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Some remarks on the Casorati–Weierstrass theorem

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Abstract. We study the growth of the classical Bieberbach maps and show that there exists a sèquence of such maps, of order ε, converging uniformly on compact sets to the identity map as ε converges to zero. Using the concept of capacity of an analytic set in C^n introduced by Stoll [7], we show that if the growth of the volume of an analytic set A is slow enough then every bounded holomorphic function on A is constant. From this we obtain a Casorati–Weierstrass theorem in terms of growth. More explicitly, let F be a non-degenerate holomorphic mapping from C^n to C^n $(n \geqslant 2)$; if the growth of F is sufficiently slow we prove that $C^n \backslash F(C^n)$ is of Lebesgue measure zero.

1. Remarks on the Bieberbach maps. It is a classical result of Bieberbach that there exist holomorphic maps $F\colon C^2 \to C^2$ with constant, non-zero, jacobian determinant and such that $C^2 \backslash \overline{F(C^2)}$ is non-empty. We will refer to such maps as *Bieberbach maps.*

Following Stehlé [6], for each $k \in C$ with $|k| > 1$, we consider the following automorphism of C^2:

$$\varphi = (u, v)\colon C^2 \to C^2$$

with

$$u(z_1, z_2) = kz_1 + j(z_2), \quad v(z_1, z_2) = kz_2 + j(kz_1 + j(z_2)), \tag{1.1}$$

where $j(x) = x^2[2(k-1)x + 3(1-k)]$ is a polynomial of one variable. Consequently (1.1) is a fixed point of Φ and $\Phi'(1, 1) = k\,\mathrm{Id}$.

THEOREM [6]. *The functional equation*

$$\varphi(F(z)) = F(kz) \tag{1.2}$$

has a unique holomorphic solution $F\colon C^2 \to C^2$ satisfying $F(0) = 0$ and $F'(0) =$ identity. Furthermore the solution F is a Bieberbach map and $(1, 1) \notin \overline{F(C^2)}$;

$$F(C^2) = \{z/z \in C^2 \ \lim_{n\to\infty} \Phi^{-n}(z) = 0\}.$$

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Using this result we want to show:

PROPOSITION 1.1. *For every $\varepsilon > 0$ there is a Bieberbach map F_ε: $C^2 \to C^2$ satisfying $\max_{|z| \leqslant r} |F_\varepsilon(z)| \leqslant c_\varepsilon \exp(r^\varepsilon)$. Moreover, there exists a sequence $\{F_\varepsilon\}$ of such maps converging uniformly, as $\varepsilon \to 0$, on compact sets to the identity map on C^2, and such that the point $(1, 1) \notin \overline{F_\varepsilon(C^2)}$ for all ε.*

Proof. Let $F = (f, g)$ be the solution of the functional equation (1.2). By definition (1.1) of φ we have

$$(1.3) \qquad f(kz) = \sum_{\alpha,\beta=0}^{N} a_{\alpha\beta} f^\alpha(z) g^\beta(z), \qquad g(kz) = \sum_{\alpha,\beta=0}^{N} b_{\alpha\beta} f^\alpha(z) g^\beta(z)$$

with $|a_{\alpha\beta}|, |b_{\alpha\beta}| \leqslant ck^5$ for some c independant of k, α and β.

Let $M_f(r) = \max_{|z| \leqslant r} \log |f(z)|$ and $M_g(r) = \max_{|z| \leqslant r} \log |g(z)|$. Fix r_0 so that $M_f(r_0)$ and $M_g(r_0)$ are strictly larger than 1. Then (1.3) implies that for $r \geqslant r_0$

$$\begin{aligned} M_f(kr) &\leqslant \log \max_{|z| \leqslant r} \Big(\sum_{\alpha,\beta=0}^{N} |a_{\alpha\beta}| \, |f(z)|^\alpha |g(z)|^\beta \Big) \\ &\leqslant N\big(M_f(r) + M_g(r)\big) + \log(N^2 ck^5) \end{aligned}$$

and

$$M_g(kr) \leqslant N\big(M_f(r) + M_g(r)\big) + \log(N^2 ck^5).$$

Using these inequalities repeatedly we have for any positive integer s

$$\begin{aligned} M_f(k^s r_0) &\leqslant (2N)^s \big(M_f(r_0) + M_g(r_0)\big) + \sum_{j=0}^{s-1} (2N)^j \log(N^2 ck^5) \\ &\leqslant (2N)^s \{M_f(r_0) + M_g(r_0) + \log(N^2 ck^5)\}. \end{aligned}$$

