Bergman completeness of complete circular domains

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Abstract. It is proved that any bounded complete circular domain of holomorphy with continuous Minkowski functional is complete w.r.t. the Bergman distance.

1. Introduction. We start with the basic notions we shall use:

DEFINITION. A domain $D \subset C^n$ is called *complete circular* if whenever $z \in D$, $\lambda \in C$, $|\lambda| \leq 1$ then $\lambda z \in D$.

We mention that any complete circular domain can be defined by its *Minkowski functional*, i.e., there exists $h: C^n \to [0, \infty)$, $h(\lambda z) = |\lambda| h(z)$ ($\lambda \in C$, $z \in C^n$) such that $D = \{z \in C^n: h(z) < 1\}$; sometimes we write $D = D_h$. By [2], we know: D_h is a domain of holomorphy iff h is a plurisubharmonic function.

Observe that any pseudoconvex Reinhardt domain D containing the origin is a complete circular domain $D = D_h$ with continuous Minkowski functional h.

Let D be a domain in C^n; then by $c_D(z', z'')$ we denote the Carathéodory pseudodistance between z', z'' (cf. [3]). Then the following is known [7]:

THEOREM. *Any bounded pseudoconvex Reinhardt domain $D \subset C^n$, containing the origin, is c_D-finitely compact, i.e., any c_D-ball is a relatively compact subset of D w.r.t. the C^n-topology.*

Remark. There is also the notion of c_D-completeness in the sense of metric spaces. But, so far, it is not known whether both notions coincide. It is clear that c_D-finitely compact always implies c_D-complete.

Hence, any such bounded pseudoconvex Reinhardt domain is the existence domain of a bounded holomorphic function. On the other hand, one can characterize the bounded complete circular domains of holomorphy which are H^∞-domains of holomorphy (cf. [8]).

THEOREM. *Let $D = D_h$ be a pseudoconvex, bounded complete circular domain in C^n. Then the following conditions are equivalent:*

(a) D is an H^∞-domain of holomorphy;

(b) $\{z \in C^n:\ h$ is not continuous at $z\}$ is pluripolar.

Hence, not every bounded complete circular domain $D = D_h$ of holomorphy is c_D-finitely compact. Moreover, it is clear (cf. [2]) that continuity of h is a necessary condition; this leads to the following question: is any bounded, complete circular domain $D = D_h$ of holomorphy with continuous Minkowski functional already c_D-finitely compact?

In view of Hahn's inequality [4], necessary for having a positive answer is that every such $D = D_h$ is Bergman complete. Here we give a proof of this result, which was found in February 1986 while the second author was a guest at Jagiellonian University, Kraków.

2. Bergman completeness. Let D be a bounded domain in C^n with its Bergman kernel $K_D:\ D \times D \to C$, holomorphic in the first, antiholomorphic in the second variable. By

$$B_D(z,\ X) := \left[\sum \frac{\partial^2}{\partial z_\nu\, \partial \bar z_\mu} \log K_D(z,\ z)\, X_\nu \bar X_\mu \right]^{1/2} \qquad (z \in D,\ x \in C^n),$$

we denote the Bergman metric in D and by

$$b_D(z',\ z'') := \inf \left\{ \int_0^1 B_D(\gamma(t),\ \dot\gamma(t))\, dt:\ \gamma \text{ piecewise } C^1\text{-curve in } D \text{ from } z' \text{ to } z'' \right\}$$

$$(z',\ z'' \in D)$$

the Bergman distance in D.

We also recall the Skwarczyński distance for $z',\ z'' \in D$ [9]:

$$\lambda_D(z',\ z'') := \left[1 - \frac{|K_D(z',\ z'')|}{\sqrt{K_D(z',\ z')}\ \sqrt{K_D(z'',\ z'')}} \right]^{1/2}$$

and observe: $\lambda_D \leqslant c b_D$ (cf. [5]), where c is a constant.

Thus, in order to prove that D is b_D-complete is suffices to show that D is λ_D-complete.

More precisely, we use the following result [9]:

THEOREM. *Let D be a bounded domain in C^n and assume:*

(i) *the bounded holomorphic functions in D lie dense in the Hilbert space $L^2_h(D)$ of square-integrable holomorphic functions,*

(ii) $\lim\limits_{z \to \zeta} K_D(z,\ z) = \infty$ *whenever $\zeta \in \partial D$.*

Then D is λ_D-complete.

Observe that, for a bounded complete circular domain $D = D_h$ with continuous h, condition (i) is always fulfilled; take $f_t(z) := f(t \cdot z)$ for $t \to 1$.

