}] = [\delta_{ij}],$$

where $\delta_{ij}$ is the Kronecker delta.

Then we get a system of nine equations of the first order with one unknown function $H$ depending on six variables $x_i$, $i = 1, \ldots, 6$:

$$
(10)\quad \begin{cases}
2x_1H_1 + 2x_2H_2 + \qquad + 2x_4H_4 + x_5H_5 + x_6H_6 = -rH, \\
2x_1H_1 + \qquad + 2x_3H_3 + x_4H_4 + 2x_5H_5 + x_6H_6 = -rH, \\
\qquad 2x_2H_2 + 2x_3H_3 + x_4H_4 + x_5H_5 + 2x_6H_6 = -rH, \\
\qquad 2x_6H_2 + \qquad + x_5H_4 + \qquad + x_3H_6 = 0, \\
2x_5H_1 + \qquad + x_6H_4 + x_3H_5 \qquad = 0, \\
\qquad 2x_6H_3 + \qquad + x_4H_5 + x_2H_6 = 0, \\
2x_4H_1 + \qquad + x_2H_4 + x_6H_5 \qquad = 0, \\
\qquad 2x_5H_3 + \qquad + x_1H_5 + x_4H_6 = 0, \\
\qquad 2x_4H_2 + \qquad + x_1H_4 + \qquad + x_5H_6 = 0.
\end{cases}
$$

We assert that (10) is a *complete system*. Denoting the left-hand sides of (10) by $X_1, X_2, \ldots, X_9$ respectively, we have

$$X_i(H) \overset{\text{df}}{=} \sum_{k=1}^{6} a_{ik} H_k, \quad i = 1, 2, \ldots, 9, \tag{11}$$

where the coefficients $a_{ik}$ are certain simple functions of $x_1, \ldots, x_6$.

Let us introduce a shorter notation for Poisson brackets:

$$(X_i, X_j) \overset{\text{df}}{=} \sum_{k=1}^{6} [X_i(a_{jk}) - X_j(a_{ik})] H_k, \quad i < j, i, j = 1, 2, \ldots, 9. \tag{12}$$

After a number of simple operations based on (10), (11) and (12) we get

$$\begin{array}{lll}
(X_1, X_2) = 0, & (X_2, X_3) = 0, & (X_3, X_5) = X_5, \\
(X_1, X_3) = 0, & (X_2, X_4) = X_4, & (X_3, X_6) = 0, \\
(X_1, X_4) = -X_4, & (X_2, X_5) = 0, & (X_3, X_7) = X_7, \\
(X_1, X_5) = -X_5, & (X_2, X_6) = 0, & (X_3, X_8) = -X_8, \\
(X_1, X_6) = X_6, & (X_2, X_7) = X_7, & (X_3, X_9) = -X_9, \\
(X_1, X_7) = 0, & (X_2, X_8) = 0, & (X_4, X_5) = 0, \\
(X_1, X_8) = X_8, & (X_2, X_9) = X_9, & (X_4, X_6) = X_2 - X_1, \\
(X_1, X_9) = 0, & (X_3, X_4) = 0, & (X_4, X_7) = X_5, \\
(X_4, X_8) = -X_9, & (X_5, X_8) = X_3 - X_1, & (X_6, X_9) = -X_8, \\
(X_4, X_9) = 0, & (X_5, X_9) = X_4, & (X_7, X_8) = X_6, \\
(X_5, X_6) = -X_7, & (X_6, X_7) = 0, & (X_7, X_9) = X_3 - X_2, \\
(X_5, X_7) = 0, & (X_6, X_8) = 0, & (X_8, X_9) = 0,
\end{array} \tag{13}$$

from which it follows that (10) is a complete system.

For integrating complete systems of the type (10) it is convenient to find a so-called *integrating direction* ([15] or [10]).

Denoting the equations of the system (10) by (10.1)-(10.9), we shall integrate them in the following direction: (10.7), (10.5), (10.6), (10.8), (10.9), (10.4), (10.1), (10.2), (10.3).

To equation (10.7) corresponds the system of the ordinary equations:

$$\frac{dx_1}{2x_4} = \frac{dx_2}{0} = \frac{dx_3}{0} = \frac{dx_4}{x_2} = \frac{dx_5}{x_6} = \frac{dx_6}{0}. \tag{14}$$

Solving this system, we obtain

$$H = \varphi(x_2, x_3, x_6, x_4^2 - x_1 x_2, x_4 x_6 - x_2 x_5) = \varphi(y_1, \ldots, y_5), \tag{15}$$

where $\varphi \in C^1$.

Substituting solution (15) in equation (10.5) we get

$$\frac{dy_1}{0} = \frac{dy_2}{0} = \frac{dy_3}{0} = \frac{dy_4}{2y_5} = \frac{dy_5}{y_3^2 - y_1 y_2}. \tag{16}$$

Hence, by the assumption that $y_3^2 - y_1 y_2 \neq 0$, we have

$$\begin{aligned} H &= \psi[x_2, x_3, x_6, x_2(x_1 x_6^2 + x_2 x_5^2 + x_3 x_6^2 - x_1 x_2 x_3 - 2x_4 x_5 x_6)] \\ &= \psi(z_1, z_2, z_3, z_4), \end{aligned} \tag{17}$$

where $\psi \in C^1$.