For $r > r_0$ there exists an integer s such that

$$k^{s-1} r_0 \leqslant r < k^s r_0.$$

This implies that

$$(2N)^{s-1} \leqslant \left(\frac{r}{r_0}\right)^\varrho, \qquad \varrho = \log 2N / \log k.$$

Therefore

$$M_f(r) \leqslant M_f(k^s r_0) \leqslant \left(\frac{r}{r_0}\right)^\varrho 2N \{M_f(r_0) + M_g(r_0) + \log(N^2 ck^5)\}.$$

Consequently

$$M_f(r) \leqslant c_k r^\varrho.$$

Since for sufficiently large k, $\varrho = \log 2N/\log k < \varepsilon$, we have

$$\exp\big(M_f(r)\big) \leqslant \exp(c_k r^\varepsilon) = c_\varepsilon \exp(r^\varepsilon)$$

and

$$\exp\left(M_g(r)\right) \leqslant c_\varepsilon \exp\left(r^\varepsilon\right).$$

Thus

$$\max_{|z| \leqslant r} |F(z)| \leqslant 2c_\varepsilon \exp\left(r^\varepsilon\right).$$

This proves the first part of the proposition.

Denote by F_k the solution of the functional equation (1.2) then $\varphi(z) = \sum_{j=1}^{N} \varphi_j(z)$ and $F_k(z) = \sum_{j=0}^{\infty} P_{j,k}(z)$, where φ_j and $P_{j,k}$ are homogeneous polynomials of degree j. Since $F_k(0) = 0$ and $F_k'(0) =$ identity, we have $P_{0,k}(z) = 0$ and $P_{1,k}(z) = z$. Then

$$|F_k(z) - z| \leqslant \sum_{j \geqslant 2} |P_{j,k}(z)| \leqslant \sum_{j \geqslant 2} \mu_{j,k} |z|^j, \tag{1.4}$$

where

$$\mu_{j,k} = \sup_{z \in C^2} \left|P_{j,k}(z)/|z|^j\right|.$$

According to Stehlé ([7], p. 122), there is a positive constant M such that

$$\mu_{j,k} \leqslant \frac{M}{k^j - k} \sum_\alpha \left(c_\alpha \prod_p \mu_{p,k}^{\alpha_p}\right), \quad c_\alpha = \frac{|\alpha|!}{\prod_p (\alpha_p)!}, \tag{1.5}$$

where the above summation ranges over all multi-indices α with $2 \leqslant |\alpha| \leqslant N = \deg \varphi$ and $\sum_p p\alpha_p = j$. Let $\bar{\mu}_{1,k} = 1$ and define inductively

$$\bar{\mu}_{j,k} = \frac{M}{k^j - k} \sum_\alpha \left(c_\alpha \prod_p \bar{\mu}_{p,k}^{\alpha_p}\right), \quad j \geqslant 2, \tag{1.6}$$

where α, p are as in (1.5).

Define

$$\bar{\mu}_j = M \sum C_\alpha \prod_p \bar{\mu}_p^{\alpha_p}.$$

Clearly for every k we have

$$\bar{\mu}_{j,k} \leqslant \frac{\bar{\mu}_j}{k^j - k}.$$

Observe that the series $\sum \bar{\mu}_j x^j$ has a positive radius of convergence, in fact if

$$G(x, y) = y - M \sum_{j=2}^{N} (x+y)^j,$$

then

$$y(x) = \sum_{j=2}^{\infty} \bar{\mu}_j x^j.$$

Then (1.4) implies

$$|F_k(z)-z| \leqslant \sum_{j\geqslant 2} \mu_{jk}|z|^j \leqslant \sum_{j\geqslant 2} \bar{\mu}_{j,k}|z|^j \leqslant \sum_{j\geqslant 2} \bar{\mu}_j \frac{|z|^j}{k^j-k}.$$

This last sum tends to zero, uniformly on bounded sets, as k tends to infinity. This shows that F_k converges uniformly to the identity on compact sets.