What remains is the verification of condition (ii). Here we use the following deep result due to Ohsawa-Takegoshi [6]:

THEOREM. *Let* Ω *be a bounded pseudoconvex domain in* C^n, ψ: $\Omega \to R \cup \{-\infty\}$ *a plurisubharmonic function and* $H \subset C^n$ *a complex hyperplane. Then, for any holomorphic function* f *on* $\Omega \cap H$ *satisfying* $\int_{\Omega \cap H} e^{-\psi} |f|^2 dV_{n-1} < \infty$, *there exists a holomorphic function* F *on* Ω *with* $F|_{\Omega \cap H} = f$ *and* $\int_{\Omega} e^{-\psi} |F|^2 dV_n \leqslant C \int_{\Omega \cap H} e^{-\psi} |f|^2 dV_{n-1}$, *where* $C = C(\mathrm{diam}\,\Omega) = 1620 \pi [1 + (\mathrm{diam}\,\Omega)^2]^2$ *and* dV_k *denotes the* $2k$-*dimensional Lebesgue measure.*

Now, we can formulate and prove our result:

THEOREM. *Let* D *be a bounded complete circular domain of holomorphy with its Minkowski functional* h *and assume* h *to be continuous. Then* D *is* λ_D-*complete and hence Bergman complete.*

Proof. We only have to verify that $\lim_{z \to \zeta} K_D(z, z) = +\infty$ whenever $\zeta \in \partial D$.

Therefore we claim: for any $D = D_h \subset C^n$ there exists $b > 0$, depending only on the diameter and the dimension, such that, for $z \in D$,

(∗) $$K_D(z, z) \geqslant b/[1 - h^2(z)]^2.$$

In order to prove (∗) we proceed by induction:

In the case $n = 1$, we always have $h(z) = \frac{1}{R}|z|$ and, if $z \in D$,

$$K_D(z, z) = \frac{1}{\pi R^2} \frac{1}{[1 - h^2(z)]^2}.$$

Now we turn to the step $(n-1)$ to n. Thus we assume $D = D_h \subset C^n$. Let $z^0 \in D = D_h$, $z^0 \neq 0$; without loss of generality we may assume $z^0 = (z_1^0, 0, \ldots, 0)$. Define $H := \{z = C^n: z_n = 0\}$. Then there exists $C_0 > 0$ only depending on the diameter of D such that every holomorphic function on the bounded complete circular pseudoconvex domain $D \cap H$ in C^{n-1}, square-integrable w.r.t. the $(2n-2)$-dimensional Lebesgue measure, extends to F holomorphic on D and satisfying $\|F\|_{L^2(D)} \leqslant C_0 \|f\|_{L^2(D \cap H)}$.

Therefore we obtain

$$K_D(z^0, z^0) \geqslant \frac{|F(z^0)|^2}{\|F\|_{L^2(D)}^2} \geqslant \frac{|f(z^0)|^2}{C_0^2 \|f\|_{L^2(D \cap H)}^2}.$$

Since $f \in L^2(D \cap H)$ was arbitrary, this inequality implies

$$K_D(z^0, z^0) \geqslant \frac{1}{C_0^2} K_{D \cap H}(\bar{z}^0, \bar{z}^0) \qquad (\bar{z}^0 = (z_1^0, 0)).$$

and, by the induction hypothesis,

$$K_D(z^0, z^0) \geqslant b/[1 - h^2(z^0)]^2.$$

Hence (∗) is verified.

Applying that h is a continuous function, we immediately conclude $\lim K_D(z, z) = \infty$, whenever $\zeta \in \partial D$ which proves our theorem.

Remarks. We like to mention that this result is also formulated in [1], Theorem 1, 11, but the arguments given there are incomplete.

We close this note by posing the following question:

Let $\tilde{h}: C^n \to [0, \infty)$ be a plurisubharmonic function with $\tilde{h}(\lambda z)$ $= |\lambda| \tilde{h}(z)$ $(z \in C^n, \lambda \in C)$ such that $\tilde{h}$ vanishes on a dense subset of C^n (cf. [8]). We define $D = D_h := \{z \in C^n: h(z) < 1\}$, where $h(z) := \tilde{h}(z) + \|z\|$. Hence D is a pseudoconvex, complete circular domain with $\hat{D} = B$, B the open Euclidean unit ball in C^n, and its Minkowski functional is far away from being continuous. Therefore we ask:

Is D also Bergman complete?

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> [7] A. James and R. Hill, An index formula, J. Differential Equations 15 (1976), 197–211.
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