Substituting solution (17) in equation (10.6) we get the system of equations

$$\frac{dz_1}{0} = \frac{dz_2}{2z_3} = \frac{dz_3}{z_1} = \frac{dz_4}{0}. \tag{18}$$

The solution of (18) is

$$\begin{aligned} H &= \theta[x_2, x_6^2 - x_2 x_3, x_2(x_1 x_6^2 + x_2 x_6^2 + x_3 x_6^2 - x_1 x_2 x_3 - 2x_4 x_5 x_6)] \\ &= \theta(u_1, u_2, u_3), \end{aligned} \tag{19}$$

where $\theta \in C^1$.

Substituting solution (19) in equation (10.8) we get

$$(x_4 x_6 - x_2 x_5)\theta_2 = 0, \tag{20}$$

hence, by assuming that $x_4 x_6 - x_2 x_5 \neq 0$, we receive

$$\theta_2 = 0,$$

so

$$\begin{aligned} H &= \omega[x_2, x_2(x_1 x_6^2 + x_2 x_5^2 + x_3 x_4^2 - x_1 x_2 x_3 - 2x_4 x_5 x_6)] \\ &= \omega(v_1, v_2), \qquad \omega \in C^1. \end{aligned} \tag{21}$$

Substituting solution (21) in equation (10.9) we get

$$\frac{dv_1}{v_1} = \frac{dv_2}{v_2}. \tag{22}$$

The solution of (22) has the form

$$H = \varkappa(x_1 x_2 x_3 + 2x_4 x_5 x_6 - x_2 x_5^2 - x_1 x_6^2 - x_3 x_4^2) = \varkappa(w), \tag{23}$$

where $\varkappa(w) \in C^1$.

Solution (23) can be rewritten in the form

$$H = \varkappa(w) = \varkappa\left(\begin{vmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & x_6 \\ x_5 & x_6 & x_3 \end{vmatrix}\right). \tag{24}$$

Substituting solution (23) in equation (10.4) we obtain an identity, and thus equation (10.4) is not independent of the proviously integrated equations.

Substituting solution (24) in (10.1) we have

$$4\varkappa'(w)w = -r\varkappa(w). \tag{25}$$

Solving the homogeneous equation (25) we obtain (similarly to the equation (21), Section 2)

$$H = C|w|^{-r/4}, \quad r \text{ — arbitrary}, \; w \neq 0, \tag{26}$$

where the integration constant $C \neq 0$ has the form

$$C = \begin{cases} C_1 & \text{for } w > 0, \\ C_2 & \text{for } w < 0, \end{cases} \tag{27}$$

and

$$w = \begin{vmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & x_6 \\ x_5 & x_6 & x_3 \end{vmatrix}. \tag{28}$$

We verify without difficulty that solution (25) fulfils equations (10.2) and (10.3).

We also verify that the symmetric determinant (28), formed from the essential components (1) of the tensor $R_{\alpha\beta\gamma\delta}$, is a Weyl density of weight (4), i.e. denoting by $w'$ the right-hand side of (28) for $x_{i'}$, $i = 1, 2, \ldots, 6$ of the form (5) we obtain (after tedious calculations)

$$w' = J^{-4}w. \tag{29}$$

The results of Section 3 can now be formulated as follows:

THEOREM 2. *In the space $V_3$ each scalar concomitant $H(x_1, \ldots, x_6)$ (of class $C^1$) of the curvature tensor $R_{\alpha\beta\gamma\delta}$ is a sonstant function $H(x_1, \ldots, x_6) = C$, while every density concomitant of weight $(-r)$ is a Weyl density of the form*

$$H(x_1, \ldots, x_6) = C|w|^{-r/4}, \quad C \neq 0, \; w \neq 0, \tag{30}$$

*where $w$ is defined by* (28) *and $C$ by* (27).

Remark 1. The above considerations have only been based on the symmetry and antisymmetry of tensor $R_{\alpha\beta\gamma\delta}$ but we have ignored the fact that $R_{\alpha\beta\gamma\delta}$ as a curvature tensor comes from the metric tensor $g_{\alpha\beta}$. The whole consideration is maintained if we assume that the tensor $R_{\alpha\beta\gamma\delta}$ has only properties (10) and is independent of the tensor $g_{\alpha\beta}$, i.e. the assumption that the space is Riemannian is not necessary.

Remark 2. The concomitant defined by formula (28) being an algebraic concomitant of the tensor $R_{\alpha\beta\gamma\delta}$, it can be called a differential concomitant of the second order of the metric tensor $g_{\alpha\beta}$ (because $R_{\alpha\beta\gamma\delta}$ are expressed by $g_{\alpha\beta}$, $\partial_\gamma g_{\alpha\beta}$, $\partial_{\gamma\delta} g_{\alpha\beta}$). However, there are other algebraic concomitants of the tensor $g$, which are densities. The simplest of those is $g = \det(g_{\alpha\beta})$, another one (for $n = 3$) is a symmetric determinant of the third order formed from the essential components of the tensor

$$G_{\alpha\beta\gamma\delta} \stackrel{\text{df}}{=} 2g_{[\alpha|\gamma|}g_{\beta]\delta}, \tag{31}$$

i.e. from the minors of the second order of $\det(g_{\alpha\beta})$. The tensor $G_{\alpha\beta\gamma\delta}$ of the form (31) is the so-called induced metric tensor of the bivector space $V_n^2$, occurring in paper [8].

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