2. Analytic sets of slow growth. The maps in the preceding proposition are of exponential growth. We will show that if a holomorphic map grows sufficiently slowly then it has dense image. We begin by establishing a sufficient condition for the non-existence of bounded holomorphic functions on an analytic subvariety A in C^n. We will need the result of Stoll in [7].

For an analytic set A in C^n of pure dimension k, let $A_r = A \cap B_r$ and $\partial A_r = A \cap \partial B_r$. Here B_r denotes the ball of center 0 and radius r in C^n. Let χ_r denote the characteristic function of B_r and let $[A]$ be the current of integration on A. Then for almost every r the current $d(\chi_r[A])$ represents the current of integration on the regular points of the real analytic set ∂A_r; denote this last current by $[\partial A_r]$.

In fact, let $A = \bigcup_{j=0}^{k} A^d$, where $\bigcup_{s=0}^{p-1} A^s$ is the set of singular points of A^p and $\dim A^p = p$. Sard's theorem implies that, for almost every r, ∂B_r is transverse to every A^p, consequently, for such r, ∂A_r is a real analytic set of dimension $2k-1$ and the set of its singular points is of dimension $2k-3$. Such r will be referred to as regular values. Therefore the current $d(\chi_r[A])-$ $-[\partial A_r]$ is supported by a set of dimension $2k-3$. By Federer [2], Theorems 4.1.15 and 4.1.20, it vanishes. This permits us to use Stokes' theorem in Lemma 2.2 below.

For simplicity we assume that $0 \notin A$ and A is irreducible. Let χ be a closed non-negative $\mathscr{C}^2$ form of type $(k-1, k-1)$ on A. Assume also that $\chi > 0$ at some point of A. Then for regular values $0 < s < r$ there exists (Stoll [7], Lemma 4.2, p. 72) a continuous function φ_{rs} on A, $\mathscr{C}^\infty$ on the simple points of $A_r - \overline{A_s}$, with the following properties:

(1) $0 \leqslant \varphi_{rs} \leqslant 1$,

(2) $\varphi_{rs} = 1$ on $\overline{A_s}$ and $\varphi_{rs} = 0$ on $A\backslash A_r$,

(3) $dd^c \varphi_{rs} \Lambda \chi \equiv 0$, where $d^c = \dfrac{i}{4\pi}(\bar{\partial}-\partial)$,

(4) $-d^c \varphi_{rs} \Lambda \chi > 0$ on $\partial A_r \cup \partial A_s$,

and

(5) $-\int_{\partial A_r} d^c \varphi_{rs} \Lambda \chi = -\int_{\partial A_s} d^c \varphi_{rs} \Lambda \chi > 0.$

In this paper χ will always denote $(dd^c \log |z|^2)^{k-1}$, where $|z|$ is the euclidian norm of $z = (z_1, \ldots, z_n)$.

Following Stoll [7], define

$$c_{rs} = -\int_{\partial A_r} d^c \varphi_{rs} \wedge \chi. \tag{2.1}$$

For fixed s, c_{rs} decreases as r increases. Hence $\lim_{r \to \infty} c_{rs}$ exists and

$$c_\infty(A) = \lim_{r \to \infty} c_{rs} \geqslant 0$$

is called the *capacity of* A.

As usual we shall define

$$n(A, r) = \frac{c_n \operatorname{vol}(A_r)}{r^{2k}} = \int_{A_r} (dd^c \log |z|^2)^k, \quad N(A, r) = \int_0^r \frac{n(A, t)}{t} dt,$$

where c_k is determined so that $n(C^k, r) \equiv 1$. We shall now prove that on an analytic set of "slow" growth there are no non-constant bounded holomorphic functions.

THEOREM 2.1. *Let $A \subseteq C^n$ be an irreducible analytic set of pure dimension k. Suppose that*

$$\lim_{r \to \infty} \inf \frac{N(A, r)}{(\log r)^2} < \infty.$$

Then every bounded holomorphic function on A is constant.

For the proof we shall need the following lemma whose proof is quite standard.

LEMMA 2.2. *Let u be a function of class $\mathscr{C}^2$ on A. Then if r and s are regular values, $0 < s < r$, we have*

$$\int_{A_r} \varphi_{rs} dd^c u \wedge \chi = \int_{\partial A_s - \partial A_r} u d^c \varphi_{rs} \wedge \chi. \tag{2.2}$$

Proof. Using Stokes' theorem and property (4) of φ_{rs} we have

$$\int_{\partial A_s - \partial A_r} u d^c \varphi_{rs} \wedge \chi = \int_{A_s - A_r} d[u d^c \varphi_{rs} \wedge \chi] = \int_{A_s - A_r} du \wedge d^c \varphi_{rs} \wedge \chi.$$

On the other hand, by type considerations, we have

$$du \wedge d^c \varphi \wedge \chi = d\varphi \wedge d^c u \wedge \chi.$$

Therefore

$$\begin{aligned} \int_{A_s - A_r} du \wedge d^c \varphi_{rs} \wedge \chi &= \int_{A_s - A_r} d\varphi_{rs} \wedge d^c u \wedge \chi \\ &= \int_{A_s - A_r} d(\varphi_{rs} d^c u \wedge \chi) - \int_{A_s - A_r} \varphi_{rs} dd^c u \wedge \chi \\ &= \int_{\partial A_s - \partial A_r} \varphi_{rs} d^c u \wedge \chi - \int_{A_s - A_r} \varphi_{rs} dd^c u \wedge \chi. \end{aligned}$$

Since $\varphi_{rs} = 1$ on A_s and $\varphi_{rs} = 0$ on $A \backslash A_r$ the last relation can be written as

$$\int_{\partial A_s} d^c u \Lambda \chi - \int_{A_s} dd^c u \Lambda \chi + \int_{A_r} \varphi_{rs} dd^c u \Lambda \chi = \int_{A_r} \varphi_{rs} dd^c u \Lambda \chi.$$

This is again a consequence of Stokes' theorem.

Proof of Theorem 2.1. Let f be a holomorphic function an A and let $M_f(r) = \max_{|z| \leqslant r} \log(1+|f|^2)(z)$. Apply the above lemma to $u = \log(1+|f|^2)$. Then by property (4) of φ_{rs}

$$\int_{A_r} \varphi_{rs} dd^c \log(1+|f|^2) \Lambda \chi = \int_{\partial A_s - \partial A_r} \log(1+|f|^2) d^c \varphi_{rs} \Lambda \chi$$
$$\leqslant - \int_{\partial A_r} \log(1+|f|^2) d^c \varphi_{rs} \Lambda \chi.$$

Therefore we have

$$\int_{A_r} \varphi_{rs} dd^c \log(1+|f|^2) \Lambda \chi \leqslant c_{rs} M_f(r). \tag{2.3}$$

Applying Lemma 2.1 to $u = \log|z|^2$ we have

$$\int_{A_r} \varphi_{rs} dd^c \log|z|^2 \Lambda \chi = \int_{A_r} \varphi_{rs} (dd^c \log|z|^2)^k \tag{2.4}$$
$$= \int_{\partial A_s - \partial A_r} \log|z|^2 d^c \varphi_{rs} \Lambda \chi = 2c_{rs} \log \frac{r}{s}.$$

Since $0 \leqslant \varphi_{rs} \leqslant 1$, (2.4) implies

$$n(A, r) \geqslant 2c_{rs} \log \frac{r}{s}. \tag{2.5}$$

Since c_{rs} decreases as r increases, we have for regular values of $t, s < t < r$,

$$2c_{rs} \log \frac{t}{s} \leqslant 2c_{ts} \log \frac{t}{s} \leqslant n(A, t). \tag{2.6}$$

Integrating (2.6) we have

$$2c_{rs} \int_s^r \frac{1}{t} \log \frac{t}{s} dt \leqslant \int_s^r \frac{n_A(t)}{t} dt = N(A, r) - N(A, s),$$

and

$$c_{rs} \leqslant \frac{4[N(A, r) - N(A, s)]}{(\log r/s)^2} \leqslant 4 \frac{N(A, r)}{(\log r/s)^2}. \tag{2.7}$$

Using (2.3) and (2.7) and the fact that φ_{rs} is positive and equals one on A_s we see that for $0 < s < r$

$$\int_{A_s} dd^c \log(1+|f|)^2 \Lambda \chi \leqslant 4 \frac{N(A, r)}{(\log r/s)^2} M_f(r). \tag{2.8}$$

If f is bounded and $\liminf_{r\to\infty} \frac{N(A,r)}{(\log r)^2} = 0$, then by (2.8) and Fatou's lemma, f is constant.

To obtain the general case we need the following lemma which is probably well known.

LEMMA 2.3. *If U is a bounded connected open set in $\mathbf{C}$, there is a holomorphic function g from U to the unit disc D such that, for almost every $a \in D$, $g^{-1}(a)$ is infinite.*

Proof. If $U = D$ the existence of such a function is well known, take any infinite Blaschke product. In the general case let h be the Ahlfors mapping of U with respect to $z_0 \in U$. Recall that h maximizes $|f'(z_0)|$ among $f \in H^\infty(U)$ satisfying $\|f\|_\infty \leqslant 1$, $f(z_0) = 0$, $\operatorname{Re} f'(z_0) > 0$. It is known that the Ahlfors function is unique and of modulus on the Shilov boundary of $H^\infty(U)$, see [3] for example. We prove that $D \backslash h(U)$ is of analytic capacity zero, i.e. every bounded holomorphic function on $D \backslash h(U)$ extends to a function in $H^\infty(D)$. In fact if E is a closed set in D of positive analytic capacity with $0 \notin E$, then the Ahlfors function f of $D \backslash E$ with respect to 0 is such that $|f'(0)| > 1$. Suppose $|f'(0)| = 1$ then since f is unique $f(z) = z$ and this function is not of modulus one on the part of the Shiloy boundary which project on E, consequently $|f'(0)| > 1$. If $D \backslash h(U)$ is not of analytic capacity 0, $f \circ h$ will be such that $|(f \circ h)'(z_0)| > |h'(z_0)|$, contradicting the extremal property of h. Since $D \backslash h(U)$ is of analytic capacity zero it is also of Lebesgue measure zero and if B is an infinite Blashke product the function $g = B \circ h$ satisfies the properties required by the lemma.

We return to the proof of the theorem. Suppose f is a non-constant holomorphic map from A to D. Choose a holomorphic function g from $U = f(A)$ to D as in Lemma 2.3. Then for almost every a in the open set $(g \circ f)(A)$, $n(A \cap (g \circ f)^{-1}(a), r)$ tends to infinity as r tends to infinity. Replacing f by $g \circ f$ we may assume that f itself has this property.

Let ω be the Standard Kähler form on P^1, normalized so that $\int_{P^1} \omega = 1$. Choose $s = \sqrt{r}$ in (2.8). Then

$$\int_{A_{\sqrt{r}}} f^* \omega \Lambda \chi \leqslant 16 \frac{N(A,r)}{(\log r)^2}. \tag{2.9}$$

Using Crofton's formula (cf. Shiffman [5], p. 79), we have

$$\int_{P^1} n(A \cap f^{-1}(a), \sqrt{r}) = \int_{A_{\sqrt{r}}} f^* \omega \Lambda \chi \leqslant 16 \frac{N(A,r)}{(\log r)^2}, \tag{2.10}$$

and by Fatou's lemma

$$\int_{P^1} \liminf_{r\to\infty} n(A \cap f^{-1}(a), \sqrt{r}) \leqslant 16 \liminf_{r\to\infty} \frac{N(A,r)}{(\log r)^2} < \infty.$$

Hence for almost all $a \in P^1$

$$\liminf_{r\to\infty} n(A \cap f^{-1}(a), \sqrt{r}) < \infty.$$

This is a contradiction. Hence f is constant.

Remark 2.4. Relation (2.7) implies that if $\liminf_{r\to\infty} \frac{N(A,r)}{(\log r)^2} = 0$, then $C_\infty(A) = 0$ and consequently every smooth plurisubharmonic function on A is constant.

3. A theorem of Casorati–Weierstrass type. Let F be a holomorphic map form C^n to C^n. We suppose F is non-degenerate that is the Jacobian J_F of F is not identically zero. We are interested in finding conditions on F which will allow us to conclude that $F(C^n)$ is dense in C^n. The only known result in this direction is apparently a theorem of Chern Stoll Wu, involving order functions, see [1] for a short proof. We have the following result.

THEOREM 3.1. *Let $F = (f_1, \ldots, f_n)$ be a holomorphic map from C^n to C^n. Suppose that F is non-degenerate and that for $1 \leqslant q \leqslant n-1$*

$$\limsup_{r\to\infty} \frac{M_{f_q}(r)}{(\log r)^{1+\varepsilon_q}} < \infty$$

with $\sum_{q=1}^{n-1} \varepsilon_q \leqslant 1$. Then $C^n \backslash F(C^n)$ is of Lebesgue measure 0.

We shall need the following lemma.

LEMMA 3.2. *Let g be a holomorphic function in C^n satisfying $g(0) = 0$ and $M_g(r) = 0((\log r)^{1+\varepsilon})$. Let A be an analytic set of dimension k with $0 \notin A$. Then if $0 < \alpha < 1$, there are positive constants C and C' such that*

$$\int_{P^1} N(A \cap g^{-1}(\lambda), r^\alpha) \leqslant \frac{C}{1-\alpha} (\log^+ r)^\varepsilon N(A, r) + C' N(A, 2e).$$

Proof. By relation (2.8) and Crofton's formula

$$\int_{P^1} n(A \cap g^{-1}(\lambda), s)\, da(\lambda) \leqslant 4 \frac{N(A,r)}{(\log r/s)^2} M_g(r).$$

Therefore

$$(3.1) \quad \int_{P^1} N(A \cap g^{-1}(\lambda), r^\alpha)\, da$$
$$\leqslant \int_0^e 4 \frac{N(A,2t)}{(\log 2)^2} M_g(2t) \frac{dt}{t} + \int_e^{r^\alpha} 4 \frac{N(A,r)}{(\log r/t)^2} M_g(r) \frac{dt}{t}.$$

But by Schwarz's lemma $M_g(2t) \leqslant Ct$, for $t \leqslant e$, and there is a constant C_1

such that $M_g(r) \leqslant \sup[C_1(\log^+ r)^{1+\varepsilon}, C_1]$. Therefore (3.1) can be majorized by

$$C_1 N(A, 2e) + 4N(A, r) C(\mathrm{Log}^+ r)^{1+\varepsilon} \int_e^{r^\alpha} \frac{1}{(\log r/t)^2} \frac{dt}{t}$$
$$\leqslant C_1 N(A, 2e) + \frac{C_2}{1-\alpha}(A, r)(\log^+ r)^\varepsilon.$$

Proof of Theorem 3.1. We shall denote by da the restriction of the standard measure on P^1 to C and by $da_1 \ldots da_k$ the product measure on C^n. For $\lambda = (\lambda_1, \ldots, \lambda_k) \in C^k$ we denote

$$N(\lambda_1, \ldots, \lambda_k, r) = N(f_1^{-1}(\lambda_1) \cap \ldots \cap f_k^{-1}(\lambda_k), r).$$

Let $F_q = (f_1, \ldots, f_q)$, $q \leqslant n$. We prove by induction that $C^q \backslash F_q(C^n)$ is of Lebesgue measure 0. The result is clear for $q = 1$. By induction suppose $C^q \backslash F_q(C^n)$ is of Lebesgue measure zero. Let $V = \{z/J_F(z) = 0\}$. By hypothesis V is either of codimension one or empty. Let $E_q = \{\lambda/\lambda = (\lambda_1, \ldots, \lambda_q) \in C^q, F_q^{-1}(\lambda) \subset V\}$. We first prove that E_q is of Lebesgue measure zero in C^q.

In fact by a very special case of an inequality of Federer [2] we have for $r > 0$

$$\int H^{2(n-q)}(V \cap F_q^{-1}(\lambda) \cap B_r) d^{2q}\lambda \leqslant C_r H^{2n}(V \cap B_r).$$

Here H^p denotes the Hausdorff measure of dimension p, $d^{2q}\lambda$ denotes the Lebesgue measure in C^q, and C_r is a constant. In fact this inequality is valid for Lipschitz maps and metric spaces. Since $H^{2n}(V \cap B_r) = 0$, it follows that, for almost every λ, $H^{2(n-q)}(V \cap F_q^{-1}(\lambda) \cap B_r) = 0$. Consequently, for such λ, $V \cap F_q^{-1}(\lambda)$ is of dimension $n-q-1$, but if $F_q^{-1}(\lambda) \subset V$, then $H^{2(n-q)}(V \cap F_q^{-1}(\lambda) \cap B_r)$ would be strictly positive. Therefore for almost every $\lambda \in C^q$ the function f_{q+1} is not locally constant on $F_q^{-1}(\lambda)$. It remains to prove that $C \backslash f_{q+1}(F_q^{-1}(\lambda))$ is of Lebesgue measure 0 for almost every $\lambda \in C^q$, an application of Fubini's theorem will then prove that $C^{q+1} \backslash F_{q+1}(C^n)$ is of Lebesgue measure 0.

We now prove that for almost every $\lambda \in C^q$

$$\liminf_{r \to \infty} \frac{N(\lambda_1, \ldots, \lambda_q, r)}{(\log r)^2} < \infty, \quad q \leqslant n-1. \tag{3.2}$$

Recall that Jensen's formula on C^n gives the following inequality when $k \leqslant q-1$,

$$\int_{P^1} N(\lambda_k, r) da_k \leqslant C_1(\log^+ r)^{1+\varepsilon_k} + C_2, \tag{3.3}$$

where C_1 and C_2 are constants. Lemma 3.2 implies

$$\int_C N(\lambda_1, \ldots, \lambda_k, r) da_k \leqslant C_3(\log^+ r)^{\varepsilon_k} N(\lambda_1, \ldots, \lambda_{k-1}, r) + C_4 N(\lambda_1, \ldots, \lambda_{k-1}, 2e).$$

By repeatedly using Fubini's theorem and then (at the last step) using (3.3) we arrive at

$$\int_{C^q} N(\lambda_1, \ldots, \lambda_q, r)\, da_1 \ldots da_q \leqslant C_5 (\log^+ r)^{1+\sum_{i=1}^{q} \varepsilon_i} + C_6.$$

Therefore.

$$\int_{C^q} \liminf_{r\to\infty} \frac{N(\lambda_1, \ldots, \lambda_q, r)}{(\log r)^2}\, da_1 \ldots da_q < \infty.$$

This proves relation (3.2). By theorem (2.1) for almost every λ, $F_q^{-1}(\lambda)$ admit no non-trivial bounded holomorphic functions. Since f_{q+1} is not locally constant on $F_q^{-1}(\lambda)$ then $C\setminus f_{q+1}(F_q^{-1}(\lambda))$ is of analytic capacity zero, i.e. every bounded holomorphic function on the open set $f_{q+1}(F_q^{-1}(\lambda))$ is constant. Therefore $C\setminus f_{q+1}(F_q^{-1}(\lambda))$ is of Lebesgue measure 0.

Remark. Using remark (2.4) it is easy to show that if

$$\limsup_{r\to\infty} \frac{M_{f_q}(r)}{(\log r)^{1+\varepsilon_q}} = 0, \quad 1 \leqslant q \leqslant n-1,$$

then $E = C^n\setminus F(C^n)$ is of zero Γ-capacity in the sense of Ronkin [4] and therefore $H^{2n-1}[C^n\setminus F(C^n)] = 0$, which is more precise than $C^n\setminus F(C^n)$ is of Lebesgue measure zero.

Bibliography

[1] J. Carlson and P. Griffiths, *The order functions for entire holomorphic mappings*, Value distribution theory, part B (edited by R. O. Kujala amd A. L. Vitter), Marcel Dekker (1973).

[2] H. Federer, *Geometric measure theory*, Die Grendlehren der Math. Band 153. Springer Verlag (1969).

[3] S. Fisher, *On Schwarz's lemma and inner functions*, Trans. Amer. Math. Soc. 138 (1968).

[4] L. I. Ronkin, *Introduction to the theory of entire functions of several complex variables*, Transl. Math. Monographs AMS 44 (1974).

[5] B. Shiffman, *Applications of geometric measure theory to the value distribution theory for meromorphic maps*, Value distribution theory, part B (edited by R. O. Kujala and A. L. Vitter), Marcel Dekker (1973).

[6] J. L. Stehlé, *Plongements du disque dans C^2*, Sém. Lelong. Lecture Notes Math. 275 (1972).

[7] W. Stoll, *The growth of the area of a transcendental analytic set* I, II, Math. Ann. 156 (1964).

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